Karadeniz Fen Bilimleri Dergisi, 13(2), 547-560, 2023. DOI: 10.31466/kfbd.1239446

KFBD

Karadeniz Fen Bilimleri Dergisi

The Black Sea Journal of Sciences

ISSN (Online): 2564-7377 https://dergipark.org.tr/tr/pub/kfbd

Araştırma Makalesi / Research Article

Unit-Weibull Distribution: Different Method of Estimations

Hasan Hüseyin GÜL[1*]

Abstract

Recently, the unit-Weibull (UW) distribution is used quite effectively in analyzing lifetime data. The main goal of this article is to investigate the performance of seven estimation methods, namely maximum likelihood (ML), least square (LS), weighted least square (WLS), Anderson-Darling (AD), right-tail Anderson-Darling (RAD), Cramer-von-Mises (CVM) and percentile (PCE) for parameter estimation. An extensive Monte Carlo simulation study is considered to compare the performances of these methods through biases and mean square errors (MSEs). The numerical results show that the PCE estimator has significantly smaller MSE value for different sample sizes and parameter values in most cases. In addition, the ML and LS estimators have lower bias values than the other estimators in general. Finally, a real data set is presented for illustrative purposes.

Keywords: Unit-Weibull distribution, Estimation methods, Maximum likelihood, Anderson-Darling, Least square.

Unit-Weibull Dağılımı: Tahmin Metotları

Öz

Son zamanlarda Unit-Weibull (UW) dağılımı yaşam zamanı verilerin analizinde oldukça etkin bir şekilde kullanılmaktadır. Bu makalenin temel amacı, en çok olabilirlik (ML), en küçük kareler (LS), ağırlıklı en küçük kareler (WLS), Anderson-Darling (AD), sağ kuyruklu Anderson-Darling (RAD), Cramer-von-Mises (CVM) and percentile (PCE) olmak üzere yedi tahmin yönteminin performansını karşılaştırmaktır. Bu yöntemlerin performanslarını yan ve hata kare ortalaması (MSE'ler) aracılığıyla karşılaştırmak için kapsamlı bir Monte Carlo simülasyon çalışması düşünülmüştür. Sayısal sonuçlar, PCE tahmin edicisinin çoğu durumda farklı örneklem büyüklükleri ve parametre değerleri için önemli ölçüde daha küçük MSE değerine sahip olduğunu göstermektedir. Ayrıca ML ve LS tahmin edicileri genel olarak diğer tahmin edicilere göre daha düşük yan değerlerine sahiptir. Son olarak, açıklama amacıyla gerçek bir veri seti sunulmuştur.

Anahtar Kelimeler: Unit-Weibull dağılımı, Tahmin metotları, En çok olabilirlik, Anderson-Darling, En küçük kareler.

[1]Giresun University, Faculty of Arts and Science, Department of Statistics, Giresun, Turkey, hasan.huseyin@giresun.edu.tr

[1]https://orcid.org/0000-0003-4280-4800

[*]**Sorumlu Yazar/Corresponding Author**

Geliş/Received: 19.01.2023 **Kabul/Accepted:** 18.04.2023

1. Introduction

The traditional Weibull distribution was originally proposed by Waloddi Weibull (1951), which includes exponential and Rayleigh as special cases. It is a popular distribution and is widely used in many fields such as engineering, quality control, medicine and biological applications, meteorology, physics and others, see Rinne (2008). The Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$ has a probability density function (pdf) and a cumulative distribution function (cdf) respectively given by

$$f(x;\alpha,\beta) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\} \tag{1}$$

$$F(x;\alpha,\beta) = 1 - exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\} \tag{2}$$

where α and β are the shape and scale parameters of the Weibull distribution, respectively.

However, this distribution does not exhibit bathtub or unimodal shapes. Many researchers have studied generalizations and modifications of the Weibull distribution to overcome this shortcoming, in the recent past. For example; the exponentiated Weibull by Mudholkar and Srivastava (1993), the extended Weibull by Marshall and Olkin (1997), the modified Weibull by Xie et al. (2002), the beta Weibull by Lee et al. (2007), the Kumaraswamy Weibull by Corderio et al. (2010), the beta modified Weibull by Silva et al. (2010), the truncated Weibull by Zhang and Xie (2011), the beta generalized Weibull by Singla et al. (2012), the transmuted modified Weibull by Khan and King (2013), the transmuted Weibull by Khan et al. (2017).

In many practical applications, a continuous distribution with a bounded interval is needed to describe the uncertainty of a bounded phenomenon. This interval happens to be (0,1) which is called the unit interval. Beta, uniform, Kumaraswamy and Topp-Leone are some well-known distributions having supports in (0,1). Unit distributions are obtained by applying the transformation of the type $X = e^{-Y}$, where Y is the baseline distribution and X is the newly obtained distribution with support on (0,1). In recent years, several works have been introduced as new distributions on the unit interval. Notable studies related to unit distributions are as follows: Mazucheli et al. (2018) proposed unit-Gamma distribution, Mazucheli et al. (2018) introduced unit-Birnbaum-Saunders distribution, Mazucheli et al. (2018) suggested unit-Weibull distribution, Mazucheli et al. (2019) studied unit-Gompertz distribution, Mazucheli et al. (2019) proposed unit-Lindley distribution.

The main aim of this article is to study how the different estimators of the unknown parameters of unit-Weibull (UW) distribution behave for different parameter values and sample sizes. The pdf and cdf of UW distribution, respectively, are expressed as

$$f(x;\alpha,\beta)=\frac{1}{x}\alpha\beta(-logx)^{\beta-1}exp\{-\alpha(-logx)^{\beta}\} \quad (3)$$

$$F(x;\alpha,\beta)=exp\{-\alpha(-logx)^{\beta}\} \quad (4)$$

where $0<x<1$ and $\alpha,\beta>0.$

In this article, the maximum likelihood estimators (MLE), least-squares estimators (LSE) and weighted least-squares estimators (WLSE), estimators based on percentile (PCE), method of Cramer-von-Mises (CVM) and the methods of Anderson-Darling (AD) and right-tail Anderson-Darling (RAD) are considered. An extensive Monte Carlo simulation study is carried out to evaluate the performances of the estimators. A real data set is used as an example for illustrative purposes.

The remaining sections go as follows. Section 2 describes seven methods of estimation. The Monte Carlo simulation results are provided in Section 3. Section 4 presents an illustrative example based on a real data set to find the seven estimators for the unknown parameters of UW distribution. Finally, Section 5 offers some concluding remarks.

2. Materials and Methods

2.1. Method of ML

Let $x_1,x_2,\dots,x_n$ be random sample from UW distribution with pdf (3), then the log-likelihood function, L, is given as follows:

$$L=n(log\alpha+log\beta)-\sum_{i=1}^{n}logx_i+(\beta-1)\sum_{i=1}^{n}log(-logx_i)-\alpha\sum_{i=1}^{n}(-logx_i)^{\beta}. \quad (5)$$

The normal equations become:

$$\frac{dL}{d\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}(-logx_i)^{\beta}=0. \quad (6)$$

$$\frac{dL}{d\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}log(-logx_i)-\alpha\sum_{i=1}^{n}(-logx_i)^{\beta}log(-logx_i)=0. \quad (7)$$

The Eq. (6) can be solved algebraically for α, say $\hat{\alpha}(\beta)$, where

$$\hat{\alpha}(\beta)=\frac{n}{\sum_{i=1}^{n}(-logx_i)^{\beta}}. \quad (8)$$

To obtain $\hat{\beta}$, $\hat{\alpha}(\beta)$ is substituted into Eq. (7) and is solved for β. Then, $g(\beta)$ is given by

$$g(\beta)=\frac{n}{\beta}+\sum_{i=1}^{n}log(-logx_i)-\frac{n\sum_{i=1}^{n}(-logx_i)^{\beta}log(-logx_i)}{\sum_{i=1}^{n}(-logx_i)^{\beta}}=0. \quad (9)$$

Therefore, MLE of β, say $\hat{\beta}_{MLE}$, can be obtained by maximizing Eq. (9) with respect to β.

2.2. Methods of LS and WLS

LSE and WLSE were introduced by Swain et al. (1998) to estimate the parameters of the Beta distribution. Assume that $Y_1, Y_2, \dots, Y_n$ is random sample of size n from a distribution function $G(\cdot)$ and $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$ denotes the order statistics of the observed sample. It is known that

$$E\left(G\left(Y_{(i)}\right)\right)=\frac{i}{n+1}.$$

The LS estimators can be obtained by minimizing Eq. (10)

$$\sum_{i=1}^{n}\left(G\left(Y_{(i)}\right)-\frac{i}{n+1}\right)^2, \quad (10)$$

with respect to unknown parameters. The LS estimators of the unknown parameters of UW distribution can be obtained by minimizing Eq. (11)

$$\sum_{i=1}^{n}\left(exp\left\{-\alpha\left(-logy_{(i)}\right)^{\beta}\right\}-\frac{i}{n+1}\right)^2, \quad (11)$$

with respect to α and β.

The WLS estimators can be obtained by minimizing Eq. (12)

$$\sum_{i=1}^{n}w_i\left(G\left(Y_{(i)}\right)-\frac{i}{n+1}\right)^2, \quad (12)$$

with respect to unknown parameters, where

$$w_i=\left(1/\left(V\left(G\left(Y_{(i)}\right)\right)\right)\right)=\frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

Similar to the LS method, the WLS estimators of the unknown parameters of UW distribution can be obtained by minimizing Eq. (13)

$$\sum_{i=1}^{n} w_i \left(exp\left\{-\alpha\left(-logy_{(i)}\right)^{\beta}\right\} - \frac{i}{n+1}\right)^2, \tag{13}$$

with respect to α and β.

2.3. Minimum distance estimators

In this section, three methods that determine the values of parameters that minimize the distance between the estimate of the cdf and the empirical distribution function, CVM, AD and RAD are presented, see D’Agostino and Stephens (1986) and Luceno (2006).

2.3.1. CVM estimator

The CVM estimators can be obtained by minimizing Eq. (14)

$$CVM = \frac{1}{12n} + \sum_{i=1}^{n} \left(G\left(Y_{(i)}\right) - \frac{2i-1}{2n}\right)^2, \tag{14}$$

with respect to unknown parameters. The CVM estimators of the unknown parameters of UW distribution can be obtained by minimizing Eq. (15)

$$CVM = \frac{1}{12n} + \sum_{i=1}^{n} \left(exp\left\{-\alpha\left(-logy_{(i)}\right)^{\beta}\right\} - \frac{2i-1}{2n}\right)^2, \tag{15}$$

with respect to α and β.

2.3.2. AD and RAD estimators

The AD estimators can be obtained by minimizing Eq. (16)

$$AD = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{logG\left(Y_{(i)}\right) + log\left(1 - G\left(Y_{(n+1-i)}\right)\right)\right\}, \tag{16}$$

with respect to unknown parameters. The AD estimators of the unknown parameters of UW distribution can be obtained by minimizing Eq. (17)

$$AD = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)$$

$$\times\left\{log\left(exp\left\{-\alpha\left(-logy_{(i)}\right)^{\beta}\right\}\right)+log\left(1-exp\left\{-\alpha\left(-logy_{(n+1-i)}\right)^{\beta}\right\}\right)\right\}, \quad (17)$$

with respect to α and β.

The RAD estimators can be obtained by minimizing Eq. (18)

$$RAD=\frac{n}{2}-2\sum_{i=1}^{n}G\left(Y_{(i)}\right)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\,log\left(1-G\left(Y_{(i)}\right)\right), \quad (18)$$

with respect to unknown parameters. Similar to the CVM method, the RAD estimators of the unknown parameters of UW distribution can be obtained by minimizing Eq. (19)

$$RAD=\frac{n}{2}-2\sum_{i=1}^{n}exp\left\{-\alpha\left(-logy_{(i)}\right)^{\beta}\right\}-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\,log\left(1-exp\left\{-\alpha\left(-logy_{(i)}\right)^{\beta}\right\}\right), \quad (19)$$

with respect to α and β.

2.4. Estimators based on percentile

The estimator based on percentile (PCE) was originally proposed by Kao (1958, 1959) to estimate the parameters by comparing the sample points with the theoretical points. The cdf and ln-cdf of UW distribution are given as follows

$$F(x;\alpha,\beta)=exp\{-\alpha(-logx)^{\beta}\},$$

$$ln[F(x;\alpha,\beta)]=-\alpha(-logx)^{\beta}.$$

Suppose $X_{(1)}<X_{(2)}<\cdots<X_{(n)}$ be the order statistics obtained by UW distribution. If p_i denotes some estimate of $F(x_{(i)};\alpha,\beta)$, then the estimators of α and β can be obtained by minimizing Eq. (20)

$$\sum_{i=1}^{n}\left(ln(p_i)-\alpha\left(-logx_{(i)}\right)^{\beta}\right)^{2}, \quad (20)$$

with respect to α and β.

3. Findings and Discussion

Since it is difficult to compare the theoretical performances of the estimators introduced in Section 2, the results are presented numerically. So, an extensive Monte Carlo simulation study is carried out to compare the performances of the various estimators (MLE, LS, WLS, CVM, AD, RAD and PCE) for different sample sizes and different parameter values with respect to their biases and mean square errors (MSEs). The data is generated from the UW distribution by taking $X = e^{-Y}$, where $Y \sim Weibull(\alpha, \beta)$. The generated samples are of size $n = 10, 20, 30, 40, 50$, from the UW distribution with parameters $\alpha, \beta = 0.5, 1.0, 2.0$. The biases and MSEs were calculated over 10000 replications for different cases. The results are reported in Tables 1-8.

From Tables 1-8, the conclusions are summarized as follows:

i) According to the bias comparisons of the estimators:

- Some of the points are quite clear from Tables 1-3 that the ML estimator presented lower bias for α, whereas the LS estimator has lower bias for β.
- From Tables 4-5, it is observed that when *n*=10, the PCE method for α and the WLS method for β have a lower bias, whereas the WLS method for α and the LS method for β have a lower bias when *n*=20. The LS method shows the best performance as the sample size *n* increases.
- The LS estimator works the best in all the cases considered for estimating both α and β, in Table 6 and Table 8.
- It is evident from Table 7 that the CVM estimator presented lower bias for α, while the LS estimator has lower bias for β.

ii) According to the bias comparisons of the estimators:

- The method of PCE shows the best performance among the other methods with the smallest MSE in all cases in Tables 4, 5, 6 and 8.
- From Tables 1-3 it can be concluded that the ML estimator has the smallest MSE values for α. While the PCE estimator has lower MSE values at the small sample sizes, the efficiency of the ML estimator increases as the sample sizes increase.
- From Table 7, it is observed that the CVM estimator has lower MSE for α, whereas the PCE estimator has lower MSE for β.

Note that, as the sample size increases, the biases and MSEs decrease. It verifies the consistency of all the estimators.

Table 1. Simulation results for $\alpha = 0.5, \beta = 0.5$.

n	Est.	ML	LS	WLS	AD	RAD	CVM	PCE
10	Bias $(\hat{\alpha})$	0.0080	0.0392	0.0416	0.0282	0.0551	0.0205	0.0680
	MSE $(\hat{\alpha})$	0.0546	0.0632	0.0555	0.0518	0.2369	0.0941	0.0156
	Bias $(\hat{\beta})$	0.0831	0.0057	0.0085	0.0283	0.0663	0.0989	0.0668
	MSE $(\hat{\beta})$	0.0369	0.0409	0.0363	0.0276	0.0601	0.0770	0.0173
20	Bias $(\hat{\alpha})$	0.0030	0.0195	0.0168	0.0139	0.0158	0.0069	0.0474
	MSE $(\hat{\alpha})$	0.0244	0.0244	0.0250	0.0240	0.0267	0.0269	0.0061
	Bias $(\hat{\beta})$	0.0377	0.0017	0.0049	0.0123	0.0297	0.0407	0.0530
	MSE $(\hat{\beta})$	0.0120	0.0143	0.0124	0.0106	0.0174	0.0192	0.0096
30	Bias $(\hat{\alpha})$	0.0016	0.0132	0.0120	0.0091	0.0123	0.0043	0.0388
	MSE $(\hat{\alpha})$	0.0154	0.0161	0.0158	0.0158	0.0168	0.0171	0.0033
	Bias $(\hat{\beta})$	0.0246	0.0009	0.0039	0.0082	0.0161	0.0264	0.0443
	MSE $(\hat{\beta})$	0.0068	0.0086	0.0074	0.0068	0.0099	0.0105	0.0068
40	Bias $(\hat{\alpha})$	0.0008	0.0110	0.0071	0.0055	0.0090	0.0042	0.0341
	MSE $(\hat{\alpha})$	0.0114	0.0119	0.0117	0.0117	0.0116	0.0125	0.0024
	Bias $(\hat{\beta})$	0.0177	0.0007	0.0032	0.0059	0.0121	0.0184	0.0396
	MSE $(\hat{\beta})$	0.0046	0.0061	0.0054	0.0048	0.0070	0.0071	0.0054
50	Bias $(\hat{\alpha})$	0.0004	0.0081	0.0057	0.0054	0.0063	0.0026	0.0306
	MSE $(\hat{\alpha})$	0.0090	0.0095	0.0092	0.0092	0.0098	0.0098	0.0018
	Bias $(\hat{\beta})$	0.0139	0.0005	0.0029	0.0048	0.0100	0.0141	0.0355
	MSE $(\hat{\beta})$	0.0036	0.0049	0.0041	0.0039	0.0057	0.0055	0.0044

Table 2. Simulation results for $\alpha = 0.5, \beta = 1.0$.

n	Est.	ML	LS	WLS	AD	RAD	CVM	PCE
10	Bias $(\hat{\alpha})$	0.0091	0.0621	0.0376	0.0304	0.0641	0.0230	0.0646
	MSE $(\hat{\alpha})$	0.0564	0.0638	0.0552	0.0497	0.7967	0.0897	0.0140
	Bias $(\hat{\beta})$	0.1701	0.0104	0.0148	0.0523	0.1342	0.1940	0.1373
	MSE $(\hat{\beta})$	0.1526	0.1587	0.1477	0.1056	0.2332	0.2877	0.0697
20	Bias $(\hat{\alpha})$	0.0034	0.0241	0.0162	0.0146	0.0188	0.0084	0.0468
	MSE $(\hat{\alpha})$	0.5034	0.0246	0.0238	0.0243	0.0276	0.0271	0.0058
	Bias $(\hat{\beta})$	0.0760	0.0045	0.0126	0.0248	0.0551	0.0770	0.1034
	MSE $(\hat{\beta})$	0.0475	0.0540	0.0515	0.0429	0.0685	0.0727	0.0378
30	Bias $(\hat{\alpha})$	0.0008	0.0128	0.0090	0.0081	0.0128	0.0040	0.0379
	MSE $(\hat{\alpha})$	0.0158	0.0162	0.0154	0.0157	0.0175	0.0173	0.0031
	Bias $(\hat{\beta})$	0.0504	0.0010	0.0101	0.0150	0.0388	0.0517	0.0891
	MSE $(\hat{\beta})$	0.0283	0.0347	0.0309	0.0274	0.0421	0.0422	0.0269
40	Bias $(\hat{\alpha})$	0.0007	0.0089	0.0077	0.0082	0.0080	0.0021	0.0339
	MSE $(\hat{\alpha})$	0.0113	0.0119	0.0119	0.0115	0.0125	0.0125	0.0024
	Bias $(\hat{\beta})$	0.0365	0.0011	0.0077	0.0109	0.0240	0.0374	0.0787
	MSE $(\hat{\beta})$	0.0186	0.0240	0.0215	0.0192	0.0283	0.0279	0.0214
50	Bias $(\hat{\alpha})$	0.0002	0.0071	0.0071	0.0052	0.0076	0.0100	0.0308
	MSE $(\hat{\alpha})$	0.0092	0.0096	0.0092	0.0092	0.0095	0.0102	0.0018
	Bias $(\hat{\beta})$	0.0283	0.0009	0.0052	0.0096	0.0190	0.0293	0.0702
	MSE $(\hat{\beta})$	0.0144	0.0190	0.0165	0.0156	0.0218	0.0214	0.0173

Table 3. Simulation results for $\alpha = 0.5, \beta = 2.0$.

n	Est.	ML	LS	WLS	AD	RAD	CVM	PCE
10	Bias $(\hat{\alpha})$	0.0096	0.0398	0.0415	0.0294	0.0449	0.0219	0.0663
	MSE $(\hat{\alpha})$	0.0584	0.0657	0.0597	0.0544	0.0949	0.1162	0.0144

	Bias $(\hat{\beta})$	0.3425	0.0237	0.0308	0.1067	0.2802	0.4018	0.2764
	MSE $(\hat{\beta})$	0.6003	0.6494	0.5767	0.4209	0.9421	1.2094	0.2837
20	Bias $(\hat{\alpha})$	0.0045	0.0213	0.0185	0.0145	0.0217	0.0090	0.0464
	MSE $(\hat{\alpha})$	0.0250	0.0251	0.0135	0.0236	0.0287	0.0279	0.0055
	Bias $(\hat{\beta})$	0.1541	0.0031	0.0243	0.0472	0.1131	0.1685	0.2089
	MSE $(\hat{\beta})$	0.1992	0.2339	0.2040	0.1726	0.2835	0.3213	0.1526
30	Bias $(\hat{\alpha})$	0.0028	0.0115	0.0117	0.0084	0.0097	0.0026	0.0383
	MSE $(\hat{\alpha})$	0.0154	0.0157	0.0154	0.0154	0.0164	0.0168	0.0032
	Bias $(\hat{\beta})$	0.1003	0.0025	0.0106	0.0297	0.0692	0.1027	0.1784
	MSE $(\hat{\beta})$	0.1106	0.1358	0.1199	0.1085	0.1610	0.1652	0.1044
40	Bias $(\hat{\alpha})$	0.0017	0.0077	0.0087	0.0065	0.0079	0.0031	0.0344
	MSE $(\hat{\alpha})$	0.0112	0.0118	0.0117	0.0118	0.0118	0.0124	0.0024
	Bias $(\hat{\beta})$	0.0736	0.0033	0.0094	0.0209	0.0513	0.0746	0.1542
	MSE $(\hat{\beta})$	0.0773	0.1005	0.0840	0.0786	0.1120	0.1165	0.0835
50	Bias $(\hat{\alpha})$	0.0016	0.0086	0.0083	0.0059	0.0094	0.0021	0.0309
	MSE $(\hat{\alpha})$	0.0093	0.0099	0.0092	0.0091	0.0096	0.0102	0.0018
	Bias $(\hat{\beta})$	0.0563	0.0016	0.0086	0.0124	0.0359	0.0602	0.1380
	MSE $(\hat{\beta})$	0.0590	0.0796	0.0644	0.0585	0.0879	0.0896	0.0679

Table 4. Simulation results for $\alpha = 1.0, \beta = 0.5$.

n	Est.	ML	LS	WLS	AD	RAD	CVM	PCE
10	Bias $(\hat{\alpha})$	0.1094	0.0683	0.0676	0.0714	0.2526	0.2070	0.0558
	MSE $(\hat{\alpha})$	0.2647	0.2947	0.3760	0.2615	3.4638	6.4504	0.0724
	Bias $(\hat{\beta})$	0.0872	0.0097	0.0088	0.0308	0.0677	0.1040	0.0151
	MSE $(\hat{\beta})$	0.0390	0.0408	0.0363	0.0285	0.0579	0.0839	0.0112
20	Bias $(\hat{\alpha})$	0.0418	0.0253	0.0222	0.0293	0.0675	0.0581	0.0515
	MSE $(\hat{\alpha})$	0.0777	0.0771	0.0791	0.0698	0.1310	0.1083	0.0479
	Bias $(\hat{\beta})$	0.0395	0.0019	0.0040	0.0112	0.0290	0.0431	0.0098
	MSE $(\hat{\beta})$	0.0124	0.0147	0.0126	0.0105	0.0179	0.0202	0.0040
30	Bias $(\hat{\alpha})$	0.0239	0.0115	0.0164	0.0148	0.0411	0.0298	0.0417
	MSE $(\hat{\alpha})$	0.0447	0.0450	0.0449	0.0424	0.0624	0.0532	0.0350
	Bias $(\hat{\beta})$	0.0228	0.0025	0.0061	0.0079	0.0171	0.0237	0.0173
	MSE $(\hat{\beta})$	0.0066	0.0087	0.0077	0.0066	0.0102	0.0105	0.0019
40	Bias $(\hat{\alpha})$	0.0179	0.0113	0.0136	0.0151	0.0304	0.0245	0.0367
	MSE $(\hat{\alpha})$	0.0318	0.0335	0.0325	0.0311	0.0415	0.0379	0.0284
	Bias $(\hat{\beta})$	0.0179	0.0011	0.0023	0.0063	0.0132	0.0196	0.0191
	MSE $(\hat{\beta})$	0.0048	0.0064	0.0052	0.0048	0.0071	0.0074	0.0013
50	Bias $(\hat{\alpha})$	0.0143	0.0087	0.0095	0.0120	0.0226	0.0189	0.0366
	MSE $(\hat{\alpha})$	0.0242	0.0255	0.0247	0.0248	0.0323	0.0281	0.0226
	Bias $(\hat{\beta})$	0.0139	0.0009	0.0024	0.0045	0.0098	0.0145	0.0198
	MSE $(\hat{\beta})$	0.0036	0.0049	0.0041	0.0038	0.0055	0.0055	0.0008

Table 5. Simulation results for $\alpha = 1.0, \beta = 1.0$.

n	Est.	ML	LS	WLS	AD	RAD	CVM	PCE
10	Bias $(\hat{\alpha})$	0.1064	0.0805	0.0653	0.0725	0.1810	0.1886	0.0599
	MSE $(\hat{\alpha})$	0.2484	0.6826	0.5010	0.2060	0.6175	0.8854	0.0727
	Bias $(\hat{\beta})$	0.1703	0.0158	0.0149	0.0608	0.1339	0.2023	0.0281
	MSE $(\hat{\beta})$	0.1485	0.1599	0.1404	0.1096	0.2293	0.3073	0.0429
20	Bias $(\hat{\alpha})$	0.0407	0.0236	0.0223	0.0301	0.0701	0.0554	0.0507

	MSE ($\hat{\alpha}$)	0.0753	0.0741	0.0705	0.0694	0.1268	0.0991	0.0473
	Bias ($\hat{\beta}$)	0.0739	0.0023	0.0091	0.0262	0.0544	0.0793	0.0171
	MSE ($\hat{\beta}$)	0.0476	0.0580	0.0499	0.0417	0.0699	0.0780	0.0171
30	Bias ($\hat{\alpha}$)	0.0216	0.0106	0.0188	0.0176	0.0402	0.0290	0.0400
	MSE ($\hat{\alpha}$)	0.0450	0.0452	0.0437	0.0433	0.0587	0.0536	0.0347
	Bias ($\hat{\beta}$)	0.0520	0.0017	0.0070	0.0146	0.0328	0.0546	0.0332
	MSE ($\hat{\beta}$)	0.0277	0.0344	0.0292	0.0261	0.0396	0.0423	0.0085
40	Bias ($\hat{\alpha}$)	0.0183	0.0111	0.0123	0.0107	0.0340	0.0244	0.0382
	MSE ($\hat{\alpha}$)	0.0313	0.0333	0.0311	0.0303	0.0428	0.0379	0.0266
	Bias ($\hat{\beta}$)	0.0369	0.0013	0.0066	0.0128	0.0288	0.0394	0.0393
	MSE ($\hat{\beta}$)	0.0189	0.0246	0.0212	0.0194	0.0288	0.0286	0.0043
50	Bias ($\hat{\alpha}$)	0.0165	0.0107	0.0122	0.0094	0.0238	0.0210	0.0212
	MSE ($\hat{\alpha}$)	0.0250	0.0262	0.0250	0.0244	0.0312	0.0289	0.0223
	Bias ($\hat{\beta}$)	0.0267	0.0009	0.0062	0.0112	0.0212	0.0270	0.0404
	MSE ($\hat{\beta}$)	0.0144	0.0196	0.0169	0.0154	0.0225	0.0219	0.0032

Table 6. Simulation results for $\alpha = 1.0, \beta = 2.0$.

n	Est.	ML	LS	WLS	AD	RAD	CVM	PCE
10	Bias ($\hat{\alpha}$)	0.1098	0.0522	0.0633	0.0691	0.2189	0.1578	0.0585
	MSE ($\hat{\alpha}$)	0.3175	0.2818	0.3613	0.1886	0.7589	0.8427	0.0728
	Bias ($\hat{\beta}$)	0.3398	0.0159	0.0365	0.1192	0.2730	0.3917	0.0681
	MSE ($\hat{\beta}$)	0.6181	0.5588	0.6025	0.4271	0.9003	1.1010	0.1890
20	Bias ($\hat{\alpha}$)	0.0379	0.0195	0.0273	0.0308	0.0712	0.0507	0.0476
	MSE ($\hat{\alpha}$)	0.0755	0.0731	0.0778	0.0696	0.1215	0.0974	0.0491
	Bias ($\hat{\beta}$)	0.1630	0.0053	0.0118	0.0534	0.1063	0.1693	0.0327
	MSE ($\hat{\beta}$)	0.1989	0.2284	0.2051	0.1719	0.2736	0.3097	0.0700
30	Bias ($\hat{\alpha}$)	0.0248	0.0152	0.0195	0.0177	0.0380	0.0340	0.0395
	MSE ($\hat{\alpha}$)	0.0451	0.0479	0.0455	0.0418	0.0629	0.0574	0.0355
	Bias ($\hat{\beta}$)	0.1033	0.0041	0.0142	0.0295	0.0628	0.1098	0.0654
	MSE ($\hat{\beta}$)	0.1121	0.1381	0.1220	0.1057	0.1627	0.1695	0.0332
40	Bias ($\hat{\alpha}$)	0.0144	0.0065	0.0123	0.0129	0.0323	0.0194	0.0381
	MSE ($\hat{\alpha}$)	0.0323	0.0335	0.0311	0.0302	0.0410	0.0378	0.0273
	Bias ($\hat{\beta}$)	0.0741	0.0013	0.0109	0.0763	0.0509	0.0765	0.0777
	MSE ($\hat{\beta}$)	0.0772	0.1007	0.0845	0.0767	0.1119	0.1166	0.0194
50	Bias ($\hat{\alpha}$)	0.0152	0.0078	0.0112	0.0115	0.0238	0.0179	0.0331
	MSE ($\hat{\alpha}$)	0.0243	0.0253	0.0253	0.0249	0.0314	0.0279	0.0226
	Bias ($\hat{\beta}$)	0.0542	0.0010	0.0138	0.0155	0.0391	0.0519	0.0804
	MSE ($\hat{\beta}$)	0.0596	0.0785	0.0663	0.0589	0.0895	0.0872	0.0134

Table 7. Simulation results for $\alpha = 2.0, \beta = 0.5$.

n	Est.	ML	LS	WLS	AD	RAD	CVM	PCE
10	Bias ($\hat{\alpha}$)	0.5156	0.2306	0.3062	0.2816	0.5736	0.2070	0.1488
	MSE ($\hat{\alpha}$)	2.7325	4.2459	4.5032	2.3505	4.1321	6.4504	0.2112
	Bias ($\hat{\beta}$)	0.0812	0.0126	0.0073	0.0285	0.0404	0.1040	0.0141
	MSE ($\hat{\beta}$)	0.0361	0.0392	0.0359	0.0279	0.0306	0.0839	0.0068
20	Bias ($\hat{\alpha}$)	0.2064	0.0836	0.0978	0.1086	0.2663	0.0581	0.1032
	MSE ($\hat{\alpha}$)	0.4845	0.6521	0.5922	0.3896	2.3620	0.1083	0.1205
	Bias ($\hat{\beta}$)	0.0383	0.0011	0.0070	0.0114	0.0262	0.0431	0.0152
	MSE ($\hat{\beta}$)	0.0124	0.0151	0.0127	0.0104	0.0171	0.0202	0.0045
30	Bias ($\hat{\alpha}$)	0.1249	0.0387	0.0502	0.0632	0.1501	0.0298	0.0804

	MSE $(\hat{\alpha})$	0.2242	0.2685	0.2662	0.2087	0.4419	0.0532	0.0837
	Bias $(\hat{\beta})$	0.0232	0.0024	0.0039	0.0069	0.0174	0.0237	0.0162
	MSE $(\hat{\beta})$	0.0067	0.0082	0.0075	0.0065	0.0101	0.0105	0.0035
40	Bias $(\hat{\alpha})$	0.0930	0.0352	0.0322	0.0478	0.1014	0.0245	0.0675
	MSE $(\hat{\alpha})$	0.1477	0.1850	0.1526	0.1454	0.2778	0.0379	0.0654
	Bias $(\hat{\beta})$	0.0176	0.0016	0.0032	0.0059	0.0105	0.0196	0.0164
	MSE $(\hat{\beta})$	0.0048	0.0063	0.0052	0.0049	0.0069	0.0074	0.0030
50	Bias $(\hat{\alpha})$	0.0704	0.0184	0.0365	0.0439	0.0921	0.0189	0.0526
	MSE $(\hat{\alpha})$	0.1099	0.1309	0.1208	0.1123	0.2105	0.0281	0.0516
	Bias $(\hat{\beta})$	0.0141	0.0009	0.0040	0.0052	0.0111	0.0145	0.0153
	MSE $(\hat{\beta})$	0.0036	0.0049	0.0041	0.0038	0.0056	0.0055	0.0026

Table 8. Simulation results for $\alpha = 2.0, \beta = 1.0$.

n	Est.	ML	LS	WLS	AD	RAD	CVM	PCE
10	Bias $(\hat{\alpha})$	0.2897	0.0850	0.1641	0.1732	0.4177	0.4137	0.1525
	MSE $(\hat{\alpha})$	0.7443	0.7806	1.5188	0.8192	3.3774	2.0900	0.2079
	Bias $(\hat{\beta})$	0.1213	0.0050	0.0111	0.0377	0.0870	0.0879	0.0298
	MSE $(\hat{\beta})$	0.0787	0.0984	0.0801	0.0551	0.1122	0.1026	0.0264
20	Bias $(\hat{\alpha})$	0.1966	0.0701	0.0872	0.1162	0.2935	0.2728	0.0993
	MSE $(\hat{\alpha})$	0.4524	0.5798	0.5766	0.3798	1.2874	1.0939	0.1176
	Bias $(\hat{\beta})$	0.0737	0.0031	0.0109	0.0280	0.0574	0.0786	0.0308
	MSE $(\hat{\beta})$	0.0471	0.0566	0.0505	0.0438	0.0687	0.0763	0.0180
30	Bias $(\hat{\alpha})$	0.1250	0.0413	0.0524	0.0705	0.1540	0.1594	0.0769
	MSE $(\hat{\alpha})$	0.2288	0.2768	0.2406	0.2117	0.5138	0.3878	0.0826
	Bias $(\hat{\beta})$	0.0498	0.0018	0.0056	0.0149	0.0338	0.0528	0.0311
	MSE $(\hat{\beta})$	0.0279	0.0356	0.0297	0.0270	0.0411	0.0435	0.0142
40	Bias $(\hat{\alpha})$	0.0881	0.0257	0.0440	0.0490	0.1166	0.1094	0.0603
	MSE $(\hat{\alpha})$	0.1481	0.1815	0.1561	0.1430	0.2846	0.2303	0.0628
	Bias $(\hat{\beta})$	0.0361	0.0013	0.0081	0.0124	0.0261	0.0376	0.0307
	MSE $(\hat{\beta})$	0.0191	0.0255	0.0218	0.0197	0.0282	0.0295	0.0120
50	Bias $(\hat{\alpha})$	0.0698	0.0218	0.0307	0.0442	0.0863	0.0871	0.0550
	MSE $(\hat{\alpha})$	0.1086	0.1341	0.1227	0.1134	0.2024	0.1621	0.0524
	Bias $(\hat{\beta})$	0.0282	0.0020	0.0034	0.0098	0.0195	0.0288	0.0313
	MSE $(\hat{\beta})$	0.0142	0.0191	0.0163	0.0154	0.0216	0.0215	0.0105

3.1. Applications of real data analysis

In this section, two real data sets are used to illustrate the proposed estimation methods presented in the previous sections. The first data set is available from Dumonceaux and Antle (1973) and refer to 20 observations of the maximum flood level for Susquehanna River at Harrisburg, Pennsylvania. The second data set can be found in Mazucheli et al. (2018), it represents 48 observations obtained from 12 core samples from petroleum reservoirs that were sampled by 4 cross-sections. Both data sets are given in Table 9.

Table 9. Flood level data and Petroleum reservoirs data.

Data Set I
0.26, 0.27, 0.30, 0.32, 0.32, 0.34, 0.38, 0.38, 0.39, 0.40, 0.41, 0.42, 0.42, 0.42, 0.45, 0.48, 0.49, 0.61, 0.65, 0.74
Data Set II
0.09, 0.11, 0.12, 0.12, 0.13, 0.14, 0.15, 0.15, 0.15, 0.15, 0.15, 0.16, 0.16, 0.16, 0.16, 0.17, 0.17, 0.18, 0.18, 0.18, 0.18, 0.19, 0.19, 0.20, 0.20, 0.20, 0.20, 0.20, 0.20, 0.23, 0.23, 0.23, 0.23, 0.24, 0.25, 0.26, 0.26, 0.28, 0.28, 0.28, 0.29, 0.31, 0.33, 0.33, 0.34, 0.42, 0.44, 0.46

The Kolmogorov-Smirnov (KS) test statistic is applied to confirm which estimator of α and β makes the UW distribution fits better to these data sets. The values of estimators of α and β and the KS test statistics for UW distribution with different estimators are reported in Table 10.

Table 10. The values of estimators of α and β and the KS test statistics for UW distribution with different estimators.

	Data Set I			Data Set II		
Method	$\hat{\alpha}$	$\hat{\beta}$	KS	$\hat{\alpha}$	$\hat{\beta}$	KS
ML	1.0249	3.9036	0.1448	0.0602	5.1131	0.1008
LS	0.9648	3.8769	0.1278	0.0608	5.0620	0.0956
WLS	0.9578	3.7089	0.1319	0.0635	5.0034	0.1011
AD	0.9854	3.8085	0.1364	0.0597	5.1236	0.0994
RTAD	0.9416	3.5112	0.1354	0.0631	4.9934	0.0969
CVM	0.9901	4.2191	0.1225	0.0560	5.2264	0.0943
PCE	0.9129	4.3156	0.1530	0.0728	4.7469	0.1063

This table shows that the values of estimators of α and β are quite close to each other. In terms of KS test statistic, the CVM estimator provides the best fit among the competing estimators for both data sets. In general, the smaller value of the KS test statistic, the better fit to the data.

4. Conclusions and Recommendations

In this study, the UW distribution introduced by Mazucheli et al. (2018) is studied in terms of point estimations. The model parameters are estimated by seven methods of estimation, namely maximum likelihood estimator, least square estimator, weighted least square estimator, Anderson-Darling estimator, right-tail Anderson-Darling estimator, Cramer-von-Mises estimator and estimator based on percentile. A Monte-Carlo simulation study is carried out to compare the performance of seven estimation methods with different sample sizes and parameter values. To show capability of the UW distribution, two real data applications are conducted. The results show that, according to efficiency, the method of PCE performs better than its counterparts and it is followed by the method of ML. In terms of bias, the LS shows better performance than the rest in most cases. Considering all the points above, it is recommended to use the LS estimator in terms of bias and the CVM estimator in terms of MSE for estimating unknown parameters of UW distribution.

Authors' Contributions

The authors contributed equally to the study.

Statement of Conflicts of Interest

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

References

Cordeiro, G. M., Ortega, E. M., and Nadarajah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. *Journal of The Franklin Institute*, 347(8), 1399-1429.

Dumonceaux, R., and Antle, C. E. (1973). Discrimination between the log-normal and the Weibull distributions. *Technometrics*, 15(4), 923-926.

D'Agostino, R. B., and Stephens, M. A. (1986). *Goodness-of-Fit Techniques*. Dekker, New York, NY.

Kao, J. H. (1958). Computer methods for estimating Weibull parameters in reliability studies. *IRE Transactions on Reliability and Quality Control*, 15-22.

Kao, J. H. (1959). A graphical estimation of mixed Weibull parameters in life-testing of electron tubes. *Technometrics*, 1(4), 389-407.

Khan, M. S., and King, R. (2013). Transmuted modified Weibull distribution: A generalization of the modified Weibull probability distribution. *European Journal of Pure and Applied Mathematics*, 6(1), 66-88.

Khan, M. S., King, R., and Hudson, I. L. (2017). Transmuted Weibull distribution: Properties and estimation. *Communications in Statistics-Theory and Methods*, 46(11), 5394-5418.

Lee, C., Famoye, F., and Olumolade, O. (2007). Beta-Weibull distribution: some properties and applications to censored data. *Journal of Modern Applied Statistical Methods*, 6(1), 17.

Luceño, A. (2006). Fitting the generalized Pareto distribution to data using maximum goodness-of-fit estimators. *Computational Statistics & Data Analysis*, 51(2), 904-917.

Marshall, A. W., and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84(3), 641-652.

Mazucheli, J., Menezes, A. F. B., and Dey, S. (2018). Improved maximum-likelihood estimators for the parameters of the unit-gamma distribution. *Communications in Statistics-Theory and Methods*, 47(15), 3767-3778.

Mazucheli, J., Menezes, A. F. B., and Dey, S. (2018). The unit-Birnbaum-Saunders distribution with applications. *Chilean Journal of Statistics*, 9(1), 47-57.

Mazucheli, J., Menezes, A. F. B., and Ghitany, M. E. (2018). The unit-Weibull distribution and associated inference. *Journal of Applied Probability and Statistics*, 13(2), 1-22.

Mazucheli, J., Menezes, A. F. B., and Dey, S. (2019). Unit-Gompertz distribution with applications. *Statistica*, 79(1), 25-43.

Mazucheli, J., Menezes, A. F. B., and Chakraborty, S. (2019). On the one parameter unit-Lindley distribution and its associated regression model for proportion data. *Journal of Applied Statistics*, 46(4), 700-714.

Mudholkar, G. S., and Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. IEEE Transactions on Reliability, 42(2), 299-302.

Rinne, H. (2008). *The Weibull distribution: a handbook*. Chapman and Hall/CRC.

Silva, G. O., Ortega, E. M., and Cordeiro, G. M. (2010). The beta modified Weibull distribution. *Lifetime Data Analysis*, 16(3), 409-430.

Singla, N., Jain, K., and Sharma, S. K. (2012). The beta generalized Weibull distribution: properties and applications. *Reliability Engineering & System Safety*, 102, 5-15.

Swain, J. J., Venkatraman, S., and Wilson, J. R. (1998). Least-squares estimation of distribution functions in Johnson's translation system. *Journal of Statistical Computation and Simulation*, 29(4), 271-297.

Weibull, W. (1951). A statistical distribution function of wide applicability. *Journal of Applied Mechanics*, 18(3), 293-297.

Xie, M., Tang, Y., and Goh, T. N. (2002). A modified Weibull extension with bathtub-shaped failure rate function. *Reliability Engineering & System Safety*, 76(3), 279-285.

Zhang, T., and Xie, M. (2011). On the upper truncated Weibull distribution and its reliability implications. *Reliability Engineering & System Safety*, 96(1), 194-200.