Konuralp Journal of Mathematics
Volume 5 No. 1 pp. 214–222 (2017) ©KJM

ON SPACELIKE BERTRAND CURVES IN MINKOWSKI 3-SPACE

KAZIM İLARSLAN AND NIHAL KILIÇ ASLAN

Abstract. In this paper, we study spacelike Bertrand curves in Minkowski 3-space. According to the casual character of the principal normal vector, we show that the Bertrand mate curve can be spacelike, timelike or null curve. Thus, we give the necessary and sufficient conditions for spacelike curves to be Bertrand curves and we also give the related examples.

1. Introduction

In the theory of curves in Euclidean space, one of the important and interesting problem is characterization of a regular curve. In the solution of the problem, the curvature functions k_1 (or κ) and k_2 (or τ) of a regular curve have an effective role. For example: if $k_1 = 0 = k_2$, then the curve is a geodesic or if k_1 =constant$\neq 0$ and $k_2 = 0$,then the curve is a circle with radius $(1/k_1)$, etc. Another way in the solution of the problem is the relationship between the Frenet vectors and Frenet planes of the curves ([8],[13],[14],[15]). *Mannheim* curves is an interesting examples for such classification. If there is exist a corresponding relationship between the space curves α and β such that, at the corresponding points of the curves, the principal normal lines of α coincides with the binormal lines of β, then α is called a Mannheim curve, β is called Mannheim partner curve of α. Mannheim partner curves was studied by *Liu* and *Wang* (see [10]) in Euclidean 3-space and Minkowski 3-space. In higher dimensional spaces, Mannheim curves were studied ([16],[17]).

Another interesting example is *Bertrand* curves. A Bertrand curve is a curve in the Euclidean space such that its principal normal is the principal normal of the second curve ([3],[23]). The study of this kind of curves has been extended to many other ambient spaces. In [12], Pears studied this problem for curves in the n-dimensional Euclidean space $\mathbb{E}^n$, $n > 3$, and showed that a Bertrand curve in $\mathbb{E}^n$ must belong to a three-dimensional subspace $\mathbb{E}^3 \subset \mathbb{E}^n$. This result is restated by Matsuda and Yorozu [11]. They proved that there was not any special Bertrand

Date: October 20, 2016 and, in revised form, November 8, 2016.
2000 *Mathematics Subject Classification.* 53C50, 53C40.
Key words and phrases. Bertrand curve, Minkowski 3-space, Cartan null curve, non-null curve.

curves in $\mathbb{E}^n$ $(n > 3)$ and defined a new kind, which is called $(1,3)$-type Bertrand curves in 4-dimensional Euclidean space. Bertrand curves and their characterizations were studied by many researchers in Minkowski 3-space and Minkowski space-time (see [1], [2], [6], [7], [18], [19]) as well as in Euclidean space. In addition, $(1,3)$-type Bertrand curves were studied in semi-Euclidean 4-space with index 2 ([20],[21]).

Following [22], in this paper, we consider spacelike Bertrand curves in Minkowski 3-space. Thus, we give the necessary and sufficient conditions for spacelike curves to be Bertrand curves and we also give the related examples.

2. Preliminaries

The Minkowski space $\mathbb{E}^3_1$ is the Euclidean 3-space $\mathbb{E}^3$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3_1$. Recall that a vector $v \in \mathbb{E}^3_1 \backslash \{0\}$ can be *spacelike* if $g(v,v) > 0$, *timelike* if $g(v,v) < 0$ and *null (lightlike)* if $g(v,v) = 0$ and $v \neq 0$. In particular, the vector $v = 0$ is a spacelike. The norm of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$, and two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}^3_1$, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null ([9]). Spacelike curve in $\mathbb{E}^3_1$ is called *pseudo null curve* if its principal normal vector N is null [4]. A null curve α is parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$. A spacelike or a timelike curve $\alpha(s)$ has unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$ ([4]).

Let $\{T, N, B\}$ be the moving Frenet frame along a curve α in $\mathbb{E}^3_1$, consisting of the tangent, the principal normal and the binormal vector fields respectively. Depending on the causal character of α, the Frenet equations have the following forms.

Case I. If α is a non-null curve, the Frenet equations are given by ([9]):

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2 k_1 & 0 \\ -\epsilon_1 k_1 & 0 & \epsilon_3 k_2 \\ 0 & -\epsilon_2 k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.1}$$

where k_1 and k_2 are the first and the second curvature of the curve respectively. Moreover, the following conditions hold:

$$g(T,T) = \epsilon_1 = \pm 1, g(N,N) = \epsilon_2 = \pm 1, g(B,B) = \epsilon_3 = \pm 1$$

and

$$g(T,N) = g(T,B) = g(N,B) = 0.$$

Case II. If α is a null curve, the Frenet equations are given by ([4])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_2 & 0 & -k_1 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.2}$$

where the first curvature $k_1 = 0$ if α is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$$g(T,T) = g(B,B) = g(T,N) = g(N,B) = 0, g(N,N) = g(T,B) = 1.$$

Case III. If α is pseudo null curve, the Frenet formulas have the form ([5])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ 0 & k_2 & 0 \\ -k_1 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.3}$$

where the first curvature $k_1 = 0$ if α is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$$g(N,N) = g(B,B) = g(T,N) = g(T,B) = 0, g(T,T) = g(N,B) = 1.$$

3. Spacelike Bertrand curves in Minkowski 3-space $\mathbb{E}_1^3$

In this section, we consider the spacelike Bertrand curves in $\mathbb{E}_1^3$. We get the necessary and sufficient conditions for the spacelike curves to be Bertrand curves in $\mathbb{E}_1^3$ and we also give the related examples.

Definition 3.1. A spacelike curve $\alpha : I \to \mathbb{E}_1^3$ with $\kappa_1(s) \neq 0$ is a Bertrand curve if there is a curve $\alpha^* : I^* \to \mathbb{E}_1^3$ such that the principal normal vectors of $\alpha(s)$ and $\alpha^*(s^*)$ at $s \in I$, $s^* \in I^*$ are equal. In this case, $\alpha^*(s^*)$ is the Bertrand mate of $\alpha(s)$.

Let $\beta : I \to \mathbb{E}_1^3$ be a spacelike Bertrand curve in $\mathbb{E}_1^3$ with the Frenet frame $\{T, N, B\}$ and the curvatures κ_1, κ_2, and $\beta^* : I \to \mathbb{E}_1^3$ be a Bertrand mate curve of β with the Frenet frame $\{T^*, N^*, B^*\}$ and the curvatures κ_1^*, κ_2^*.

Theorem 3.1. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed spacelike curve with spacelike principal normal and the non-zero curvatures κ_1, κ_2. Then the curve β is a Bertrand curve with Bertrand mate β^* if and only if one of the following conditions holds:*
(i) there exist constant real numbers λ and h satisfying

$$h^2 < 1, \quad 1 - \lambda\kappa_1 = -h\lambda\kappa_2, \quad \kappa_2 - h\kappa_1 \neq 0, \quad h\kappa_2 - \kappa_1 \neq 0. \tag{3.1}$$

In this case, β^* is a timelike curve in $\mathbb{E}_1^3$.

(ii) there exist constant real numbers λ and h satisfying

$$h^2 > 1, \quad 1 - \lambda\kappa_1 = -h\lambda\kappa_2, \quad \kappa_2 - h\kappa_1 \neq 0, \quad h\kappa_2 - \kappa_1 \neq 0. \tag{3.2}$$

In this case, β^* is a spacelike curve with spacelike principal normal in $\mathbb{E}_1^3$.

Proof. Assume that β is a spacelike Bertrand curve with spacelike principal normal, parametrized by arc-length s with non-zero curvatures κ_1, κ_2 and the curve β^* is the Bertrand mate curve of the curve β parametrized by with arc-length or pseudo arc s^*.

(i) Let β^* be a timelike curve. Then, we can write the curve β^* as

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda(s)N(s) \tag{3.3}$$

for all $s \in I$ where $\lambda(s)$ is $C^\infty-$function on I. Differentiating (3.3) with respect to s and using (2.1), we get

$$T^* f' = (1 - \lambda\kappa_1)T + \lambda' N - \lambda\kappa_2 B. \tag{3.4}$$

By taking the scalar product of (3.4) with N, we have

$$\lambda' = 0. \tag{3.5}$$

Substituting (3.5) in (3.4), we find

$$T^* f' = (1 - \lambda\kappa_1)T - \lambda\kappa_2 B. \tag{3.6}$$

By taking the scalar product of (3.6) with itself, we obtain

$$(f')^2 = (1 + \lambda\kappa_1)^2 - (\lambda\kappa_2)^2 . \tag{3.7}$$

If we denote

$$\delta = \frac{1 - \lambda\kappa_1}{f'} \text{ and } \gamma = \frac{-\lambda\kappa_2}{f'}, \tag{3.8}$$

we get

$$T^* = \delta T + \gamma B_1. \tag{3.9}$$

Differentiating (3.9) with respect to s and using (2.1), we find

$$f'\kappa_1^* N^* = \delta' T + (\delta\kappa_1 - \gamma\kappa_2)\, N + \gamma' B. \tag{3.10}$$

By taking the scalar product of (3.10) with itself, we get

$$\delta' = 0 \text{ and } \gamma' = 0. \tag{3.11}$$

Since $\gamma \neq 0$, we have $1 - \lambda\kappa_1 = -h\lambda\kappa_2$ where $h = \delta/\gamma$. Substituting (3.11) in (3.10) , we find

$$f'\kappa_1^* N^* = (\delta\kappa_1 - \gamma\kappa_2)\, N \tag{3.12}$$

By taking the scalar product of (3.12) with itself, using (3.7) and (3.8), we have

$$(f')^2 \,(\kappa_1^*)^2 = \frac{(\kappa_2 - h\kappa_1)^2}{1 - h^2} \tag{3.13}$$

where $\kappa_2 - h\kappa_1 \neq 0$ and $h^2 < 1$. If we put $v = \frac{\delta\kappa_1 - \gamma\kappa_2}{f'\kappa_1^*}$, we get

$$N^* = vN. \tag{3.14}$$

Differentiating (3.14) with respect to s and using (2.1), we find

$$f'\kappa_2^* B^* = -v\kappa_1 T - v\kappa_2 B - f'\kappa_1^* T^* \tag{3.15}$$

where $v' = 0$. Rewriting (3.15) by using (3.6) , we get

$$f'\kappa_2^* B^* = P\,(s)\, T + Q\,(s)\, B$$

where

$$\begin{aligned} P\,(s) &= \frac{\lambda\kappa_2\,(\kappa_2 - h\kappa_1)}{(f')^2\kappa_1^*\,(1 - h^2)}\,(h\kappa_2 - \kappa_1)\,, \\ Q\,(s) &= \frac{\lambda\kappa_2\,(\kappa_2 - h\kappa_1)\,h}{(f')^2\kappa_1^*\,(1 - h^2)}\,(h\kappa_2 - \kappa_1) \end{aligned}$$

which implies that $h\kappa_2 - \kappa_1 \neq 0$.

Conversely, assume that β is a spacelike curve with spacelike principal normal, parametrized by arc-length s with non-zero curvatures κ_1, κ_2 and the conditions of (3.1) holds for constant real numbers λ and h. Then, we can define a curve β^* as

$$\beta^*(s^*) = \beta(s) + \lambda N(s). \tag{3.16}$$

Differentiating (3.16) with respect to s and using (2.1), we find

$$\frac{d\beta^*}{ds} = -\lambda\kappa_2\{hT + B\} \tag{3.17}$$

which leads to that

$$f' = \sqrt{\left| g\left(\frac{d\beta^*}{ds}, \frac{d\beta^*}{ds}\right) \right|} = m_1 \lambda \kappa_2 \sqrt{1-h^2}$$

where $m_1 = \pm 1$ such that $m_1 \lambda \kappa_2 > 0$. Rewriting (3.17), we obtain

$$T^* = \frac{-m_1}{\sqrt{1-h^2}} \{hT + B\}, \quad g(T^*, T^*) = -1. \tag{3.18}$$

Differentiating (3.18) with respect to s and using (2.1), we get

$$\frac{dT^*}{ds^*} = \frac{-m_1 (h\kappa_1 - \kappa_2)}{f'\sqrt{1-h^2}} N$$

which causes that

$$\kappa_1^* = \left\| \frac{dT^*}{ds^*} \right\| = \frac{m_2 (h\kappa_1 - \kappa_2)}{f'\sqrt{1-h^2}} \tag{3.19}$$

where $m_2 = \pm 1$ such that $m_2 (h\kappa_1 - \kappa_2) > 0$. Now, we can find N^* as

$$N^* = -m_1 m_2 N, \quad g(N^*, N^*) = 1. \tag{3.20}$$

Differentiating (3.20) with respect to s, using (3.18) and (3.19), we get

$$\frac{dN^*}{ds^*} - \kappa_1^* T^* = \frac{m_1 m_2 (\kappa_1 - h\kappa_2)}{f'(1-h^2)} \{T + hB\}$$

which bring about that

$$\kappa_2^* = \frac{m_3 (\kappa_1 - h\kappa_2)}{f'\sqrt{1-h^2}},$$

where $m_3 = \pm 1$ such that $m_3 (\kappa_1 - h\kappa_2) > 0$. Lastly, we define B^* as

$$B^* = \frac{m_1 m_2 m_3}{\sqrt{1-h^2}} \{T + hB\}, \quad g(B^*, B^*) = 1.$$

Then β^* is a timelike curve and the Bertrand mate curve of β. Thus β is a Bertrand curve.

(ii) Let β^* be a spacelike curve with spacelike principal normal in $\mathbb{E}_1^3$. Then the proof can be done similarly to (i). □

Theorem 3.2. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed spacelike curve with spacelike principal normal and the non-zero curvatures κ_1, κ_2, and $\beta^* : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a Cartan null curve with curvatures $\kappa_1^* = 1$, κ_2^*. If the curve β^* is a Bertrand mate curve of the curve β, then there exist constant real numbers λ and $h = \pm 1$ satisfying $1 - \lambda\kappa_1 = h\lambda\kappa_2$ and $h\kappa_1 + \kappa_2 \neq 0$.*

Proof. We assume that β is a spacelike Bertrand curve with spacelike principal normal, parametrized by arc-length s and the curve β^* is the Cartan null Bertrand mate curve with arc-length s^* of the curve β. Then, we can write the curve β^* as

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda(s) N(s) \tag{3.21}$$

for all $s \in I$ where $\lambda(s)$ is $C^\infty-$function on I. Differentiating (3.21) with respect to s and using (2.1), we get

$$T^* f' = (1 - \lambda\kappa_1) T + \lambda' N - \lambda\kappa_2 B. \tag{3.22}$$

By taking the scalar product of (3.22) with N, we have

$$\lambda' = 0. \tag{3.23}$$

Substituting (3.23) in (3.22), we find

$$T^* f' = (1 - \lambda\kappa_1)T - \lambda\kappa_2 B. \tag{3.24}$$

By taking the scalar product of (3.6) with itself, we obtain

$$(1 - \lambda\kappa_1)^2 = (\lambda\kappa_2)^2. \tag{3.25}$$

Rewriting (3.24) by using (3.25), we get

$$T^* f' = \lambda\kappa_2(hT - B), \quad h = \pm 1. \tag{3.26}$$

Putting $\delta = \lambda\kappa_2 / f'$ and differentiating (3.26) with respect to s by using (2.1), we find

$$f' N^* = \delta\left(h\kappa_1 + \kappa_2\right) N$$

which means that $h\kappa_1 + \kappa_2 \neq 0$. □

Theorem 3.3. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed spacelike curve with spacelike principal normal and non-zero constant curvatures κ_1, κ_2, and $\beta^* : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a Cartan null curve with curvatures $\kappa_1^* = 1$, κ_2^*. Then the curve β^* is a Bertrand mate curve of the curve β if and only if there exist constant real numbers λ and $h = \pm 1$ satisfying $1 - \lambda\kappa_1 = h\lambda\kappa_2$ and $h\kappa_1 + \kappa_2 \neq 0$.*

Proof. Assume that β is a spacelike Bertrand curve with spacelike principal normal, parametrized by arc-length s with non-zero constant curvatures κ_1, κ_2 and the curve β^* is the Cartan null Bertrand mate curve of the curve β parametrized by with pseudo arc s^* with curvatures $\kappa_1^* = 1$, κ_2^*. Then from above theorem, there exist constant real numbers λ and $h = \pm 1$ satisfying $1 - \lambda\kappa_1 = h\lambda\kappa_2$ and $h\kappa_1 + \kappa_2 \neq 0$.

Conversely, assume that β is a spacelike curve with spacelike principal normal, parametrized by arc-length s with non-zero constant curvatures κ_1, κ_2 and there exist constant real numbers λ and $h = \pm 1$ satisfying $1 - \lambda\kappa_1 = h\lambda\kappa_2$ and $h\kappa_1 + \kappa_2 \neq 0$. Then, we can define a curve β^* as

$$\beta^*(s^*) = \beta(s) + \lambda N(s). \tag{3.27}$$

Differentiating (3.27) with respect to s and using (2.1), we find

$$\frac{d\beta^*}{ds} = \lambda\kappa_2(hT - B) \tag{3.28}$$

Differentiating (3.27) with respect to s and using (2.1), we find

$$\frac{d^2\beta^*}{ds^2} = \lambda\kappa_2\left(h\kappa_1 + \kappa_2\right) N \tag{3.29}$$

which leads to that

$$f' = \left(\left| g\left(\frac{d\alpha^*}{ds}, \frac{d\alpha^*}{ds}\right)\right|\right)^{1/4} = \sqrt{m_1 m_2 \lambda\kappa_2\left(h\kappa_1 + \kappa_2\right)} \tag{3.30}$$

where $m_1 = \pm 1$ such that $m_1\lambda\kappa_2 > 0$ and $m_2 = \pm 1$ such that $m_2\left(h\kappa_1 + \kappa_2\right) > 0$. Rewriting (3.28) and (3.29), we obtain

$$T^* = \frac{\lambda\kappa_2}{\sqrt{m_1 m_2 \lambda\kappa_2\left(h\kappa_1 + \kappa_2\right)}}\{hT - B\}, \quad g\left(T^*, T^*\right) = 0, \tag{3.31}$$

$$N^* = m_1 m_2 N, \quad g\left(N^*, N^*\right) = 1 \quad \text{and} \quad \kappa_1^* = 1. \tag{3.32}$$

We know that $\kappa_2^* = -\frac{1}{2}g\left(\frac{dN^*}{ds^*}, \frac{dN^*}{ds^*}\right)$. Thus we have

$$\kappa_2^* = \frac{\kappa_2^2 - \kappa_1^2}{2m_1m_2\lambda\kappa_2\left(h\kappa_1 + \kappa_2\right)}. \tag{3.33}$$

Lastly, we can define B^* as

$$B^* = \kappa_2^* T^* - \frac{dN^*}{ds^*} = \frac{\lambda\kappa_2 h\left(h\kappa_1 + \kappa_2\right)^2}{2\left(m_1m_2\lambda\kappa_2\left(h\kappa_1 + \kappa_2\right)\right)^{3/2}}\left\{T + hB\right\}, \quad g\left(B^*, B^*\right) = 0$$

which completes the proof. □

In the following theorem, we give th conditions for spacelike curves with timelike principal normal to be Bertrand curves. We omit the proof which can be similarly done.

Theorem 3.4. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a spacelike curve with timelike principal normal N with non-zero curvatures κ_1, κ_2. Then, the curve β is a Bertrand curve if and only if there exist constant real numbers λ and h ($h^2 < 1$) satisfying*

$$1 - \lambda\kappa_1 = h\lambda\kappa_2, \tag{3.34}$$

$$\kappa_2 - h\kappa_1 \neq 0 \quad \text{and} \quad h\kappa_2 + \kappa_1 \neq 0. \tag{3.35}$$

Example 1. Let us consider a spacelike curve with spacelike principal normal in $\mathbb{E}_1^3$ parametrized by

$$\beta(s) = \left(\sinh s, \cosh s, \sqrt{2}s\right)$$

with

$$\begin{aligned} T(s) &= \left(\cosh s, \sinh s, \sqrt{2}\right), \\ N(s) &= \left(\sinh s, \cosh s, 0\right), \\ B(s) &= \left(-\sqrt{2}\cosh s, -\sqrt{2}\sinh s, -1\right) \\ \kappa_1(s) &= 1 \quad \text{and} \quad \kappa_2(s) = \sqrt{2}. \end{aligned}$$

If we take $h = -\sqrt{2}/2$ and $\lambda = 1/2$ in (i) of theorem 3.1, then we get the curve β^* as follows:

$$\beta^*(s) = \beta(s) + \frac{1}{2}N(s) = \left(\frac{3}{2}\sinh s, \frac{3}{2}\cosh s, \sqrt{2}s\right)$$

By straight calculations, we get

$$\begin{aligned} T^*(s) &= \left(3\cosh s, 3\sinh s, 2\sqrt{2}\right), \\ N^*(s) &= \left(\sinh s, \cosh s, 0\right), \\ B^*(s) &= \left(-2\sqrt{2}\cosh s, -2\sqrt{2}\sinh s, -3\right), \\ \kappa_1^*(s) &= 6, \quad \kappa_2^*(s) = 4\sqrt{2}. \end{aligned}$$

It can be easily seen that the curve β^* is a timelike Bertrand mate curve of the curve β.

Example 2. For the same spacelike curve β in Example 1, if we take $h = -\sqrt{2}$ and $\lambda = 1/3$ in (ii) of theorem 3.1, then we get the curve β^* as follows:

$$\beta^*(s) = \beta(s) + \frac{1}{3}N(s) = \left(\frac{4}{3}\sinh s, \frac{4}{3}\cosh s, \sqrt{2}s\right)$$

By straight calculations, we get

$$\begin{aligned}
T^*(s) &= \left(2\sqrt{2}\cosh s, 2\sqrt{2}\sinh s, 3\right), \\
N^*(s) &= (\sinh s, \cosh s, 0), \\
B^*(s) &= \left(3\cosh s, 3\sinh s, 2\sqrt{2}\right), \\
\kappa_1^*(s) &= 6, \quad \kappa_2^*(s) = -9/\sqrt{2}.
\end{aligned}$$

It can be easily seen that the curve β^* is a spacelike Bertrand mate curve of the curve β.

Example 3. For the same spacelike curve β in Example 1, if we take $\lambda = -1 - \sqrt{2}$ in theorem 3.3, then we get the curve β^* as follows:

$$\beta^*(s) = \left(-\sqrt{2}\sinh s, -\sqrt{2}\cosh s, \sqrt{2}s\right)$$

By straight calculations, we get

$$\begin{aligned}
T^*(s) &= \left(-\sqrt[4]{2}\cosh s, -\sqrt[4]{2}\sinh s, \sqrt[4]{2}\right), \\
N^*(s) &= (-\sinh s, -\cosh s, 0), \\
B^*(s) &= \frac{1}{2\sqrt[4]{2}}(\cosh s, \sinh s, 1), \\
\kappa_1^*(s) &= 1, \quad \kappa_2^*(s) = \frac{1}{2\sqrt{2}}.
\end{aligned}$$

It can be easily seen that the curve β^* is a Cartan null Bertrand mate curve of the curve β.

Example 4. Let us consider a spacelike curve with timelike principal normal in $\mathbb{E}_1^3$ parametrized by

$$\beta(s) = \left(\frac{\cosh s}{\sqrt{2}}, \frac{\sinh s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right).$$

If we choose $\lambda = 1$ and $h = 1 + \sqrt{2}$ in Theorem 3.4, then we get the following curve

$$\beta^*(s) = \left(\frac{1+\sqrt{2}}{\sqrt{2}}\cosh s, \frac{1+\sqrt{2}}{\sqrt{2}}\sinh s, \frac{s}{\sqrt{2}}\right).$$

It can be similarly seen that β^* is a spacelike curve with timelike principal normal and the Bertrand mate curve of β.

References

[1] H. Balgetir, M. Bektaş and M. Ergüt, Bertrand curves for non-null curves in 3-dimensional Lorentzian space, Hadronic J. 27 (2004), no. 2, 229-236.

[2] H. Balgetir, M. Bektaş and J. Inoguchi, Null Bertrand curves in Minkowski 3-space and their characterizations, Note Mat. 23 (2004/05), no. 1, 7-13.

[3] J. M. Bertrand, Mémoire sur la théorie des courbes á double courbure, Comptes Rendus, 36, (1850).

[4] W. B. Bonnor, Null curves in a Minkowski space-time, Tensor 20 (1969), 229-242.

[5] W. B. Bonnor, Curves with null normals in Minkowski space-time. A random walk in relativity and cosmology, Wiley Easten Limited (1985), 33-47.

[6] N. Ekmekci and K. İlarslan, On Bertrand curves and their characterization, Differ. Geom. Dyn. Syst. 3 (2001), no. 2, 17-24.

[7] D. H. Jin, Null Bertrand curves in a Lorentz manifold, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. 15 (2008), no. 3, 209-215.

[8] S. Özkaldı Karakuş, K. İlarslan and Y. Yaylı, A new approach for characterization of curve couples in Euclidean 3-space, Honam Mathematical J. 36(2014), no.1, 113-129.
[9] W. Kuhnel, Differential geometry: curves-surfaces-manifolds, Braunschweig, Wiesbaden, (1999).
[10] H. Liu and F. Wang, Mannheim partner curves in 3-space, Journal of Geometry, 88 (2008), 120-126.
[11] H. Matsuda and S. Yorozu, Notes on Bertrand curves, Yokohama Math. J. 50 (2003), no. 1-2, 41-58.
[12] L. R. Pears, Bertrand curves in Riemannian space, J. London Math. Soc. Volumes 1-10, Number 2, 180-183 , July 1935.
[13] A. Uçum, K. İlarslan and S. Özkaldı Karakuş, Curve couples and spacelike Frenet planes in Minkowski 3-space, Honam Mathematical J. 36 (2014), No. 3, 475-492.
[14] A. Uçum, K. İlarslan and S. Özkaldı Karakuş, On curve couples with joint lightlike Frenet planes in Minkowski 3-space, Acta Univ. Apulensis Math. Inform. 41 (2015), 111-129.
[15] A. Uçum, K. İlarslan and S. Özkaldı Karakuş, On curve couples with joint timelike Frenet planes in Minkowski 3-space, Acta Univ. Apulensis Math. Inform. 44 (2015), 87-108.
[16] A. Uçum, E. Nesovic and K. İlarslan, On generalized timelike Mannheim curves in Minkowski space-time, Journal of Dynamical Systems & Geomatric Theories, 13(1) (2015), 71-94.
[17] A. Uçum, E. Nesovic and K. İlarslan, On generalized spacelike Mannheim curves in Minkowski space-time, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, Volume 86, Issue 2 (2016), pp 249-258.
[18] A. Uçum, K. İlarslan and M. Sakaki, On $(1, 3)$-Cartan Null Bertrand curves in Semi-Euclidean 4-Space with index 2, J. Geom., Volume 107, Issue 3 (2015), pp 579591.
[19] A. Uçum, O. Keçilioğlu and K. İlarslan, Generalized Pseudo Null Bertrand curves in Semi-Euclidean 4-Space with index 2, Rendiconti del Circolo Matematico di Palermo, 65, Issue 3 (2016), pp 459-472.
[20] A. Uçum, O. Keçilioğlu and K. İlarslan, Generalized Bertrand curves with spacelike $(1, 3)$-normal plane in Minkowski space-time, Turk. J. Math. 40 (2016), 487-505.
[21] A. Uçum, O. Keçilioğlu and K. İlarslan, Generalized Bertrand curves with timelike $(1, 3)$-normal plane in Minkowski space-time, Kuwait J. Sci. 42 (2015), 10-27.
[22] A. Uçum and K. İlarslan, On timelike Bertrand Curves in Minkowski 3-space, Honam Mathematical Journal, Vol.38, No.3 (2016), pp.467-477.
[23] B. Saint Venant, Mémoire sur les lignes courbes non planes, Journal de l'Ecole Polytechnique, vol. 18, pp.1-76, 1845.

Kirikkale University, Faculty of Sciences and Arts, Department of Mathematics, Kirikkale-Turkey
E-mail address: `kilarslan@yahoo.com`

Kirikkale University, Faculty of Sciences and Arts, Department of Mathematics, Kirikkale-Turkey
E-mail address: `nhlklc71@gmail.com`