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Apostol Bernoulli-Fibonacci Polynomials, Apostol Euler-Fibonacci Polynomials and Their Generating Functions

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Abstract. In this article, the Apostol Bernoulli-Fibonacci polynomials are defined and various properties of Apostol Bernoulli-Fibonacci polynomials are obtained. Furthermore, Apostol Euler-Fibonacci numbers and polynomials are found. In addition, harmonic-based F exponential generating functions are defined for Apostol Bernoulli-Fibonacci numbers and Apostol Euler-Fibonacci numbers.

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1. Introduction

Due to the ease with which polynomials can be used to compute and operate mathematically, special numbers and polynomials have many valuable applications not only in mathematics but also in physics and engineering. As well as solving real-world problems, polynomials are used by physicists, engineers and biologists. The Bernoulli polynomials $B_n(x)$ are defined as follows in [1]:

$$\frac{te^{tx}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

For $x=0$, $B_n(0)$ is called the n-th Bernoulli number and denoted B_n. Some Bernoulli numbers and Bernoulli polynomials are:

$$B_0=1,\quad B_1=-\frac{1}{2},\quad B_2=\frac{1}{6},\quad B_3=0,\quad B_4=-\frac{1}{30},\quad B_5=0,$$

$$B_0(x)=1,\quad B_1(x)=x-\frac{1}{2},\quad B_2(x)=x^2-x+\frac{1}{6},\quad B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2},\quad B_4(x)=x^4-2x^3+x^2-\frac{1}{30}.$$

Bernoulli polynomials and Bernoulli numbers hold the following identities [1]

$$B_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k(x)y^{n-k},$$

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k},\quad n\geq 0,$$

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$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k, \quad n \geq 2.$$

The Euler polynomials $E_n(x)$ are defined by the generating function:

$$\frac{2e^{tx}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

For $x = 0$, $E_n(0)$ is called the n-th Euler number, which is denoted by E_n. Some Euler numbers and Euler polynomials are

$$E_0 = 1, \quad E_1 = 0, \quad E_2 = -1, \quad E_3 = 0, \quad E_4 = 5, \quad E_5 = -61,$$

$$E_0(x) = 1, \quad E_1(x) = x - \frac{1}{2}, \quad E_2(x) = x^2 - x, \quad E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}.$$

Euler polynomials and Euler numbers hold the following properties [1]:

$$E_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) y^{n-k}.$$

Srivastava and Choi worked on Euler polynomials and numbers [15].

Some generalized forms of the Bernoulli polynomials and Euler polynomials have already appeared in literature. In [6], the Bernoulli polynomials and the Euler polynomials of order α are defined by the following generating functions

$$\left(\frac{z}{e^z - 1}\right)^{\alpha} e^{zx} = \sum_{n=0}^{\infty} B_n^{\alpha}(x) \frac{z^n}{n!},$$

$$\left(\frac{2}{e^z + 1}\right)^{\alpha} e^{zx} = \sum_{n=0}^{\infty} E_n^{\alpha}(x) \frac{z^n}{n!}.$$

Note that, for $x = 0$, the Bernoulli and the Euler polynomials of order α reduce to the Bernoulli and the Euler numbers of order α. Another generalization is Apostol Bernoulli polynomial of order α and Apostol Euler polynomials of order α respectively for arbitrary reel or complex parameters α and λ (see, for details [2, 3, 8]).

Similar definitions of classical Bernoulli numbers and polynomials were given by T. M. Apostol, who worked on Lipchitz-Lerch Zeta functions [2]. These polynomials, called Apostol-Bernoulli, were later studied by Srivastava, and some generalizations of Apostol-Bernoulli polynomials were also explored by Luo and Srivastava [9, 13]. Thus, Luo and Srivastava came together to reveal the α-expanded Apostol-Euler and Apostol-Bernoulli polynomials, and by studying these polynomials systematically, they finally gave the product formulas of Apostolic type polynomials and the recurrence relations of Apostol-Euler polynomials [6, 7, 9, 10, 13]. In addition to these, the sum theorem for Euler and Bernoulli polynomials was also explored by Srivastava [14].

The Apostol Bernoulli polynomials and the Apostol Euler polynomials are defined as

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} B_n^{\alpha}(x, \lambda) \frac{t^n}{n!},$$

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} E_n^{\alpha}(x, \lambda) \frac{t^n}{n!}.$$

Some identities of these polynomials are ([2, 3, 8]):

$$B_n^{\alpha}(x + y) = \sum_{k=0}^{n} \binom{n}{k} B_k^{\alpha}(x) y^{n-k},$$

$$B_n^{\alpha} = \sum_{k=0}^{n} \binom{n}{k} B_k^{\alpha},$$

$$E_n^{\alpha}(x + y) = \sum_{k=0}^{n} \binom{n}{k} E_k^{\alpha}(x) y^{n-k},$$

$$B_n^{\alpha+\beta}(x+y,\lambda) = \sum_{k=0}^{n} \binom{n}{k} B_n^{\alpha}(x,\lambda) B_{n-k}^{\beta}(y,\lambda),$$

$$E_n^{\alpha+\beta}(x+y,\lambda) = \sum_{k=0}^{n} \binom{n}{k} E_n^{\alpha}(x,\lambda) E_{n-k}^{\beta}(y,\lambda).$$

Note that, for $\lambda = 1$ the Apostol Bernoulli and the Apostol Euler polynomials of order α reduce to the Bernoulli and the Euler polynomials of order α. If $\lambda = \alpha = 1$ are taken, then classical Bernoulli and Euler polynomials are obtained.

Now, we give some definitions (for these definitions see [11, 12]) that we will use throughout the article.

The $F-$factorial is defined as

$$F_n! = F_n \cdot F_{n-1} \cdot F_{n-2} \cdots F_1, \quad F_0! = 1,$$

where F_n is n-th Fibonacci numbers. The Fibonomial coefficients are defined as $(0 \le k \le n)$ as

$$\binom{n}{k}_F = \frac{F_n!}{F_{n-k}! F_k!}$$

with $\binom{n}{0}_F = \binom{n}{n}_F = 1$ and $\binom{n}{k}_F = 0$ for $n < k$. The F–binomial theorem is given by

$$(x +_F y)^n = \sum_{k=0}^{n} \binom{n}{k}_F x^k y^{n-k}.$$

The F-exponential functions e_F^x and E_F^x are defined as:

$$e_F^x = \sum_{n=0}^{\infty} \frac{x^n}{F_n!},$$

$$E_F^x = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{x^n}{F_n!}.$$

The following identity holds

$$e_F^x E_F^y = e_F^{(x+y)_F}. \tag{1.1}$$

The authors [5, 11] defined generating functions for Bernoulli-Fibonacci polynomials $B_n^F(x)$ and Euler-Fibonacci polynomials as follow

$$\frac{t e_F^{tx}}{e_F^t - 1} = \sum_{n=0}^{\infty} B_n^F(x) \frac{t^n}{F_n!}, \tag{1.2}$$

$$\frac{2 e_F^{tx}}{e_F^t + 1} = \sum_{n=0}^{\infty} E_n^F(x) \frac{t^n}{F_n!}, \tag{1.3}$$

where F_n is nth Fibonacci number.

In [5], Bernoulli F-polynomials were defined and various properties were obtained. In the light of the Eq. (1.2) and Eq. (1.3), we define the Apostol Bernoulli-Fibonacci polynomials and obtain various properties of these polynomials. Moreover, we define the Apostol Euler-Fibonacci polynomials and numbers. Finally, we give the harmonic-based F-exponential generating function for Apostol Bernoulli-Fibonacci numbers and Apostol Euler-Fibonacci numbers.

2. Apostol Bernoulli-Fibonacci Polynomials and Apostol Euler-Fibonacci Polynomials

In this section, we define some new generalizations of the Bernoulli polynomials and the Euler polynomials, the Apostol Bernoulli polynomials and the Apostol Euler polynomials with the help of golden exponential generating function. Then, we obtain some properties of these newly established polynomials.

Definition 2.1. The exponential generating function for Apostol Bernoulli-Fibonacci polynomials of order α, $B_n^{F,\alpha}(x,\lambda)$ and Apostol Euler-Fibonacci polynomials of order α, $E_n^{F,\alpha}(x,\lambda)$ are defined by

$$\left(\frac{t}{\lambda e_F^t - 1}\right)^{\alpha} e_F^{tx} = \sum_{n=0}^{\infty} B_n^{F,\alpha}(x,\lambda) \frac{t^n}{F_n!},$$

$$\left(\frac{2}{\lambda e_F^t + 1}\right)^{\alpha} e_F^{tx} = \sum_{n=0}^{\infty} E_n^{F,\alpha}(x,\lambda)\frac{t^n}{F_n!}. \tag{2.1}$$

If we take $\alpha = 1$, Apostol Bernoulli-Fibonacci polynomials of order α and Apostol Euler-Fibonacci polynomials of order α reduce to Apostol Bernoulli-Fibonacci polynomials and Apostol Euler-Fibonacci polynomials, respectively. These polynomials are

$$\frac{t}{\lambda e_F^t - 1} e_F^{tx} = \sum_{n=0}^{\infty} B_n^F(x,\lambda)\frac{t^n}{F_n!}, \tag{2.2}$$

and

$$\frac{2}{\lambda e_F^t + 1} e_F^{tx} = \sum_{n=0}^{\infty} E_n^F(x,\lambda)\frac{t^n}{F_n!}, \tag{2.3}$$

respectively. If we write $x = 0$ in equations (2.2) and (2.3), we obtain the Apostol Bernoulli-Fibonacci numbers and Apostol Euler-Fibonacci numbers as follows;

$$\frac{t}{\lambda e_F^t - 1} = \sum_{n=0}^{\infty} B_n^F(\lambda)\frac{t^n}{F_n!},$$

and

$$\frac{2}{\lambda e_F^t + 1} = \sum_{n=0}^{\infty} E_n^F(\lambda)\frac{t^n}{F_n!},$$

Theorem 2.2. *Apostol Bernoulli-Fibonacci polynomials $B_n^F(x,\lambda)$ satisfy the following relation:*

$$B_n^F(x+y) = \sum_{n=0}^{n} \binom{n}{k}_F (-1)^{\frac{(n-k)(n-k-1)}{2}} y^{n-k} B_n^F(x,\lambda).$$

Proof. Using Eq. (2.2) and Eq. (1.1), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n^F(x+y,\lambda)\frac{t^n}{F_n!} &= \left(\frac{t}{\lambda e_F^t - 1}\right) e_F^{t(x+y)} \\
&= \left(\frac{t}{\lambda e_F^t - 1}\right) e_F^{tx} E_F^{ty} \\
&= \sum_{n=0}^{\infty} B_n^F(x,\lambda)\frac{t^n}{F_n!} \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} y^n \frac{t^n}{F_n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}_F (-1)^{\frac{(n-k)(n-k-1)}{2}} y^{n-k} B_k^F(x,\lambda) \right) \frac{t^n}{F_n!}.
\end{aligned}$$

If the coefficients are compared, the desired result is obtained. □

Theorem 2.3. *The following formula for Apostol Euler-Fibonacci polynomials $E_n^F(x,\lambda)$ is correct*

$$E_n^F(x+y) = \sum_{n=0}^{n} \binom{n}{k}_F (-1)^{\frac{(n-k)(n-k-1)}{2}} y^{n-k} E_n^F(x,\lambda).$$

Proof. By utilizing Using Eq. (2.3) and Eq. (1.1), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} E_n^F(x+y,\lambda)\frac{t^n}{F_n!} &=(\frac{2}{\lambda e_F^t+1})e_F^{t(x+y)}\\ &=\left(\frac{2}{\lambda e_F^t+1}\right)e_F^{tx}E_F^{ty}\\ &=\sum_{n=0}^{\infty} E_n^F(x,\lambda)\frac{t^n}{F_n!}\sum_{n=0}^{\infty}(-1)^{\frac{n(n-1)}{2}}y^n\frac{t^n}{F_n!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F(-1)^{\frac{(n-k)(n-k-1)}{2}}y^{n-k}E_k^F(x,\lambda)\right)\frac{t^n}{F_n!}.\end{aligned}$$

Thus, the desired result is rielived when the coefficients are equalized. □

Theorem 2.4. *The following formula for Apostol Euler-Fibonacci polynomials, $E_n^F(x,\lambda)$, is correct*

$$E_n^F\left(\frac{x}{2},\lambda\right)=\sum_{k=0}^{n}\binom{n}{k}_F(-1)^{\frac{(n-k)(n-k+1)}{2}}\left(\frac{x}{2}\right)^{n-k}E_k^F(x,\lambda).$$

Proof. From Eq. (2.3) and Eq. (1.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} E_n^F\left(\frac{x}{2},\lambda\right)\frac{t^n}{F_n!} &=\left(\frac{2}{\lambda e_F^t+1}\right)e_F^{\frac{x}{2}t}\\ &=\left(\frac{2}{\lambda e_F^t+1}\right)e_F^{xt}E_F^{-\frac{x}{2}t}\\ &=\sum_{n=0}^{\infty} E_n^F(x,\lambda)\frac{t^n}{F_n!}\sum_{n=0}^{\infty}(-1)^{\frac{n(n-1)}{2}}\left(\frac{-x}{2}\right)^n\frac{t^n}{F_n!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F(-1)^{\frac{(n-k)(n-k+1)}{2}}\left(\frac{x}{2}\right)^{n-k}E_k^F(x,\lambda)\right)\frac{t^n}{F_n!}.\end{aligned}$$

So the proof is completed. □

Definition 2.5 ([12])**.** The Golden Derivative operator D_F^x acts on arbitrary function $f(x)$

$$D_F^x\left(f(x)\right)=\frac{f(\vartheta x)-f\left(-\frac{x}{\vartheta}\right)}{\left(\vartheta-\left(-\frac{1}{\vartheta}\right)x\right)},$$

where $\vartheta=\frac{1+\sqrt{5}}{2}$.

Note that, the Golden derivatives of Golden exponential functions are hold [12]:

$$D_F^x\left(e_F^{tx}\right)=te_F^{tx}. \tag{2.4}$$

Theorem 2.6. *Apostol-Bernoulli-Fibonacci polynomials satisfy the formula*

$$B_n^F(x,\lambda)=\frac{D_F^x\left(B_{n+1}^F(x,\lambda)\right)}{F_{n+1}!}.$$

Proof. By virtue of Eq. (2.1), we find that

$$\begin{aligned} D_F^x\left(\frac{t}{\lambda e_F^t-1}e_F^{tx}\right) &= \frac{t}{\lambda e_F^t-1}D_F^x\left(e_F^{tx}\right) \\ &= t\frac{te_F^{tx}}{\lambda e_F^t-1} \\ &= t\sum_{n=0}^{\infty}B_n^F(x,\lambda)\frac{t^n}{F_n!} \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} D_F^x\left(\sum_{n=0}^{\infty}B_n^F(x,\lambda)\frac{t^n}{F_n!}\right) &= \sum_{n=1}^{\infty}D_F^x\left(B_n^F(x,\lambda)\right)\frac{t^n}{F_n!} \\ &= \sum_{n=0}^{\infty}D_F^x\left(B_{n+1}^F(x,\lambda)\right)\frac{t^{n+1}}{F_{n+1}!} \\ &= \sum_{n=0}^{\infty}D_F^x\left(B_{n+1}^F(x,\lambda)\right)\frac{t}{F_{n+1}}\frac{t^n}{F_n!}. \end{aligned} \tag{2.6}$$

If the coefficients in the Eq. (2.5) and Eq. (2.6) are compared, the desired result is obtained. □

Theorem 2.7. *For $n \geq 1$, Apostol Bernoulli-Fibonacci polynomials can be calculated recursively by*

$$\lambda\sum_{l=0}^{n}\binom{n}{l}_F B_l^F(x,\lambda) = F_n x^{n-1} + B_n^F(x,\lambda).$$

Proof. We know that, the generating function of Apostol Bernoulli-Fibonacci polynomials is

$$\frac{t}{\lambda e_F^t-1}e_F^{tx} = \sum_{n=0}^{\infty}B_n^F(x,\lambda)\frac{t^n}{F_n!}.$$

If we multiply this equality by λe_F^t, we have

$$\frac{te_F^{tx}}{\lambda e_F^t-1}\lambda e_F^t = \sum_{n=0}^{\infty}B_n^F(x,\lambda)\frac{t^n}{F_n!}\lambda e_F^t.$$

If we subtract the above two equations side by side, we get

$$\frac{te_F^{tx}}{\lambda e_F^t-1}\left(\lambda e_F^t-1\right) = \sum_{n=0}^{\infty}\left(\lambda e_F^t B_n^F(x,\lambda)-B_n^F(x,\lambda)\right)\frac{t^n}{F_n!}.$$

Thus,

$$\begin{aligned} te_F^{tx} &= \sum_{n=0}^{\infty}\left(\lambda e_F^t B_n^F(x,\lambda)-B_n^F(x,\lambda)\right)\frac{t^n}{F_n!} \\ &= \sum_{l=0}^{\infty}\lambda e_F^t B_l^F(x,\lambda)\frac{t^l}{F_l!} - \sum_{n=0}^{\infty}B_n^F(x,\lambda)\frac{t^n}{F_n!}. \end{aligned}$$

For the first sum of the RHS of this equation, we find that

$$\begin{aligned} \lambda\sum_{l=0}^{\infty}e_F^t B_l^F(x,\lambda)\frac{t^l}{F_l!} &= \lambda\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}B_l^F(x,\lambda)\frac{t^k}{F_k!}\frac{t^l}{F_l!} \\ &= \lambda\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}B_l^F(x,\lambda)\frac{t^{k+l}}{F_k!F_l!}. \end{aligned}$$

For $k+l=n$, we get

$$\begin{aligned}\lambda\sum_{l=0}^{\infty}e_F^tB_l^F(x,\lambda)\frac{t^l}{F_l!}=&\lambda\sum_{n=0}^{\infty}\frac{1}{F_n!}\sum_{l=0}^{n}B_l^F(x,\lambda)\frac{t^nF_n!}{F_{n-l}!F_l!}\\=&\lambda\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{k}_F B_l^F(x,\lambda)\right)\frac{t^n}{F_n!}.\end{aligned}$$

By virtue of Eq. (2.4), we obtain

$$\begin{aligned}D_F^x\left(e_F^{tx}\right)=&D_F^x\left(\sum_{n=0}^{\infty}\frac{(tx)^n}{F_n!}\right)\\=&\sum_{n=1}^{\infty}D_F^x\left(x^n\right)\frac{t^n}{F_n!}\\=&\sum_{n=1}^{\infty}F_nx^{n-1}\frac{t^n}{F_n!}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{n=1}^{\infty}F_nx^{n-1}\frac{t^n}{F_n!}=&\lambda\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{k}_F B_l^F(x,\lambda)\right)\frac{t^n}{F_n!}-\sum_{n=0}^{\infty}B_n^F(x,\lambda)\frac{t^n}{F_n!}\\=&\lambda B_0^F(x,\lambda)\binom{0}{0}_F+\lambda\sum_{n=1}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{k}_F B_l^F(x,\lambda)\right)\frac{t^n}{F_n!}-B_0^F(x,\lambda)-\sum_{n=1}^{\infty}B_n^F(x,\lambda)\frac{t^n}{F_n!}\\=&\lambda\sum_{n=1}^{\infty}\frac{t^n}{F_n!}\left(\sum_{l=0}^{n}\binom{n}{k}_F B_l^F(x,\lambda)-B_n^F(x,\lambda)\right).\end{aligned}$$

Therefore, we have

$$\lambda\sum_{l=0}^{n}\binom{n}{k}_F B_l^F(x,\lambda)-B_n^F(x,\lambda)=F_nx^{n-1}.$$

□

3. Harmonic-Based Apostol Euler-Fibonacci and Apostol Bernoulli-Fibonacci Numbers

In this section, we obtain harmonic-based F-exponential generating functions of the Apostol Euler-Fibonacci and Apostol Bernoulli-Fibonacci numbers. The harmonic numbers defined by

$$h_n=\sum_{k=1}^{n}\frac{1}{k},$$

where $h_0=0$.

In [4], Dattoli studied vacuum amplification of harmonic numbers operator, with the help of harmonic-based exponential generator functions (see [4, 16]) defined the hyperharmonic Fibonacci numbers as follows

$$\mathbb{F}_n=\sum_{k=1}^{n}\frac{1}{F_k},$$

where $\mathbb{F}_0=0$.

Kuş et al. [5] obtained harmonic-based F-exponential generators of Euler-Fibonacci numbers, Euler-Fibonacci polynomials and Bernoulli Fibonacci polynomials

$$e^{f_{\mathbb{F}}t}=1+\sum_{n=1}^{\infty}\mathbb{F}_n\frac{t^n}{F_n!},$$

is a harmonic-based F-exponential function, where

$$\partial_{F,t}\, e^{f_{\mathbb{F}} t} = 1 + \sum_{n=1}^{\infty} \mathbb{F}_{n+1} \frac{t^n}{F_n!}.$$

Theorem 3.1. *Harmonic-based F-exponential generating function of Apostol Bernoulli-Fibonacci numbers are*

$$\sum_{n=0}^{\infty} B_n^F(\lambda) \frac{t^n}{F_n!} = \frac{t}{\lambda - 1 + \lambda t(\partial_{F,t} e^{f_{\mathbb{F}} t} - e^{f_{\mathbb{F}} t} + 1)}.$$

Proof. By using the harmonic-based F-exponential function $e^{f_{\mathbb{F}} t}$ and the exponential generating functions for Apostol Bernoulli-Fibonacci numbers, we get

$$\begin{aligned}
\lambda e_F^t - 1 &= \lambda \sum_{n=0}^{\infty} \frac{t^n}{F_n!} - 1 \\
&= \lambda + \lambda \sum_{n=1}^{\infty} \frac{t^n}{F_n!} - 1 \\
&= \lambda - 1 + \lambda t \sum_{n=0}^{\infty} \frac{1}{F_{n+1}} \frac{t^n}{F_n!} \\
&= \lambda - 1 + \lambda t \sum_{n=0}^{\infty} (\mathbb{F}_{n+1} - \mathbb{F}_n) \frac{t^n}{F_n!} \\
&= \lambda - 1 + \lambda t \left(1 + \sum_{n=1}^{\infty} \mathbb{F}_{n+1} \frac{t^n}{F_n!} - \sum_{n=1}^{\infty} \mathbb{F}_n \frac{t^n}{F_n!} \right) \\
&= \lambda - 1 + \lambda t(\partial_{F,t} e^{f_{\mathbb{F}} t} - e^{f_{\mathbb{F}} t} + 1).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n^F(\lambda) \frac{t^n}{F_n!} &= \frac{t}{\lambda e_F^t - 1} \\
&= \frac{t}{\lambda - 1 + \lambda t(\partial_{F,t} e^{f_{\mathbb{F}} t} - e^{f_{\mathbb{F}} t} + 1)}.
\end{aligned}$$

□

Theorem 3.2. *Harmonic-based F-exponential generating function of Apostol-Euler-Fibonacci numbers is given by:*

$$\sum_{n=0}^{\infty} E_n^F(\lambda) \frac{t^n}{F_n!} = \frac{2}{1 + \lambda + \lambda t\, (\partial_{F,t}\, e^{f_{\mathbb{F}} t} - e^{f_{\mathbb{F}} t} + 1)}.$$

Proof. By using the F-exponential function $e^{f_{\mathbb{F}}t}$ and the exponential generating functions for Apostol Euler-Fibonacci numbers, we get

$$\begin{aligned}
\lambda e_F^t + 1 =& 1 + \lambda + \lambda \sum_{n=1}^{\infty} \frac{t^n}{F_n!} \\
=& 1 + \lambda + \lambda t \sum_{n=0}^{\infty} \frac{1}{F_{n+1}} \frac{t^n}{F_n!} \\
=& 1 + \lambda + \lambda t \sum_{n=0}^{\infty} (\mathbb{F}_{n+1} - \mathbb{F}_n) \frac{t^n}{F_n!} \\
=& 1 + \lambda + \lambda t \left(\sum_{n=0}^{\infty} \mathbb{F}_{n+1} \frac{t^n}{F_n!} - \sum_{n=0}^{\infty} \mathbb{F}_n \frac{t^n}{F_n!} \right) \\
=& 1 + \lambda + \lambda t \left(1 + \sum_{n=1}^{\infty} \mathbb{F}_{n+1} \frac{t^n}{F_n!} - \sum_{n=1}^{\infty} \mathbb{F}_n \frac{t^n}{F_n!} \right) \\
=& 1 + \lambda + \lambda t \left(\partial_{F,t}\, e^{f_{\mathbb{F}}t} - e^{f_{\mathbb{F}}t} + 1 \right).
\end{aligned}$$

□

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The authors have read and agreed to the published version of the manuscript.

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