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Research Paper

Synthesis of Four-Bar Linkages by Four Infinitely Close Relative Positions and Pressure Angle

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Abstract: A computer-applicable linear mathematical model has been developed to determine the Burmester curves for infinitely close relative positions (cubic of stationary curvature), which indirectly uses the Carter-Hall circle. By varying a free parameter and using elements of kinematic and analytical geometry, an incomparably simpler solution is achieved than that obtained by the third-degree equations of the Burmester curves for stationary curvature. The mathematical model for the synthesis of four-bar linkages includes and a condition for the pressure angle, whereupon is uniquely defined the kinematic diagram of the mechanism. Of the pressure angle, the reactions of the forces in the kinematic pairs and the force sizing of the mechanism depend. The model would facilitate the engineers in the synthesis of four-bar linkages by generating a function approximating a given function in the vicinity of a given position, where the two functions have four infinitely close common points (3rd-order approximation). An example of the synthesis of a four-bar linkage illustrates the application of the model, which is linear - it includes only equations of straight lines written in Cartesian coordinates, which is why it is convenient for computer calculations.

Keywords: Carter-Hall Circle, Burmester Curves (Cubic of Stationary Curvature), Pressure Angle, Synthesis of Four-Bar Mechanism

1. Introduction

The synthesis of four-bar linkages by given positions is one of the tasks of kinematic geometry, which began its development with the fundamental work of Burmester [1]. Burmester curve for infinitely close four positions (3rd-order derivative approximation) play an important role in the synthesis of linkages. Kotelnikov [2] and Mueller [3] called these curves the Burmester curves in honor of the founder of kinematic geometry. Curves are known as *cubic of stationary curvatures* or *circling-point curves also* [4]. They are circular curves of the third degree with two cyclic points, a double point defined by an instantaneous center of velocity (ICV), for which the polar normal and the polar tangent are tangents to the curves.

Various graphical and graph-analytical methods for the synthesis of these curves are known [5-10]. Analytical methods in polar and Cartesian coordinates have also been developed, focal axes of curves, focal points and asymptotes have been determined [11-16]. Mitchiner and Mobie [17] use the Newton-Rafson method to computer-determine stationary curves.

In a series of articles, Eren K. and Ersoy, S. reveal new possibilities for: revisiting the Burmester theory with complex forms [18]; cardan positions in the Lorentzian plane [19]; Burmester theory in Cayley–Klein planes with affine base [20]; a comparison of original and inverse motion in the Minkowski plane [21].

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The Burmester curves for relative positions on infinitesimal distances (further briefly referred to as Burmester curves only) are uniquely determined by the first three *transfer functions*, the geometric derivatives of the mechanism *displacement function*. These curves are the geometric locations of points on the planes of simple motion units, which determine the positions of the movable centers equivalent to the third derivative four-bar linkages with a common frame.

From Carter's article [22] and from the Hall's comment in the appendix to the article, it is clear that it is possible to synthesize four-bar linkages at set Values of the first three transfer functions, but the use of segment lengths and the introduction of characters after the designer's judgment, makes it difficult to automate computer synthesis [23]. This is evidenced by the opinion of Carter, published in the publication [22]: „The methods of linkage synthesis presented in this paper are based on the collineation-axis equations and are partly graphical in nature. As a consequence, they have certain ads and disadvantage inherent to graphical work. The methods presented here are not suitable for computing linkage design, but are of value as an aid in work and in other synthesis problems". On the other hand, graphical synthesis is very difficult when the relative and/or absolute instantaneous center of velocities is significantly away from the frame. Probably that is why the Carter synthesis method has not gained development and popularity.

The aim of the research is to derive a computer-applied linear mathematical model for the synthesis of four-bar mechanisms in infinitely close four positions and pressure angle [24] by the Burmester curves means, generated through the Carter-Hall circle to achieve an incomparably simpler solution to this generally nonlinear problem.

2. Experimental Procedure and Analysis

In this section, a sample use of the computer-applicable linear mathematical model has been developed to determine the Burmester curves for infinitely close relative positions (cubic of stationary curvature), which indirectly uses the Carter-Hall circle.

2.1. Kinematic Invariants up to the Third Row

With rotating the input and output links, introduced coordinate system *Oxy* with an axis $x \equiv OC$, determining the position of the frame with length $L = l_{OC}$, and set values of geometric transfer functions to the third sequence $\psi'(\varphi)$, $\psi''(\varphi)$ and $\psi'''(\varphi)$, the kinematic invariants are determined (Fig.1):

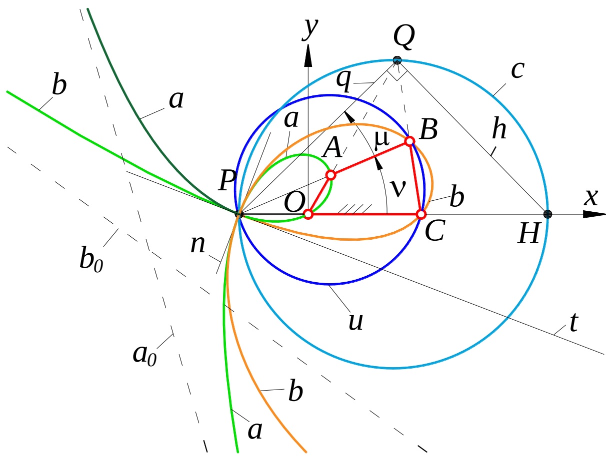

Figure 1. The Burmester curves ***a*** and ***b*** with asymptotes $\boldsymbol{a_0}$ and $\boldsymbol{b_0}$, respectively, obtained by means of the Carter-Hall circle ***c***. Four-bar linkage, synthesized at a given transmission angle $\gamma = \angle ABC$

- the abscissa of the relative ICV *P* (according to the Aronhold-Kennedy and of Willis theorems)

$$x_P = L\psi' / (\psi' - 1)\,; \quad (y_P = 0) \tag{1}$$

located on the *Ox* axis;

- the angle μ locked between the direction of the connecting rod (straight *BP*) and the collineation axis $q \equiv PQ$ according to the of Bobillier (1970) and of Freudenstein theorems [25] (Bobillier's and Freudenstein's theorems) [4] (*Q* is the absolute ICV)

$$tg\,\mu = \psi'\left(1-\psi'\right)/\psi''\,; \tag{2}$$

- the abscissa of the intersection point *H* (x_H, y_H = 0) of the Carter-Hall circle ***c*** (diameter $d_c \equiv l_{PH} = \left|x_H - y_P\right|$) with the *Ox* axis [15,16]

$$x_H = x_P + \frac{3L[\psi'^2(1-\psi')^2 + \psi''^2]}{[\psi''' + \psi'(1-\psi'^2)](1-\psi')^2 + 3\psi''^2(1-\psi')}\,. \tag{3}$$

2.2. Mathematical Model for Determining the Burmester Curves

The mathematical model includes only the abscissa x_H of equation (3) and the equation of lines, without including the equation of the Carter-Hall circle ***c***. A suitable variable parameter is the angular coefficient

$$k_\nu = tg\nu \tag{4}$$

of the rights determining the position of the connecting rod.

The angular coefficient of the collineation axis *q* is

$$k_q = tg(\nu + \mu)\,, \tag{5}$$

where the angle μ is determined by (2).

The Burmester curves are obtained entirely in the range $\nu \in [0, 180°]$. This interval can be significantly reduced if we search for the Burmester curves in a predetermined range β of the allowable maximum value $|\theta_P|$ of the pressure angle θ [20].

The line $h \perp q$ with angular coefficient

$$k_h = k_q^{-1} \tag{6}$$

passes through point *H* and intersects the axis *q* in the absolute ICV *Q* with coordinates

$$x_Q = \left(k_q x_P - k_h x_H\right)/\left(k_q - k_h\right),\; y_Q = k_q\left(x_Q - x_P\right). \tag{7}$$

After calculation the angular coefficients of the lines *OQ* and *CQ*

$$k_{OQ} = y_Q / x_Q \tag{8}$$

$$k_{CQ} = y_Q / (x_Q - x_C) \tag{9}$$

determining the positions of the units with rotational motion, the coordinates are located

$$x_A = k_v x_P / (k_v - k_{OQ}); y_A = k_{OQ} x_A \tag{10}$$

$$x_B = (k_{CQ} L - k_v x_P) / (k_{CQ} - k_v); y_B = k_{CQ}(x_B - x_C) \tag{11}$$

of the centers *A* and *B* of the moving joints as intersections of line passing through the center *P* at an angle *v*, respectively with the lines *OQ* and *CQ*. By changing the parameter *v*, the Burmester curves ***a*** and ***b*** are obtained simultaneously.

These curves can be considered as geometric places in the planes of the two links of the respective relative trajectories of the centers *A* and *B*, whose evolution has cuspate point. The osculating circles have centers in the cuspate points of the evolution of the mentioned relative trajectories.

The final solution for determining the parameters of a four-bar mechanism can be obtained by the intersection of the Carter-Hall circle ***c*** with a circle ***u***. It is the geometric location of points from which the segment *PC* is visible at the transmission angle $\gamma = 90° - |\theta|$, where the pressure angle

$$|\theta| = \left| arctg \frac{1 + k_v k_{OQ}}{k_v - k_{OQ}} \right| \tag{12}$$

has a preset value [20], on which the reactions of the forces in the kinematic pairs and the force dimension of the mechanism depend.

3. Results and Discussion

A computer-applicable linear mathematical model for determining the Burmester curves for infinitesimal close relative positions of four-bar mechanisms is derived, in which the Carter-Hall circle is indirectly used.

An example of the synthesis of a four-bar mechanism illustrates the application of the mathematical model. For this synthesis, the programs MathCAD (for the mathematical model) and AutoCAD (for visualizing the obtained results) were used.

3.1. Example

To be synthesized a four-bar mechanism that approximates the function $\Delta\psi = -0.25\Delta\varphi + 0.5\Delta\varphi^2 - (0.5/6)\Delta\varphi^3$ in the vicinity of the project position $\varphi_P = \varphi_0 + \Delta\varphi_P$ of the mechanism at $\Delta\varphi_P = 1$ *rad* in the interval $\varphi \in (\varphi_o = \varphi_p - \Delta\varphi_p, \varphi_f = \varphi_p + \Delta\varphi_p)$**,** at a pressure angle $|\theta| = 4°$**,** and at $L = x_C = 100$ *mm*. Determine the deviation $\delta\psi = \psi - \psi$ of the generated function ψ

from the set ψ in the interval $\varphi(\varphi_o^*, \varphi_f^*)$ for the boundaries of which $(\delta\psi)_{\max} = 0.5°$ and $(\delta\psi)_{\max} = 0.25°$.

3.2. Synthesis Results Based on the Developed Mathematical Model

As said above mathematical model is linear - it includes only equations of straight lines written in Cartesian coordinates, which is why it is convenient for computer calculations.

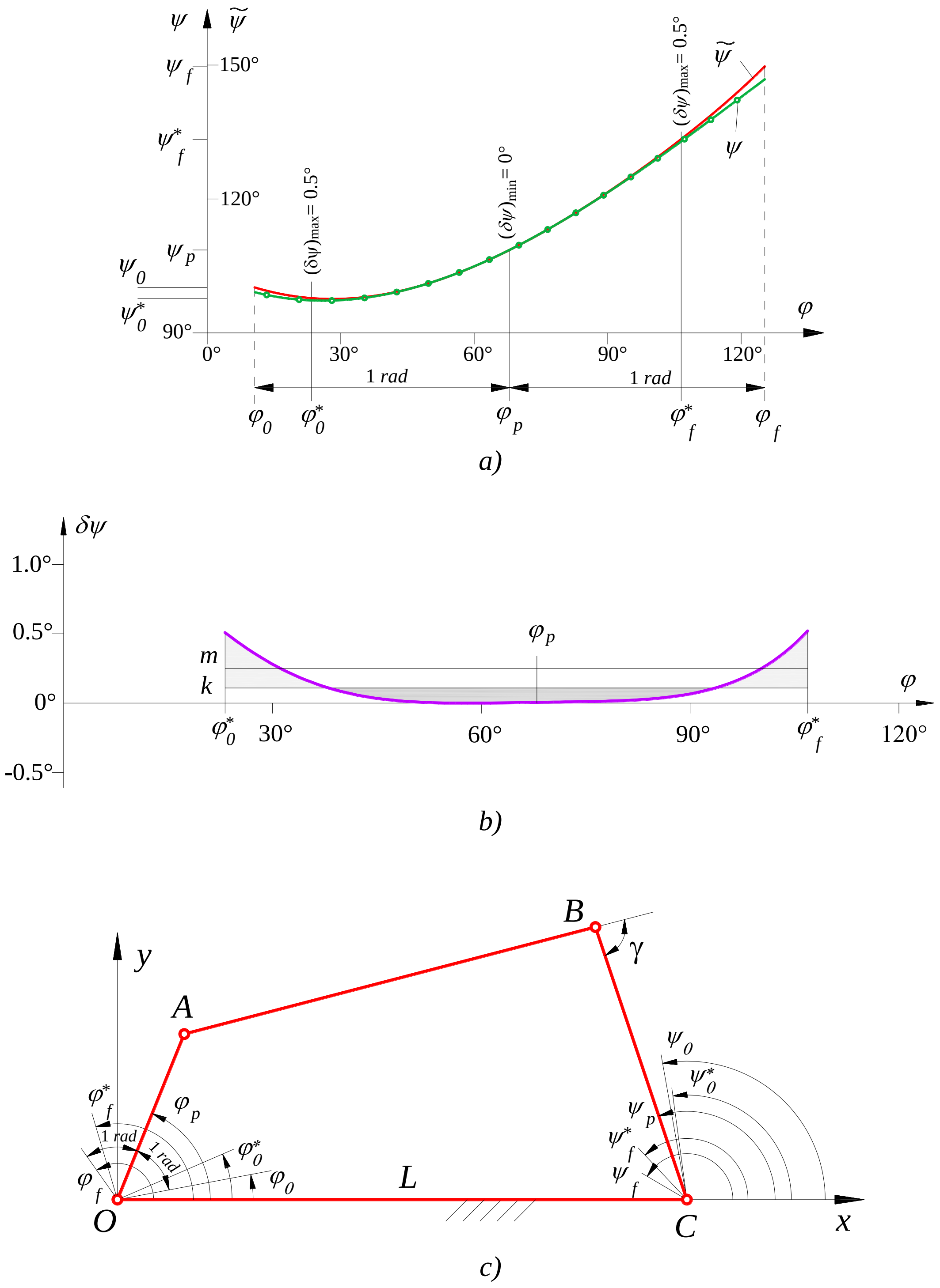

Fig. 2. Synthesis results according to the example: a) given function ψ and generated function $\tilde{\psi}$; b) deviation $\delta\psi$ – MiniMax deviation in relation to rights ***m*** and standard deviation in relation to rights ***k***; c) the synthesized four-bar mechanism

In the project position $\varphi_P = \varphi_0 + \Delta\varphi_P$ of the mechanism at $\Delta\varphi_P = 1$ *rad* for derivatives of $\Delta\psi$ it's working: $\psi' = 0.5$, $\psi'' = 0.5$ and $\psi''' = -0.5$. From equations (1), (2) and (3) are calculated respectively: $x_P = -100mm$; $tg\mu = 0.5$ ($\mu = 26.565°$) and $x_H = 172.727mm$. By changing the value of the parameter k_v and using equations (5) to (11), calculate the Burmester curves ***a*** and ***b***, shown in Fig. 1. The kinematic diagram of the mechanism at $k_q = 0.8$ ($k_v = 0.545$) and transmission angle $\gamma = 90° - |\theta| = 86°$ is determined: the center $B = u \cap b$ with coordinates $x_B = 83.941mm$, $y_B = 47.805mm$ and the center $A = u \cap a$ with coordinates $x_A = 11.739mm$, $y_A = 29.041mm$, whence it follows: $l_{OA} = 31.323mm$; $l_{AB} = 74.600mm$; $l_{BC} = 50.431mm$; $\varphi_P = arctg(y_A/x_A) = 67.990°$; $\psi_P = arctg(y_B/y_B - L) = 108.569°$; $\varphi_o = 10.665°$ и $\psi_o = 100.124°$; $\varphi_f = 125.315°$ and $\psi_f = 149.622°$.

The deviation $\delta\psi = \psi - \psi$ of the generated function ψ from the mechanism $\psi(\varphi) = 180^\circ - \alpha - \beta = 180^\circ - \arccos\frac{R^2 + d^2}{2Rd} - \arcsin\frac{y_A}{d}$, where $d = \sqrt{L^2 + r^2 - 2Lx_A}$ from the given function ψ, in the vicinity of the project position $\varphi_P = \varphi_0 + \Delta\varphi_P$ of the mechanism is shown in Fig. 2. The boundaries have been set $\varphi_o^* = 23.427°$, $\varphi_f^* = 106.521°$ at intervals $\varphi(\varphi_o^*, \varphi_f^*)$ and boundaries $\psi_o^* = 97.726°$, $\psi_f^* = 133.328°$ at intervals $\psi(\psi_o^*, \psi_f^*)$ in case of deviation $(\delta\psi)_{\max} = 0.5°$, as well as and boundaries $\varphi_o^* = 31.153°, \varphi_f^* = 100.276°$ at intervals $\varphi(\varphi_o^*, \varphi_f^*)$ and boundaries $\psi_o^* = 97.613°$, $\psi_f^* = 128.628°$ at the same interval in case of deviation $(\delta\psi)_{\max} = 0.25°$. The synthesized four-bar linkage is presented in Fig. 2c.

At the Fig. 2b shows the MiniMax (best uniform) deviation of the generated from the set function $\delta\psi = \psi - \psi$, whereby $|\delta\psi|_{\max}$ it is reduced 2 times, and the standard deviation at which $|\delta\psi|_{\max}$ decreases 4.587 times in the infinitesimal vicinity of φ_P at the account of 1.216 times increase of $|\delta\psi|_{\max}$ for the interval limits $\varphi(\varphi_o^*, \varphi_f^*)$.

4. Conclusions

A computer-applicable linear mathematical model for determining the Burmester curves for infinitesimal close relative positions of four-bar mechanisms is derived, in which the Carter-Hall circle is indirectly used. By varying only one free parameter and using elements of kinematic and analytical geometry, an incomparably simpler solution is obtained than that obtained by the third-exponent equations of the *Burmester theory* for the cubic of stationary curvature. The kinematic diagram of a four-bar mechanism whereupon is uniquely determined by the condition for the pressure angle, on which the reactions of the forces in the kinematic pairs and the force sizing of the mechanism depend.

The proposed mathematical model is linear - it includes only equations of straight lines written in Cartesian coordinates, which is why it is convenient for computer calculations. The model would facilitate the engineers in the synthesis of articulated four-bar linkages by generating a function approximating a given function in the vicinity of a given position, where the two functions have four infinitely close common points (3rd-order approximation).

An example of the synthesis of a four-bar mechanism illustrates the application of the mathematical model.

Authors' Contributions

VG (Vitan Galabov) developed the method and the mathematical model for synthesis, RR (Roumen Roussev) used the MathCAD program to obtain the synthesis results and check the adequacy of the model, and BP-K (Blagoyka Paleva-Kadiyska) used the AutoCAD program for graphical rechecking and visualization of the results.

The authors read and approved the final manuscript.

Competing Interests

The authors declare that they have no competing interests.

References

[1]. L. Burmester, *Textbook of kinematics*, Leipzig, published by Arthur Felix, 1888, (in German).
[2]. A. P. Kotelnikov, "Burmester points, their properties and construction," *Mathematical Handbook "Mathematical Society"*, vol. 24, no. 3-4, pp. 205-348, 1927, (in Russan).
[3]. R. Mueller, *Introduction to theoretical kinematics*, Berlin, Springer, 1932, (in German).
[4]. J. Hirschorn, *Kinematics and Dynamics of Plane Mechanisms*, McGraw - Hill, New York, 1972.
[5]. C. Rodenberg, "Determination of the circling-points curves by four plane positions," *Mathematics and Physics*, vol. 36, pp. 267-277, 1891, (in German).
[6]. R. Beyer, *Kinematic synthesis of mechanisms*, Berlin, Springer-Verlag, 1953, (in German).
[7]. Ya. L. Geronimus, *Geometric apparatus of the theory of synthesis of plane linkages*, Moscow, Fizmatgiz, 1962, (in Russan).
[8]. W. Lichtenheldt, K. Luck, *Synthesis of mechanisms*, 5th edited and expanded edition, Akademie-Verlag, Berlin 1979, (in German).
[9]. E. A. Dijksman and J. M. Herve, "Instantaneous four-bar kinematics based on the Carter-Hall circle," *Proceedings of the first international applied mechanical systems design conference*: with tutorial workshops; June 11-14, 1989, Nashville, Tennesee, Technological University, vol. 1, pp. P1.1-P1.12.
[10]. J. J. Uicker, G. R. Pennock and J. E. Shigley, *Theory of Machines and Mechanisms*, 5th ed., Oxford University Press, New York, 2017.
[11]. G. R. Veldkamp, "Some Remarks on Higher Curvature Theory," American Society of Mechanical Engineers (ASME), *Journal of Engineering for Industry*, February, 1967, vol. 87B, pp. 84-86.
[12]. B. Roth and A. T. Yang, "The Application of the Instantaneous Invariants to the Analysis and Synthesis of Mechanisms," *American Society of Mechanical Engineers (ASME), Journal of Engineering for Industry*, vol. 99B, no. 1, pp. 97-103, 1977.
[13]. J. Haschek, "For the Isogonal kinematics of the circling-point curve," *Mechanism and Machine Theory*, vol. 21, pp. 167-172, 1986, (in German).
[14]. J. Angeles, A. Alivizatos and A. Akhras, "An Unconstrained Nonlinear Least-Square Method of Optimization of RRRR Planar Path Generator," *Mechanisms and Machine Theory*, vol. 23, no. 5, pp. 343-353, 1988.
[15]. V. Galabov, *Synthesis of mechanisms in robotics*, Technical University - Sofia, 1992, (in Bulgarian).
[16]. V. Galabov, "Structural-metric synthesis of linkages," DSc-technical sciences dissertation, Technical University - Sofia, 1998, (in Bulgarian).

[17]. R. G. Mitchiner and H. H. Mobie, “The synthesis of 4-bar linkage coupler curves using derivatives of the radius of curvature-I. Straight path procedure”, *Mechanism and Machine Theory*, vol. 12, pp. 133-146, 1977.

[18]. K. Eren and S. Ersoy, “Revisiting Burmester theory with complex forms of Bottema’s instantaneous invariants,” *Complex Variables and Elliptic Equations*, vol. 62, no. 4, pp. 431-437, 2017.

[19]. K. Eren and S. Ersoy, “Cardan Positions in the Lorentzian Plane,” *Honam Mathematical Journal*, vol. 40, no. 1, pp. 187-198, 2018.

[20]. K. Eren and S. Ersoy, “Burmester theory in Cayley–Klein planes with affine base,” *Journal of Geometry*, vol. 109, no. 3, art. 45, pp. 12, 2018.

[21]. K. Eren and S. Ersoy, “A Comparison of Original and Inverse Motion in Minkowski Plane,” *Applications and Applied Mathematics*: *An International Journal*, vol. 40, Special Issue no. 5, pp. 56-67, 2019.

[22]. W. J. Carter, “Kinematic Analysis and Synthesis Using Colleneation-Axis Equation,” *Transactions of the American Society of Mechanical Engineers (Trans. ASME)*, vol. 79, no. 6, pp. 1305-1312, 1957, (with discussion by A. S. Hall, Jr.).

[23]. K. Lakshminarayana, “On the Carter-Hall Circle and its Application,” *Journal of Mechanisms*, vol. 6, pp. 517-532, 1971.

[24]. V. Galabov and I. Andonov, “Determining the area of the initial position of the initial unit in the synthesis of cam, belt and lever mechanisms. Part I - Mechanisms with rotation of the inlet and rocker at the outlet,” *Mechanics of Machines*, vol. 105, pp. 69-75, 2014.

[25]. F. Freudenstein, “On the Maximum and Minimum Velocities and Accelerations in Four-Link Mechanisms,” *Transactions of the American Society of Mechanical Engineers (Trans. ASME)*, vol. 78, no. 4, pp. 779-787, 1956.