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Research Article

Branched continued fraction representations of ratios of Horn's confluent function H_6

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ABSTRACT. In this paper, we derive some branched continued fraction representations for the ratios of the Horn's confluent function H_6. The method employed is a two-dimensional generalization of the classical method of constructing of Gaussian continued fraction. We establish the estimates of the rate of convergence for the branched continued fraction expansions in some region Ω (here, region is a domain (open connected set) together with all, part or none of its boundary). It is also proved that the corresponding branched continued fractions uniformly converge to holomorphic functions on every compact subset of some domain Θ, and that these functions are analytic continuations of the ratios of double confluent hypergeometric series in Θ. At the end, several numerical experiments are represented to indicate the power and efficiency of branched continued fractions as an approximation tool compared to double confluent hypergeometric series.

Keywords: Hypergeometric function, branched continued fraction, convergence.

2020 Mathematics Subject Classification: 33C65, 32A17, 40A99.

1. INTRODUCTION

This paper deals with branched continued fraction representations for the ratios of the Horn's confluent function H_6, which occurs in [27] (see also [24, Subsection 5.7.1]) of second-order hypergeometric series of two variables. The branched continued fraction representations under consideration will be two-dimensional generalization of the classical Gaussian continued fraction, or rather its confluent case. Necessarily, due to the convergence of branched continued fractions, this requires restrictions on the allowed values of the parameters of the Horn's confluent function H_6.

J. Horn [27] listed all convergent hypergeometric series of the second order: 14 complete series, including Appell's hypergeometric series F_1, F_2, F_3, and F_4, dating back to 1880 [6], and 20 of their confluent cases. In [24, Section 5.9], for each function in Horn's list a system of two partial differential equations is given, which has this function as a solution. For the basics of hypergeometric functions of two variables, see, for instance, [7, Chapter 9], [24, Section 5.9–2.12], and [25, Chapter 1].

In order for a branched continued fraction to be a representation of a function, it is required to solve such problems: to construct the branched continued fraction expansion, to prove the convergence of the constructed expansion, and last, more important, to prove the convergence of the branched continued fraction to the function of which it is an expansion.

For Appell's hypergeometric functions, branched continued fraction representations were derived in [8, pp. 244–252] for F_1, in [15] for F_3, and in [16, 26] for F_4. A branched continued

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fraction expansion for function F_2 was constructed in [13], but the problem of its convergence remains open. In [1], it is represented a branched continued fraction representations for the Horn's function H_3. At last, in [18], it is indicated which three- and four-term recurrent relations give similar expansions for the Horn's function H_4. Some interesting and different branched continued fraction representations of other hypergeometric series can be found in [2, 3, 14, 28, 29, 31], and some special analytic functions of one or several variables in [19, 20, 21, 22, 23, 30, 32].

The contents of this paper are as follows. In Section 2, we derive three different formal branched continued fraction expansions for three different ratios of the Horn's confluent function H_6. In Section 3, we establish the estimates of the rate of convergence for the branched continued fractions mentioned above. We also prove that the branched continued fraction expansions converge to the functions, which are analytic continuations of Horn's confluent function H_6 ratios in some domain (here, domain is an open connected set), i.e., our main result is formulated in the Theorem 3.3. Finally, in Section 4, we present some numerical experiments to indicate the power and efficiency of branched continued fractions as an approximation tool compared to double confluent hypergeometric series.

2. Expansions

The Horn's confluent function H_6 [27] is defined as double power series by

$$\mathrm{H}_6(a,c;\mathbf{z})=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}}{(c)_{m+n}}\frac{z_1^m z_2^n}{m!\,n!},\quad |z_1|<1/4, \tag{2.1}$$

where a,c are complex numbers, $c\notin\{0,-1,-2,\dots\}$, $(\cdot)_k$ is the Pochhammer symbol, $\mathbf{z}=(z_1,z_2)\in\mathbb{C}^2$.

Throughout the paper, let $[\cdot]$ be an integer part of a number. We set $\mathcal{I}_0=\{1,2,3\}$ and for $k\in\mathbb{N}$ we introduce the following sets of multiindices

$$\mathcal{I}_k=\{i(k)=(i_0,i_1,i_2,\dots,i_k):\ i_0\in\mathcal{I}_0,\ 2-[(i_{r-1}-1)/2]\le i_r\le 3-[(i_{r-1}-1)/2],\ 1\le r\le k\}.$$

Using the idea of combining several branched continued fraction expansions into one form using the Kronecker delta symbol, proposed in [1], we will prove the following theorem.

Theorem 2.1. *Let for all $i_0\in\mathcal{I}_0$*

$$R_{i_0}(a,c;\mathbf{z})=\frac{\mathrm{H}_6(a,c;\mathbf{z})}{\mathrm{H}_6(a+\delta_{i_0}^1+\delta_{i_0}^2,c+\delta_{i_0}^2+\delta_{i_0}^3;\mathbf{z})}, \tag{2.2}$$

where δ_i^j is the Kronecker delta. Then for each $i_0\in\mathcal{I}_0$, the ratio $R_{i_0}(a,c;\mathbf{z})$ has a formal branched continued fraction expansion of the form

$$1-\frac{a}{2c}\delta_{i_0}^3+\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\frac{P_{i(1)}(\mathbf{z})}{Q_{i(1)}}+\sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]}\frac{P_{i(2)}(\mathbf{z})}{Q_{i(2)}}+\dots, \tag{2.3}$$

where for $i(1) \in \mathcal{I}_1$

$$P_{i(1)}(\mathbf{z}) = p_{i_0,i_1}(a,c;\mathbf{z})$$

$$= \begin{cases} -2\dfrac{a+1}{c}z_1, & \text{if } i_0=1,\ i_1=2, \\ -\dfrac{z_2}{c}, & \text{if } i_0=1,\ i_1=3, \\ -\dfrac{(2c-a)(a+1)}{c(c+1)}z_1, & \text{if } i_0=2,\ i_1=2, \\ -\dfrac{c-a}{c(c+1)}z_2, & \text{if } i_0=2,\ i_1=3, \\ \dfrac{a}{2c}, & \text{if } i_0=3,\ i_1=1, \\ \dfrac{a}{2c(c+1)}z_2, & \text{if } i_0=3,\ i_1=2, \end{cases} \tag{2.4}$$

for $i(k+1) \in \mathcal{I}_{k+1}$, $k \geq 1$,

$$P_{i(k+1)}(\mathbf{z}) = p_{i_k,i_{k+1}}\left(a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3, c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1;\mathbf{z}\right)$$

$$= \begin{cases} -\dfrac{2(a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3+1)}{c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1}z_1, & \text{if } i_k=1,\ i_{k+1}=2, \\ -\dfrac{z_2}{c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1}, & \text{if } i_k=1,\ i_{k+1}=3, \\ -\dfrac{(2c-a+k+\sum_{r=0}^{k-1}(\delta_{i_r}^3-2\delta_{i_r}^1))(a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3+1)}{(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1+1)}z_1, & \text{if } i_k=2,\ i_{k+1}=2, \\ -\dfrac{c-a+\sum_{r=0}^{k-1}(\delta_{i_r}^3-\delta_{i_r}^1)}{(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1+1)}z_2, & \text{if } i_k=2,\ i_{k+1}=3, \\ \dfrac{a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)}, & \text{if } i_k=3,\ i_{k+1}=1, \\ \dfrac{a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+k+1-\sum_{r=0}^{k-1}\delta_{i_r}^1)}z_2, & \text{if } i_k=3,\ i_{k+1}=2, \end{cases} \tag{2.5}$$

and for $i(k) \in \mathcal{I}_k$, $k \geq 1$,

$$Q_{i(k)} = q_{i_k}\left(a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3, c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1\right)$$

$$= 1 - \frac{a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)}\delta_{i_k}^3. \tag{2.6}$$

Proof. The formal identities

$$\mathrm{H}_6(a,c;\mathbf{z}) = \mathrm{H}_6(a+1,c;\mathbf{z}) - \frac{2(a+1)}{c}z_1\mathrm{H}_6(a+2,c+1;\mathbf{z}) - \frac{1}{c}z_2\mathrm{H}_6(a+1,c+1;\mathbf{z}), \tag{2.7}$$

$$\mathrm{H}_6(a,c;\mathbf{z}) = \mathrm{H}_6(a+1,c+1;\mathbf{z}) - \frac{(a+1)(2c-a)}{c(c+1)}z_1\mathrm{H}_6(a+2,c+2;\mathbf{z})$$
$$- \frac{c-a}{c(c+1)}z_2\mathrm{H}_6(a+1,c+2;\mathbf{z}), \tag{2.8}$$

and

$$\mathrm{H}_6(a,c;\mathbf{z}) = \frac{a}{2c}\mathrm{H}_6(a+1,c+1;\mathbf{z}) + \frac{2c-a}{2c}\mathrm{H}_6(a,c+1;\mathbf{z}) + \frac{a}{2c(c+1)}z_2\mathrm{H}_6(a+1,c+2;\mathbf{z}) \tag{2.9}$$

are easily verified from (2.1). Dividing (2.7) by $\mathrm{H}_6(a+1,c;\mathbf{z})$, (2.8) by $\mathrm{H}_6(a+1,c+1;\mathbf{z})$, and (2.9) by $\mathrm{H}_6(a,c+1;\mathbf{z})$, we get

$$R_1(a,c;\mathbf{z}) = 1 - \frac{2(a+1)}{c}z_1\frac{1}{R_2(a+1,c;\mathbf{z})} - \frac{1}{c}z_2\frac{1}{R_3(a+1,c;\mathbf{z})}, \tag{2.10}$$

$$R_2(a,c;\mathbf{z}) = 1 - \frac{(a+1)(2c-a)}{c(c+1)}z_1\frac{1}{R_2(a+1,c+1;\mathbf{z})} - \frac{c-a}{c(c+1)}z_2\frac{1}{R_3(a+1,c+1;\mathbf{z})} \tag{2.11}$$

and

$$R_3(a,c;\mathbf{z}) = \frac{2c-a}{2c} + \frac{a}{2c}\frac{1}{R_1(a,c+1;\mathbf{z})} + \frac{a}{2c(c+1)}z_2\frac{1}{R_2(a,c+1;\mathbf{z})}, \tag{2.12}$$

respectively. It is obvious that for $i\in\mathcal{I}_0$ the identities (2.10)–(2.12) can be written as

$$R_i(a,c;\mathbf{z}) = 1 - \frac{a}{2c}\delta_i^3 + \sum_{j=2-[(i-1)/2]}^{3-[(i-1)/2]}\frac{p_{i,j}(a,c;\mathbf{z})}{R_j(a+1-\delta_i^3,c+1-\delta_i^1;\mathbf{z})}, \tag{2.13}$$

where $p_{i,j}(a,c;\mathbf{z})$, $(i,j)\in\mathcal{I}_1$ are defined as (2.4).

Now, we can construct branched continued fractions for ratios $R_{i_0}(a,c;\mathbf{z})$ for all $i_0\in\mathcal{I}_0$. Setting $i=i_0$, on the first step, from (2.13) for $i_0\in\mathcal{I}_0$, we obtain

$$\begin{aligned} R_{i_0}(a,c;\mathbf{z}) &= 1 - \frac{a}{2c}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\frac{p_{i_0,i_1}(a,c;\mathbf{z})}{R_{i_1}(a+1-\delta_{i_0}^3,c+1-\delta_{i_0}^1;\mathbf{z})} \\ &= 1 - \frac{a}{2c}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\frac{P_{i(1)}(\mathbf{z})}{R_{i_1}(a+1-\delta_{i_0}^3,c+1-\delta_{i_0}^1;\mathbf{z})}. \end{aligned}$$

It follows from (2.13) that for $i_1\in\mathcal{I}_0$

$$\begin{aligned} &R_{i_1}(a+1-\delta_{i_0}^3,c+1-\delta_{i_0}^1;\mathbf{z}) \\ =&q_{i_1}(a+1-\delta_{i_0}^3,c+1-\delta_{i_0}^1) + \sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]}\frac{p_{i_1,i_2}(a+1-\delta_{i_0}^3,c+1-\delta_{i_0}^1;\mathbf{z})}{R_{i_2}(a+2-\sum_{r=0}^1\delta_{i_r}^3,c+2-\sum_{r=0}^1\delta_{i_r}^1;\mathbf{z})} \\ =&Q_{i(1)} + \sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]}\frac{P_{i(2)}(\mathbf{z})}{R_{i_2}(a+2-\sum_{r=0}^1\delta_{i_r}^3,c+2-\sum_{r=0}^1\delta_{i_r}^1;\mathbf{z})}, \end{aligned}$$

where $P_{i(2)}(\mathbf{z})$, $i(2)\in\mathcal{I}_2$, and $Q_{i(1)}$, $i(1)\in\mathcal{I}_1$, are defined by (2.5) and (2.6), respectively. Then, on the second step for $i_0\in\mathcal{I}_0$, we have

$$\begin{aligned} R_{i_0}(a,c;\mathbf{z}) = 1 - \frac{a}{2c}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\frac{P_{i(1)}(\mathbf{z})}{Q_{i(1)}} \\ + \sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]}\frac{P_{i(2)}(\mathbf{z})}{R_{i_2}(a+2-\sum_{r=0}^1\delta_{i_r}^3,c+2-\sum_{r=0}^1\delta_{i_r}^1;\mathbf{z})}. \end{aligned}$$

Next, applying (2.13) after nth steps, for $i_0 \in \mathcal{I}_0$ we get

$$R_{i_0}(a,c;\mathbf{z}) = 1 - \frac{a}{2c}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{P_{i(1)}(\mathbf{z})}{Q_{i(1)}} + \sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]} \frac{P_{i(2)}(\mathbf{z})}{Q_{i(2)}} + \dots + \sum_{i_{n-1}=2-[(i_{n-2}-1)/2]}^{3-[(i_{n-2}-1)/2]} \frac{P_{i(n-1)}(\mathbf{z})}{Q_{i(n-1)}} + \sum_{i_n=2-[(i_{n-1}-1)/2]}^{3-[(i_{n-1}-1)/2]} \frac{P_{i(n)}(\mathbf{z})}{R_{i_n}(a+n-\sum_{r=0}^{n-1}\delta_{i_r}^3, c+2-\sum_{r=0}^{n-1}\delta_{i_r}^1;\mathbf{z})},$$

where $P_{i(1)}(\mathbf{z})$, $i(1) \in \mathcal{I}_1$, $P_{i(k)}(\mathbf{z})$, $i(k) \in \mathcal{I}_k$, $2 \le k \le n$, and $Q_{i(k)}$, $i(k) \in \mathcal{I}_k$, $1 \le k \le n-1$, are defined by (2.4), (2.5), and (2.6), respectively. Finally, by (2.13), we obtain the formal branched continued fraction expansions (2.3) for ratios (2.2) for all $i_0 \in \mathcal{I}_0$. □

3. Convergence

In this section, we consider some question of convergence of the branched continued fractions (2.3). We refer the readers to [1, 5, 12] for the notations and definitions used below.

Let i_0 be an arbitrary index from the set $\mathcal{I}_0$. For the 'tails' of the approximants of the branched continued fraction (2.3), we set

$$G_{i(r)}^{(r)}(\mathbf{z}) = Q_{i(r)}, \quad i(r) \in \mathcal{I}_r,\ r \ge 1, \tag{3.14}$$

and

$$G_{i(k)}^{(r)}(\mathbf{z}) = Q_{i(k)} + \sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]} \frac{P_{i(k+1)}(\mathbf{z})}{Q_{i(k+1)}} + \sum_{i_{k+2}=2-[(i_{k+1}-1)/2]}^{3-[(i_{k+1}-1)/2]} \frac{P_{i(k+2)}(\mathbf{z})}{Q_{i(k+2)}} + \dots + \sum_{i_r=2-[(i_{r-1}-1)/2]}^{3-[(i_{r-1}-1)/2]} \frac{P_{i(r)}(\mathbf{z})}{Q_{i(r)}},$$

where $i(k) \in \mathcal{I}_k,\ 1 \le k \le r-1,\ r \ge 2$. Then, it is easily seen that relations

$$G_{i(k)}^{(r)}(\mathbf{z}) = Q_{i(k)} + \sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]} \frac{P_{i(k+1)}(\mathbf{z})}{G_{i(k+1)}^{(r)}(\mathbf{z})}, \quad i(k) \in \mathcal{I}_k,\ 1 \le k \le r-1,\ r \ge 2 \tag{3.15}$$

hold. It follows that for each $n \ge 1$ the nth approximant

$$f_n^{(i_0)}(\mathbf{z}) = 1 - \frac{a}{2c}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{P_{i(1)}(\mathbf{z})}{Q_{i(1)}} + \sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]} \frac{P_{i(2)}(\mathbf{z})}{Q_{i(2)}} + \dots + \sum_{i_n=2-[(i_{n-1}-1)/2]}^{3-[(i_{n-1}-1)/2]} \frac{P_{i(n)}(\mathbf{z})}{Q_{i(n)}}$$

can be written as

$$f_n^{(i_0)}(\mathbf{z}) = 1 - \frac{a}{2c}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{P_{i(1)}(\mathbf{z})}{G_{i(1)}^{(n)}(\mathbf{z})}.$$

In addition, it can be shown (see [12, p. 28]) that for $m > n$ and $n \geq 1$

$$f_m^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z}) = (-1)^n \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]} \ldots \sum_{i_{n+1}=2-[(i_n-1)/2]}^{3-[(i_n-1)/2]} \frac{\prod_{k=1}^{n+1} P_{i(k)}(\mathbf{z})}{\prod_{k=1}^{n+1} G_{i(k)}^{(m)}(\mathbf{z}) \prod_{k=1}^{n} G_{i(k)}^{(n)}(\mathbf{z})},$$

provided $G_{i(k)}^{(r)}(\mathbf{z}) \neq 0$ for all $i(k) \in \mathcal{I}_k$, $1 \leq k \leq r$, $r \in \{m, n\}$, which for convenience we will write as

$$f_m^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z}) = (-1)^n \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \ldots \sum_{i_{n+1}=2-[(i_n-1)/2]}^{3-[(i_n-1)/2]} \frac{P_{i(1)}(\mathbf{z})}{G_{i(1)}^{(q)}(\mathbf{z})}$$

$$\times \prod_{k=1}^{[(n+1)/2]} \frac{P_{i(2k)}(\mathbf{z})}{G_{i(2k-1)}^{(r)}(\mathbf{z}) G_{i(2k)}^{(r)}(\mathbf{z})} \prod_{k=1}^{[n/2]} \frac{P_{i(2k+1)}(\mathbf{z})}{G_{i(2k)}^{(q)}(\mathbf{z}) G_{i(2k+1)}^{(q)}(\mathbf{z})}, \tag{3.16}$$

where $q = m$, $r = n$, if n is even, and $q = n$, $r = m$, if n is odd.

To prove the main result, we will state the following theorem.

Theorem 3.2. *Let a and c be real constants such that*

$$a \geq 0, \quad c \geq a + 1 + \delta_{i_0}^1 \quad \text{for all} \quad i_0 \in \mathcal{I}_0. \tag{3.17}$$

Then for each $i_0 \in \mathcal{I}_0$:

(**A**) *The branched continued fraction (2.3) converges to a finite value $f^{(i_0)}(\mathbf{z})$ for each $\mathbf{z} \in \Omega$, where*

$$\Omega = \{\mathbf{z} \in \mathbb{R}^2 : \ -L_1 \leq z_1 \leq 0, \ -L_2 \leq z_2 \leq 0\}, \tag{3.18}$$

L_1 and L_2 are positive constants such that $2L_2 < c + 1$, and it converges uniformly on every compact subset of an interior of Ω.

(**B**) *If $f_n^{(i_0)}(\mathbf{z})$ denotes the nth approximant of the branched continued fraction (2.3), then for each $\mathbf{z} \in \Omega$ and $n \geq 1$*

$$|f^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z})| \leq M_{i_0} \left(\frac{\eta}{\eta + 1} \right)^n, \tag{3.19}$$

where

$$M_{i_0} = \begin{cases} \dfrac{2(a+1)L_1}{c} + \dfrac{2(c+1)L_2}{c(c+1-L_2)}, & \text{if } i_0 = 1, \\ \dfrac{(2c-a)(a+1)L_1}{c(c+1)} + \dfrac{2(c-a)L_2}{c(c+1-L_2)}, & \text{if } i_0 = 2, \\ \dfrac{a}{2c} + \dfrac{aL_2}{2c(c+1)}, & \text{if } i_0 = 3, \end{cases} \tag{3.20}$$

and

$$\eta = \max\left\{ 2L_1 + \frac{2L_2(c+1)}{c(c+1-L_2)}, \ \frac{c+1+L_2}{c+1-2L_2} \right\}. \tag{3.21}$$

Proof. The proof is similar to that of Theorem 1 in [1]. In this case, it follows directly from (2.6) that for all $i(k) \in \mathcal{I}_k$, $k \geq 1$, the elements $Q_{i(k)} = 1$ if $i_k \neq 3$. When $i_k = 3$ from (2.6), we have

$$Q_{i(k)} = 1 - \frac{a + k - \sum_{r=0}^{k-1} \delta_{i_r}^3}{2c + 2k - 2\sum_{r=0}^{k-1} \delta_{i_r}^1} \geq \frac{1}{2} \quad \text{for all} \quad i(k) \in \mathcal{I}_k, \ k \geq 1. \tag{3.22}$$

Since

$$-\sum_{r=0}^{k-1}\delta_{i_r}^3=\sum_{r=0}^{k-1}(\delta_{i_r}^1-\delta_{i_r}^3)-\sum_{r=0}^{k-1}\delta_{i_r}^1\quad\text{for all}\quad i(k)\in\mathcal{I}_k,\ k\ge 1,$$

then to prove the validity of (3.22), provided (3.17), it suffices to show that

$$\sum_{r=0}^{k-1}(\delta_{i_r}^1-\delta_{i_r}^3)\le\delta_{i_0}^1\quad\text{for all}\quad i(k)\in\mathcal{I}_k,\ k\ge 1. \tag{3.23}$$

Indeed, if $k=1$, then for any $i_0\in\mathcal{I}_0$ inequalities (3.23) are obvious. If $i(k)$ is a fixed arbitrary multiindex in $\mathcal{I}_k$, $k\ge 2$, then for any r, $1\le r\le k-1$, there is a possible pair of indices (i_{r-1},i_r), such as $(1,2)$, $(1,3)$, $(2,2)$, $(2,3)$, $(3,1)$, or $(3,2)$. It clearly follows that (3.23) is valid in these cases.

Let $\mathbf{z}$ be an arbitrary fixed point in (3.18) and n be an arbitrary natural number. It is easy to see from (2.4)–(2.6), (3.14), (3.15), and (3.18) that inequalities

$$G_{i(k)}^{(n)}(\mathbf{z})\ge 1\quad\text{for all}\quad i(k)\in\mathcal{I}_k,\ 1\le k\le n, \tag{3.24}$$

hold for all $i_k\ne 3$. By induction on k, we show that the following inequalities

$$G_{i(k)}^{(n)}(\mathbf{z})\ge\frac{c+1-L_2}{2(c+1)}\quad\text{for all}\quad i(k)\in\mathcal{I}_k,\ 1\le k\le n, \tag{3.25}$$

valid for $i_k=3$.

For $k=n$ and for each $i(n)\in\mathcal{I}_n$, inequalities (3.25) are obvious. By induction hypothesis that (3.25) hold for $k=r+1$, where $r+1\le n$, and for each $i(r+1)\in\mathcal{I}_{r+1}$, using (2.4), (2.5), (3.14), (3.18), and (3.22) for any $i(r)\in\mathcal{I}_r$ we get

$$\begin{aligned}G_{i(r)}^{(n)}(\mathbf{z})&=Q_{i(r)}+\frac{P_{i(r),1}(\mathbf{z})}{G_{i(r),1}^{(n)}(\mathbf{z})}+\frac{P_{i(r),2}(\mathbf{z})}{G_{i(r),2}^{(n)}(\mathbf{z})}\\&\ge Q_{i(r)}-\frac{|P_{i(r),2}(\mathbf{z})|}{G_{i(r),2}^{(n)}(\mathbf{z})}\\&\ge\frac{1}{2}-\frac{a+r-\sum_{p=0}^{r-1}\delta_{i_p}^3}{2(c+r-\sum_{p=0}^{r-1}\delta_{i_p}^1)(c+r+1-\sum_{p=0}^{r-1}\delta_{i_p}^1)}|z_2|\\&\ge\frac{c+1-L_2}{2(c+1)}.\end{aligned}$$

Next, we prove that

$$\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]}\frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})G_{i(k)}^{(n)}(\mathbf{z})|}\le\frac{\eta}{\eta+1}\quad\text{for all}\quad i(k)\in\mathcal{I}_k,\ k\ge 1, \tag{3.26}$$

where η is defined by (3.21), which are equivalent to

$$\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]}\frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})|}\le\eta\left(|G_{i(k)}^{(n)}(\mathbf{z})|-\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]}\frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})|}\right)$$

for all $i(k)\in\mathcal{I}_k$, $k\ge 1$. Again, let n be an arbitrary natural number. Since it follows from (2.4)–(2.6), (3.14), (3.15), (3.18), (3.22), (3.24), and (3.25) that, for any k, $1\le k\le n$, and $i(k)\in\mathcal{I}_k$, and for any $\mathbf{z} \in \Omega$

$$|G_{i(k)}^{(n)}(\mathbf{z})| - \sum_{i_{k+1}=2}^{3} \frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})|} = Q_{i(k)} = 1,$$

if $i_k \neq 3$, and

$$\begin{aligned}
|G_{i(k)}^{(n)}(\mathbf{z})| - \sum_{i_{k+1}=1}^{2} \frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})|} &\geq Q_{i(k)} - 2\frac{|P_{i(k),2}(\mathbf{z})|}{|G_{i(k),2}^{(n)}(\mathbf{z})|} \\
&\geq \frac{1}{2} - \frac{a+k-1-\sum_{r=0}^{k-2}\delta_{i_r}^3}{(c+k-1-\sum_{r=0}^{k-2}\delta_{i_r}^1)(c+k-\sum_{r=0}^{k-2}\delta_{i_r}^1)}|z_2| \\
&\geq \frac{1}{2} - \frac{|z_2|}{c+1} \\
&\geq \frac{c+1-2L_2}{2(c+1)},
\end{aligned}$$

if $i_k = 3$, then we obtain

$$\begin{aligned}
\sum_{i_{k+1}=2}^{3} \frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})|} &\leq \frac{2(a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3+1)}{c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1}|z_1| + \frac{2(c+1)}{(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+1-L_2)}|z_2| \\
&\leq 2L_1 + \frac{2L_2(c+1)}{c(c+1-L_2)} \\
&\leq \eta,
\end{aligned}$$

if $i_k = 1$,

$$\begin{aligned}
\sum_{i_{k+1}=2}^{3} \frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})|} &\leq \frac{(2c-a+k+\sum_{r=0}^{k-1}(\delta_{i_r}^3-2\delta_{i_r}^1))(a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3+1)}{(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1+1)}|z_1| \\
&\quad + \frac{(c-a+\sum_{r=0}^{k-1}(\delta_{i_r}^3-\delta_{i_r}^1))(2(c+1))}{(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1+1)(c+1-L_2)}|z_2| \\
&\leq 2L_1 + \frac{2L_2}{c+1-L_2} \\
&\leq \eta,
\end{aligned}$$

if $i_k = 2$, and

$$\begin{aligned}
\sum_{i_{k+1}=1}^{2} \frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})|} &\leq \frac{a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)} + \frac{a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1+1)}|z_2| \\
&\leq \frac{1}{2} + \frac{L_2}{2(c+1)} \\
&= \frac{c+1+L_2}{c+1-2L_2}\,\frac{c+1-2L_2}{2(c+1)} \\
&\leq \frac{c+1-2L_2}{2(c+1)}\eta,
\end{aligned}$$

if $i_k = 3$. Now, it is easy see from (2.4), (3.18), (3.20), (3.24) and (3.25) that

$$\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{|P_{i(1)}(\mathbf{z})|}{|G_{i(1)}^{(q)}(\mathbf{z})|} \le M_{i_0} \quad \text{for all} \quad i_0 \in \mathcal{I}_0 \quad \text{and} \quad q \ge 1. \tag{3.27}$$

From (3.18), (3.24), and (3.25) it follows that $G_{i(k)}^{(q)}(\mathbf{z}) \ne 0$ for all $i(k) \in \mathcal{I}_k$, $1 \le k \le q$, $q \ge 1$, and for all $\mathbf{z} \in \Omega$. Hence, applying (3.26) and (3.27) to (3.16) for any $m > n \ge 1$ and for any $\mathbf{z} \in \Omega$, we obtain

$$\begin{aligned} |f_m^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z})| &\le \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{|P_{i(1)}(\mathbf{z})|}{|G_{i(1)}^{(q)}(\mathbf{z})|} \left(\frac{\eta}{\eta+1}\right)^n \\ &\le M_{i_0} \left(\frac{\eta}{\eta+1}\right)^n, \end{aligned}$$

where $q = m$, if n is even, and $q = n$, if n is odd. From this (A) follows if $n \to \infty$. At last, passing to the limit as $m \to \infty$, we get (B). □

Now, we prove our main result.

Theorem 3.3. *Let a and c be real constants satisfying the inequalities (3.17), and $\nu_1, \nu_2, \nu_3, \mu_1, \mu_2, \mu_3$ be positive numbers such that*

$$\frac{2\nu_1}{\mu_2} \le \min\left\{1 - \mu_1 - \frac{\nu_2}{c\mu_3},\ 1 - \mu_2 - \frac{\nu_2}{(c+1)\mu_3}\right\}, \quad \frac{\nu_3}{(c+1)\mu_2} \le \frac{1}{2} - \mu_3. \tag{3.28}$$

Then for each $i_0 \in \mathcal{I}_0$:

(**A**) *The branched continued fraction (2.3) converges uniformly on every compact subset of*

$$\Theta = \{\mathbf{z} \in \mathbb{C}^2 :\ |z_1| + \mathrm{Re}(z_1) < 2\nu_1,\ |z_2| + \mathrm{Re}(z_2) < 2\nu_2,\ |z_2| - \mathrm{Re}(z_2) < 2\nu_3\} \tag{3.29}$$

to the function $f^{(i_0)}(\mathbf{z})$ holomorphic in Θ.

(**B**) *The function $f^{(i_0)}(\mathbf{z})$ is an analytic continuation of (2.2) in the domain (3.29).*

Proof. The proof of (A) is similar to the proof of Theorem 2 [1]. Let $\mathbf{z}$ be an arbitrary fixed point in (3.29). Since a and c satisfy (3.17), it follows from the proof of Theorem 2 that inequalities (3.22) hold for $i_k = 3$, and that for all $i(k) \in \mathcal{I}_k$, $k \ge 1$, the elements $Q_{i(k)} = 1$ if $i_k \ne 3$. Now, for any $i(k) \in \mathcal{I}$, $k \ge 1$, from (2.4)–(2.6) and (3.29) with $i_k = 1$, we have

$$\begin{aligned} |P_{i(k),2}(\mathbf{z})| - \mathrm{Re}(P_{i(k),2}(\mathbf{z})) &= \frac{2(a + k - \sum_{r=0}^{k-1} \delta_{i_r}^3 + 1)}{c + k - \sum_{r=0}^{k-1} \delta_{i_r}^1}(|z_1| + \mathrm{Re}(z_1)) \\ &< 4\nu_1, \\ |P_{i(k),3}(\mathbf{z})| - \mathrm{Re}(P_{i(k),3}(\mathbf{z})) &= \frac{|z_2| + \mathrm{Re}(z_2)}{c + k - \sum_{r=0}^{k-1} \delta_{i_r}^1} \\ &< \frac{2\nu_2}{c}, \end{aligned}$$

and, thus,

$$\sum_{i_{k+1}=2}^{3}\frac{|P_{i(k+1)}(\mathbf{z})|-\mathrm{Re}(P_{i(k+1)}(\mathbf{z}))}{\mu_{i_{k+1}}}<\frac{4\nu_1}{\mu_2}+\frac{2\nu_2}{c\mu_3}$$
$$\le 2(1-\mu_1)$$
$$=2(\mathrm{Re}(Q_{i(k)})-\mu_1).$$

If $i_k=2$, we obtain

$$|P_{i(k),2}(\mathbf{z})|-\mathrm{Re}(P_{i(k),2}(\mathbf{z}))$$
$$=\frac{(2c-a+k+\sum_{r=0}^{k-1}(\delta_{i_r}^3-2\delta_{i_r}^1))(a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3+1)}{(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1+1)}(|z_1|+\mathrm{Re}(z_1))$$
$$<4\nu_1,$$
$$|P_{i(k),3}(\mathbf{z})|-\mathrm{Re}(P_{i(k),3}(\mathbf{z}))$$
$$=\frac{c-a+\sum_{r=0}^{k-1}(\delta_{i_r}^3-\delta_{i_r}^1)}{(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1+1)}(|z_2|+\mathrm{Re}(z_2))$$
$$<\frac{2\nu_2}{c+1},$$

and, thus,

$$\sum_{i_{k+1}=2}^{3}\frac{|P_{i(k+1)}(\mathbf{z})|-\mathrm{Re}(P_{i(k+1)}(\mathbf{z}))}{\mu_{i_{k+1}}}<\frac{4\nu_1}{\mu_2}+\frac{2\nu_2}{(c+1)\mu_3}$$
$$\le 2(\mathrm{Re}(Q_{i(k)})-\mu_2).$$

At last, if $i_k=3$ we get

$$|P_{i(k),1}(\mathbf{z})|-\mathrm{Re}(P_{i(k),1}(\mathbf{z}))=\frac{a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)}-\frac{a+k-\sum_{p=r}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)}$$
$$=0,$$
$$|P_{i(k),2}(\mathbf{z})|-\mathrm{Re}(P_{i(k),2}(\mathbf{z}))=\frac{a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+k+1-\sum_{r=0}^{k-1}\delta_{i_r}^1)}(|z_2|-\mathrm{Re}(z_2))$$
$$<\frac{2\nu_3}{c+1},$$

and, thus,

$$\sum_{i_{k+1}=1}^{2}\frac{|P_{i(k+1)}(\mathbf{z})|-\mathrm{Re}(P_{i(k+1)}(\mathbf{z}))}{\mu_{i_{k+1}}}<2\left(\frac{1}{2}-\mu_3\right)$$
$$\le 2(\mathrm{Re}(Q_{i(k)})-\mu_3).$$

Thus, by Lemma 1 [4], for all $i(k)\in\mathcal{I}_k$, $1\le k\le n$, $n\ge 1$, and for all $\mathbf{z}\in\Theta$ the following inequalities hold

$$\mathrm{Re}(G_{i(k)}^{(n)}(\mathbf{z}))\ge\mu_k,$$

where $G_{i(k)}^{(n)}(\mathbf{z})$, $i(k)\in\mathcal{I}_k$, $1\le k\le n$, $n\ge 1$, are defined by (3.14) and (3.15). The approximants $f_n^{(i_0)}(\mathbf{z})$, $n\ge 1$, of (2.3) form a sequence of functions holomorphic in (3.29).

At last, it remains to show that the branched continued fraction (2.3) converges uniformly on compact subsets of Θ. Let $\mathcal{K}$ is an arbitrary compact subset of (3.29). Then there exists an open ball around the origin with radius R, containing $\mathcal{K}$. By (2.4), for the any $\mathbf{z} \in \mathcal{K}$ and for any $n \geq 1$, we get

$$\begin{aligned}|f_n^{(i_0)}(\mathbf{z})| &\leq 1 + \frac{a}{2c}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{|P_{i(1)}(\mathbf{z})|}{\mu_{i(1)}} \\ &= C_{i_0}(\mathcal{K}),\end{aligned}$$

where

$$C_{i_0}(\mathcal{K}) = \begin{cases} \dfrac{2(a+1)R}{c\mu_2} + \dfrac{R}{c\mu_3}, & \text{if } i_0 = 1, \\ \dfrac{(2c-a)(a+1)R}{c(c+1)\mu_2} + \dfrac{(c-a)R}{c(c+1)\mu_3}, & \text{if } i_0 = 2, \\ \dfrac{a}{2c\mu_1} + \dfrac{aR}{2c(c+1)\mu_2}, & \text{if } i_0 = 3. \end{cases}$$

It follows that for each $i_0 \in \mathcal{I}_0$ the sequence $\{f_n^{(i_0)}(\mathbf{z})\}$ is uniformly bounded on $\mathcal{K}$, and hence it is uniformly bounded on every compact subset of the domain (3.29). We set $\delta = \min\{c/4, \nu_1, \nu_3\}$. Then, by Theorem 2, the sequence $\{f_n^{(i_0)}(\mathbf{z})\}$ converges in

$$\Delta = \{\mathbf{z} \in \mathbb{C}^2 : \ -\delta < \mathrm{Re}(z_k) < 0, \ \mathrm{Im}(z_k) = 0, \ k = 1, 2\},$$

which is the real neighborhood of the point $\mathbf{z}^{(0)} = (-\delta/2, -\delta/2)$ in Θ. Furthermore, it is clear that $\Delta \subset \Theta$. Thus, by Theorem 3 [1] (see also Theorem 2.17 [12]), for each $i_0 \in \mathcal{I}_0$ the branched continued fraction (2.3) converges uniformly on compact subsets of Θ to the function $f^{(i_0)}(\mathbf{z})$ holomorphic in Θ. This proves (A).

Finally, the proof of (B) is analogous to the proof of Theorem 2 [1]; hence it is omitted. $\square$

Setting $a = 0$ and $i_0 = 1$ (or $i_0 = 2$ and replacing c by $c - 1$) in Theorem 3.3, we get a corollary.

Corollary 3.1. *Let c be real constant such that $c \geq 2$, and $\nu_1, \nu_2, \nu_3, \mu_1, \mu_2, \mu_3$ be positive numbers satisfying the inequalities (3.28). Then for $i_0 = 1$ (or $i_0 = 2$):*

(**A**) *The branched continued fraction*

(3.30) $$\frac{1}{1} + \sum_{i_1=2}^{3} \frac{P_{i(1)}(\mathbf{z})}{Q_{i(1)}} + \sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]} \frac{P_{i(2)}(\mathbf{z})}{Q_{i(2)}} + \cdots + \sum_{i_k=2-[(i_{k-1}-1)/2]}^{3-[(i_{k-1}-1)/2]} \frac{P_{i(k)}(\mathbf{z})}{Q_{i(k)}} + \cdots,$$

where for $i(1) \in \mathcal{I}_1$

(3.31) $$P_{i(1)}(\mathbf{z}) = \begin{cases} -\dfrac{2}{c}z_1, & \text{if } i_1 = 2, \\ -\dfrac{z_2}{c}, & \text{if } i_1 = 3, \end{cases}$$

for $i(k+1)\in\mathcal{I}_{k+1}$, $k\geq 1$,

$$
(3.32)\quad P_{i(k+1)}(\mathbf{z}) = \begin{cases}
-\dfrac{2(k-\sum_{r=1}^{k-1}\delta_{i_r}^3+1)}{c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1-1}z_1, & \text{if } i_k=1,\ i_{k+1}=2,\\
-\dfrac{z_2}{c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1-1}, & \text{if } i_k=1,\ i_{k+1}=3,\\
-\dfrac{(2(c-1)+k+\sum_{r=1}^{k-1}(\delta_{i_r}^3-2\delta_{i_r}^1))(k-\sum_{r=1}^{k-1}\delta_{i_r}^3+1)}{(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1-1)(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1)}z_1, & \text{if } i_k=2,\ i_{k+1}=2,\\
-\dfrac{c+\sum_{r=1}^{k-1}(\delta_{i_r}^3-\delta_{i_r}^1)-1}{(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1-1)(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1)}z_2, & \text{if } i_k=2,\ i_{k+1}=3,\\
\dfrac{k-\sum_{r=1}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1-1)}, & \text{if } i_k=3,\ i_{k+1}=1,\\
\dfrac{k-\sum_{r=1}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1-1)(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1)}z_2, & \text{if } i_k=3,\ i_{k+1}=2,
\end{cases}
$$

and for $i(k)\in\mathcal{I}_k$, $k\geq 1$,

$$
(3.33)\quad Q_{i(k)} = 1-\frac{k-\sum_{r=1}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1-1)}\delta_{i_k}^3,
$$

converges uniformly on every compact subset of (3.29) to the function $f(\mathbf{z})$ *holomorphic in* Θ.

(**B**) *The function* $f(\mathbf{z})$ *is an analytic continuation of* $\mathrm{H}_6(1,c;\mathbf{z})$ *in the domain (3.29).*

4. Numerical Experiments

From [24, Formula (37), p. 236], it follows that Horn's confluent function $\mathrm{H}_6(1,2;\mathbf{z})$ satisfies the system of two partial differential equations

$$
(4.34)\quad \begin{cases}
z_1(1-4z_1)\dfrac{\partial^2 u}{\partial z_1^2}+z_2(1-4z_1)\dfrac{\partial^2 u}{\partial z_1\partial z_2}-z_2^2\dfrac{\partial^2 u}{\partial z_2^2}+(2-10z_1)\dfrac{\partial u}{\partial z_1}-4z_2\dfrac{\partial u}{\partial z_2}-2u=0,\\
z_1\dfrac{\partial^2 u}{\partial z_1\partial z_2}+z_2\dfrac{\partial^2 u}{\partial z_2^2}-2z_1\dfrac{\partial u}{\partial z_1}+(2-z_2)\dfrac{\partial u}{\partial z_2}-u=0,
\end{cases}
$$

where $u=u(\mathbf{z})$ is an unknown function of independent variables z_1 and z_2. If the conditions of Corollary 3.1 are satisfied, the branched continued fraction (3.30) satisfies (4.34).

Setting $c=2$, $\nu_1=\nu_2=\nu_3=1/20$, and $\mu_1=\mu_2=\mu_3=1/5$ it is easy to see that the conditions (3.28) are satisfied. Thus, by Corollary 3.1, the approximations of (3.30) with $c=2$ can be used to approximate the solution of (4.34) in the domain (3.29). From (3.31)–(3.32), we have such the approximations as

$$
f_1(\mathbf{z})=1,\quad f_2(\mathbf{z})=\frac{3}{3-3z_1-2z_2},\quad \text{etc.}\quad .
$$

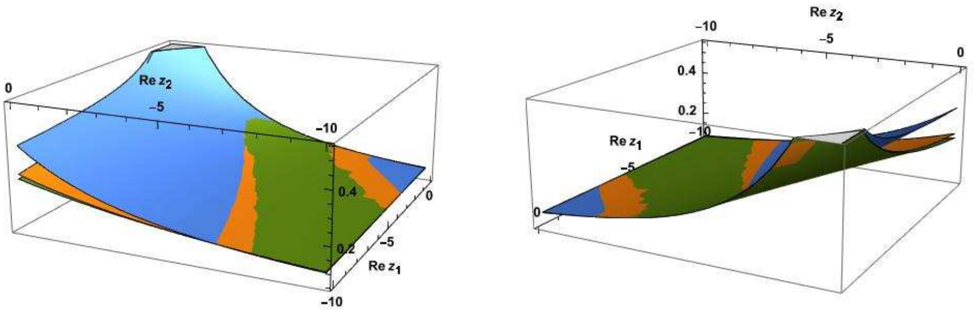
The values of these approximations $f_n(\mathbf{z})$ are given in Table 1 together with the values of the partial sums $S_n(\mathbf{z})$ of $\mathrm{H}_6(1,2,\mathbf{z})$ for $1\leq n\leq 10$ and for the various values of $\mathbf{z}$. This table shows the rate of convergence of $f_n(\mathbf{z})$ and $S_n(\mathbf{z})$ to $u(\mathbf{z})$ as n increases. We also see that the branched continued fraction gives better approximations of the solution of (4.34) than double confluent hypergeometric series.

TABLE 1. Approximation of the solution of (4.34) by branched continued fraction (3.30) with $c = 2$ and confluent hypergeometric series $\mathrm{H}_6(1, 2, \mathbf{z})$

n	$f_n(-0.2, -0.04)$	$S_n(-0.2, -0.04)$	$f_n(0.04, 0.04)$	$S_n(0.04, 0.04)$
1	1	0.78	1	1.06
2	0.8152173913043479	0.8682666666666666	1.0714285714285714	1.0650666666666666
3	0.8436283082662936	0.824104	1.066142202005891	1.0655813333333333
4	0.8390655552756958	0.8488421546666667	1.0656278844624396	1.0656393813333334
5	0.8397705605715909	0.8339969065244445	1.0656448968469723	1.0656463745422222
6	0.8396627464248548	0.8433291477625905	1.065647430637354	1.0656472558998347
7	0.8396790254380795	0.8372627668078485	1.0656473978749492	1.0656473706761305
8	0.8396765860675122	0.8413072281608152	1.0656473883827595	1.0656473859992222
9	0.8396769494594616	0.8385568865940254	1.0656473883916724	1.0656473880852033
10	0.839676895549416	0.8404571814141544	1.0656473884206237	1.0656473883736706

From [17, §3.4], it follows that

$$\mathrm{H}_6(1,2,\mathbf{z}) = \int_0^1 \left(\frac{(1-4tz_1)^{-1/2}}{B(1,1)}\, {}_1F_2\left(\frac{1}{2}; \frac{1}{2}, 1; \frac{t(1-t)z_2^2}{1-4tz_1}\right) + \frac{2(t-t^2)^{1/2} z_2}{(1-4tz_1)B(1/2,3/2)}\, {}_1F_2\left(1; \frac{3}{2}, \frac{3}{2}; \frac{t(1-t)z_2^2}{1-4tz_1}\right)\right) dt. \tag{4.35}$$



(A) ■ – 5th, ■ – (4.35), ■ – 10th (B) ■ – 5th, ■ – (4.35), ■ – 10th

FIGURE 1. The plots of values of the nth approximants of (3.30)

In Figure 1 (A)–(B), we can see the plots of the values of 5th and 10th approximations of (3.30) approaches to the plot of the function (4.35). Figure 2 (A)–(D) shows the plots where the 10th approximants of (3.30) guarantees certain truncation error bounds for function (4.35). Finally, in Table 2, we can see that the 5th approximant of (3.30) is eventually a better approximation to (4.35) than the corresponding 5th partial sum of (2.2).

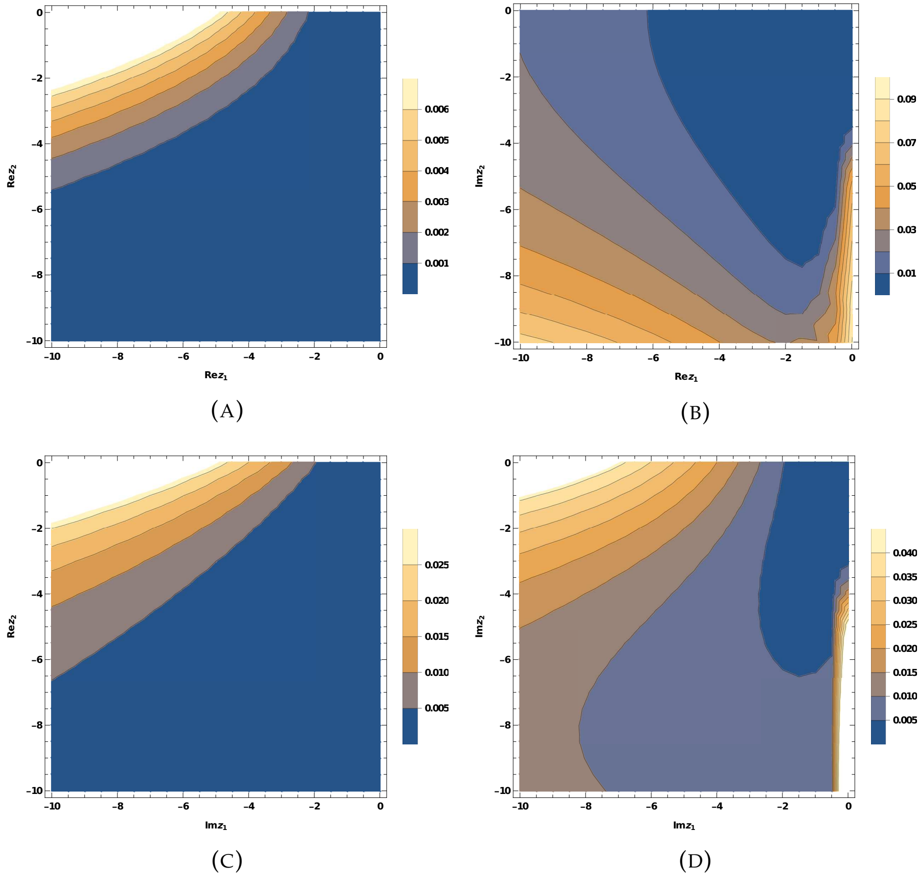


FIGURE 2. The plots where the 10th approximants of (3.30) guarantees certain truncation error bounds for (4.35)

TABLE 2. Relative errors of 5th partial sum and 5th approximant for the Horn's confluent function $\mathrm{H}_6(1, 2, \mathbf{z})$

$\mathbf{z}$	**(4.35)**	**(2.2)**	**(3.30)**
$(-0.01, 0.01)$	0.9951138277	3.8606×10^{-08}	8.8026×10^{-09}
$(-0.1, 0.1)$	0.9593510752	6.2346×10^{-05}	9.4458×10^{-06}
$(-0.1, -0.01)$	0.9118965224	1.1498×10^{-04}	6.5181×10^{-06}
$(0.09, 0.05)$	1.1425549298	1.1470×10^{-04}	5.0158×10^{-06}
$(-0.15, -0.2)$	0.8094560924	2.3880×10^{-03}	2.0638×10^{-04}
$(0.2, 0.2)$	1.5918307333	2.6823×10^{-02}	2.7319×10^{-03}
$(0.2, -5.0)$	0.1998004145	$2.0382 \times 10^{+00}$	2.5676×10^{-03}
$(-5.0, 0.3)$	0.3782185176	$3.1579 \times 10^{+05}$	2.0912×10^{-01}
$(-10.0, -10.0)$	0.0932899388	$7.0858 \times 10^{+07}$	3.8248×10^{-02}
$(-25.0, -25.0)$	0.0395665845	$1.6635 \times 10^{+10}$	6.6127×10^{-01}

5. Conclusions

The paper considers the problem of representing the ratios of the confluent hypergeometric Horn's function H_6 by branched continued fractions. It is proved that the branched continued fractions converge to the ratios of the confluent hypergeometric series of which they are expansions, but the conditions of their convergence impose additional restrictions on the parameters of the function. The expediency and effectiveness of using branched continued fractions as an approximation tool are confirmed by numerical experiments. Nevertheless, the problems of improving and developing new methods of researching the convergence of such and similar branched continued fractions are open. Along the way, let us note the recent interesting and promising ideas regarding the study of the convergence of branched continued fractions proposed in papers [9, 10, 11].

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