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Gauss, Mean and Total Curvature Formulae of Rational Bezier Curves in Minkowski 4-Space

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Highlights:
- Bezier curves
- Rational Bezier curves
- Minovski
- Space

Keywords:
- Gauss Curvature
- Mean Curvature
- Total Curvature
- Minkowski 4-Space
- Rational Bezier Curves

ABSTRACT:

In this paper Gaussian, Mean and total curvature formulae of Rational Bezier Curves with asymptotic frame field are calculated by using its curvatures in 3- dimensional lightlike cone in Minkowski 4- Space. Our main intention is to introduce and investigate some differential geometric properties of the the Rational Bezier Curves with asymptotic frame field in 3-dimensional lightlike cone in Minkowski 4- Space by using its curvatures.

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INTRODUCTION

In differential geometry curvatures of curves are easily calculated by using the curve Frenet frame and also it takes special situations where space it is calculated. The first curvature κ (or shown as k_1) and second curvature τ (or shown as k_2) of a curve was known in differential geometry. Also Gauss curvature, Mean curvature and total curvatures can be calculated easily. These curvatures are special properties which belong to curve and these are the basic subjects about curve in differential geometry.

The first study of Bezier curves in Minkowski space was made by Georgiev in 2008 (Georgiev, 2008). Later, Spacelike and Timelike Bezier curves and surfaces in Minkowski space were studied in more detail by Kuşak Samancı and Çelik (Çelik,2017; Kuşak Samancı, 2018; Kuşak Samancı and Çelik, 2018; Kuşak Samancı, Kalkan and Celik, 2019). Bezier surfaces in Minkowski space were studied by Ugail (Ugail, Márquez and Yılmaz, 2011) and Kuşak Samancı and Çelik calculated the shape operator of the Bezier curve in Minkowski space (Kuşak Samancı, Çelik and İncesu, 2020).

Some Geometric Properties of the Spacelike Bézier Curve with a Timelike Principal Normal are studied by Kuşak Samancı in Minkowski 3-space (Kuşak Samancı, 2018). Asymtotic frame field Rational Bezier curves is a paper that presents the curvatures in Minkowski space which belong to (Özkan Tükel et al., 2021, Yılmaz Ceylan et al., 2021, Yılmaz Ceylan et al., 2020). Elastic curves in a Two-dimensional Lightlike Cone are studied by Özkan Tükel and Yücesan (Özkan Tükel and Yücesan, 2015). More details see all references in this paper (Farin, 2002; Liu, 2004; Marsh, 2005; Liu and Qingxian, 2011; López, 2014; Özkan Tükel and Yücesan, 2015; Kuşak Samancı, 2018; Yılmaz Ceylan et al., 2020; Özkan Tükel et al., 2021; Yılmaz Ceylan et al., 2021; Turhan et al., 2021; Ertem Kaya, 2022).

Now paper presents Gauss, Mean and Total curvatures of the spacelike quadratic rational Bezier curves in Minkowski 4-space.

MATERIALS AND METHODS

Preliminaries

Let K , H and T be the Gauss, Mean and Total curvatures of a curve in Euclidean 3-space . Their formulae in differential geometry are shown as respectively:

$$K = k_1.k_2 = \kappa.\tau\,,$$

$$H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(\kappa + \tau)$$

and

$$\mathrm{T} = \sqrt{k_1^2 + k_2^2} = \sqrt{\kappa^2 + \tau^2}\ .$$

A rational Bézier curve of degree n with control points $b_0, b_1, \ldots, b_n$ and corresponding scalar weights ω_i, $0 \le i \le n,$ is known as:

$$R(t) = \frac{\sum_{k=0}^{n} \omega_k b_k B_{k,n}(t)}{\sum_{k=0}^{n} \omega_k B_{k,n}(t)} \quad t \in [0,1]$$

where

$$B_{k,n}(t)=\begin{cases}\dfrac{n!}{(n-k)!k!}, & if\, 0\le k\le n\\ 0, & otherwise\end{cases}$$

are called the Bernstein polynomials for n = 2 (Farin, 2002; Marsh, 2005; Özkan Tükel et al., 2021; Yılmaz Ceylan et al., 2021).

Let $\kappa(s)$ and $\tau(s)$ be the curvatures. Their formulae are defined by,

$$\kappa(s)=-\frac{1}{2}\frac{\sum_{k=0}^{5} d_k B_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle}{\left(\sum_{k=0}^{2}\omega_k B_{k,2}(s)\right)^4\left(\sum_{j=0}^{2}\omega_j B_{j,2}(s)\right)^4},$$

and

$$\tau(s)=\frac{1}{\left(\sum_{k=0}^{2}\omega_k B_{k,2}(s)\right)^4}\left(\left(\sum_{k,j=0}^{12}B_{k,12}(s)B_{j,12}(s)\langle d_k,d_j\rangle\right)-\left(\sum_{k,j=0}^{5}B_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle\right)^2\right)^{1/2}$$

(Özkan Tükel et al., 2021; Yılmaz Ceylan et al., 2021).

Quadratic rational Bezier curves in 3- dimensional lightlike cone

Let M be a submanifold E_1^4 (4-dimensional pseudo-Euclean space. Let c is a fixed in E_1^4. The pseudo-Riemannian lightlike cone (quadric cone) is shown as

$Q_1^3(c)=\{x\in E_1^4 : g(x-c,x-c=0\}$,

that c is the centre of $Q_1^3(c)$ and g is a pseudo Riemannien metric (a degenerate quadratic form). When c=0, we denote $Q_1^3(c)$ by $Q^3(c)$ and call it null cone. Let E_1^4 be 4-dimensional minkowski space and $Q^3(c)$ be the lightlike cone(Liu, 2004, Liu and Qingxian, 2011, Kulahci, Bektas, and Ergüt, 2007, Kulahcı and Almaz, 2017, Almaz, Külahcı and Yılmaz, 2020).

Theorem. If $\kappa(s)$ and $\tau(s)$ be the curvatures, then Mean, Gauss and total curvatures of Bernstein polynomials are obtained as follows:

Gauss curvature is calculated by

$$K=-\frac{1}{2}\frac{\sum_{k=0}^{5} d_k B_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle}{\left(\sum_{k=0}^{2}\omega_k B_{k,2}(s)\right)^8\left(\sum_{j=0}^{2}\omega_j B_{j,2}(s)\right)^4}\cdot\left(\left(\sum_{k,j=0}^{12}B_{k,12}(s)B_{j,12}(s)\langle d_k,d_j\rangle\right)-\left(\sum_{k,j=0}^{5}B_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle\right)^2\right)^{1/2},$$

and we have Mean curvature as follows:

$$H=\frac{1}{2\left(\sum_{k=0}^{2}\omega_k B_{k,2}(s)\right)^4}\left[\left(-\frac{1}{2}\frac{\sum_{k=0}^{5} d_k B_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle}{\left(\sum_{j=0}^{2}\omega_j B_{j,2}(s)\right)^4}\right)+\left(\begin{array}{l}\left(\sum_{k,j=0}^{12}B_{k,12}(s)B_{j,12}(s)\langle d_k,d_j\rangle\right)\\ -\left(\sum_{k,j=0}^{5}B_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle\right)^2\end{array}\right)^{1/2}\right]$$

and the total curvature is obtained as:

$$\mathrm{T}=\sqrt{\left(\frac{1}{4}\frac{\left(\sum_{k=0}^{5}d_kB_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle\right)^2}{\left(\sum_{k=0}^{2}\omega_kB_{k,2}(s)\right)^8\left(\sum_{j=0}^{2}\omega_jB_{j,2}(s)\right)^8}\right)+\left(\frac{1}{\left(\sum_{k=0}^{2}\omega_kB_{k,2}(s)\right)^8}\left(\left(\sum_{k,j=0}^{12}B_{k,12}(s)B_{j,12}(s)\langle d_k,d_j\rangle\right)-\left(\sum_{k,j=0}^{5}B_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle\right)^2\right)\right)}.$$

Proof. Firstly let we calculate Gauss curvature, then we have

$$K=k_1.k_2=\kappa.\tau$$

$$=-\frac{1}{2}\frac{\sum_{k=0}^{5}d_kB_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle}{\left(\sum_{k=0}^{2}\omega_kB_{k,2}(s)\right)^4\left(\sum_{j=0}^{2}\omega_jB_{j,2}(s)\right)^4}.$$

$$\frac{1}{\left(\sum_{k=0}^{2}\omega_kB_{k,2}(s)\right)^4}\left(\left(\sum_{k,j=0}^{12}B_{k,12}(s)B_{j,12}(s)\langle d_k,d_j\rangle\right)-\left(\sum_{k,j=0}^{5}B_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle\right)^2\right)^{1/2},$$

$$K=-\frac{1}{2}\frac{\sum_{k=0}^{5}d_kB_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle}{\left(\sum_{k=0}^{2}\omega_kB_{k,2}(s)\right)^8\left(\sum_{j=0}^{2}\omega_jB_{j,2}(s)\right)^4}.\left(\left(\sum_{k,j=0}^{12}B_{k,12}(s)B_{j,12}(s)\langle d_k,d_j\rangle\right)-\left(\sum_{k,j=0}^{5}B_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle\right)^2\right)^{1/2},$$

Now we calculate Mean curvature easily, so

$$H=\frac{1}{2}(k_1+k_2)=\frac{1}{2}(\kappa+\tau),$$

$$H=\frac{1}{2}\left(-\frac{1}{2}\frac{\sum_{k=0}^{5}d_kB_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle}{\left(\sum_{k=0}^{2}\omega_kB_{k,2}(s)\right)^4\left(\sum_{j=0}^{2}\omega_jB_{j,2}(s)\right)^4}+\frac{1}{\left(\sum_{k=0}^{2}\omega_kB_{k,2}(s)\right)^4}\left(\left(\sum_{k,j=0}^{12}B_{k,12}(s)B_{j,12}(s)\langle d_k,d_j\rangle\right)-\left(\sum_{k,j=0}^{5}B_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle\right)^2\right)^{1/2}\right)$$

$$H=\frac{1}{2\left(\sum_{k=0}^{2}\omega_kB_{k,2}(s)\right)^4}\left[\left(-\frac{1}{2}\frac{\sum_{k=0}^{5}d_kB_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle}{\left(\sum_{j=0}^{2}\omega_jB_{j,2}(s)\right)^4}\right)+\left(\left(\sum_{k,j=0}^{12}B_{k,12}(s)B_{j,12}(s)\langle d_k,d_j\rangle\right)-\left(\sum_{k,j=0}^{5}B_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle\right)^2\right)^{1/2}\right]$$

and total curvature is calculated as:

$$\mathrm{T}=\sqrt{k_1^2+k_2^2}=\sqrt{\kappa^2+\tau^2}$$

$$\mathrm{T}=\sqrt{\left(-\frac{1}{2}\frac{\sum_{k=0}^{5} d_k B_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle}{\left(\sum_{k=0}^{2}\omega_k B_{k,2}(s)\right)^4\left(\sum_{j=0}^{2}\omega_j B_{j,2}(s)\right)^4}\right)^2+\left(\frac{1}{\left(\sum_{k=0}^{2}\omega_k B_{k,2}(s)\right)^4}\left(\left(\sum_{k,j=0}^{12} B_{k,12}(s)B_{j,12}(s)\langle d_k,d_j\rangle\right)-\left(\sum_{k,j=0}^{5} B_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle\right)^2\right)^{1/2}\right)^2}$$

$$\mathrm{T}=\sqrt{\left(\frac{1}{4}\frac{\left(\sum_{k=0}^{5} d_k B_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle\right)^2}{\left(\sum_{k=0}^{2}\omega_k B_{k,2}(s)\right)^8\left(\sum_{j=0}^{2}\omega_j B_{j,2}(s)\right)^8}\right)+\left(\frac{1}{\left(\sum_{k=0}^{2}\omega_k B_{k,2}(s)\right)^8}\left(\left(\sum_{k,j=0}^{12} B_{k,12}(s)B_{j,12}(s)\langle d_k,d_j\rangle\right)-\left(\sum_{k,j=0}^{5} B_{k,5}(s)B_{j,5}(s)\langle c_k,c_j\rangle\right)^2\right)\right)}$$

Corollary 1. The cone Gauss curvature function $K(0)=-\frac{1}{2}\frac{\|c_0\|^2\left(\|d_0\|^2-\|c_0\|^4\right)^{1/2}}{\omega_0^{12}}$, cone Mean curvature function $H(0)=-\frac{\|c_0\|^2+2\omega_0^4\left(\|d_0\|^2-\|c_0\|^4\right)^{1/2}}{4\omega_0^8}$ and cone total curvature $\mathrm{T}(0)=\frac{1}{2\omega_0^8}\sqrt{\|c_0\|^4+4\omega_0^8\|d_0\|^2-4\omega_0^8\|c_0\|^4}$ at $s=0$.

Proof. The cone curvature function $\kappa(0)=-\frac{1}{2}\frac{\|c_0\|^2}{\omega_0^8}$ and cone torsion $\tau(0)=\frac{\left(\|d_0\|^2-\|c_0\|^4\right)^{1/2}}{\omega_0^4}$ at $s=0$ (Özkan Tükel et al., 2021; Yılmaz Ceylan et al., 2021). Thus we can take respectively, Gauss, Mean and Total curvatures are calculated at $s=0$ are defined by

$$K(0)=\kappa(0).\tau(0)$$

$$K(0)=-\frac{1}{2}\frac{\|c_0\|^2}{\omega_0^8}.\frac{\left(\|d_0\|^2-\|c_0\|^4\right)^{1/2}}{\omega_0^4}$$

$$K(0)=-\frac{1}{2}\frac{\|c_0\|^2\left(\|d_0\|^2-\|c_0\|^4\right)^{1/2}}{\omega_0^{12}}$$

and

$$H(0)=\frac{1}{2}\left(\kappa(0)+\tau(0)\right)$$

$$H(0)=\frac{1}{2}\left[-\frac{1}{2}\frac{\|c_0\|^2}{\omega_0^8}+\frac{\left(\|d_0\|^2-\|c_0\|^4\right)^{1/2}}{\omega_0^4}\right]$$

$$H(0)=-\frac{1}{4}\frac{\|c_0\|^2}{\omega_0^8}+\frac{\left(\|d_0\|^2-\|c_0\|^4\right)^{1/2}}{2\omega_0^4}$$

$$H(0)=-\frac{1}{4}\frac{\|c_0\|^2}{\omega_0^8}+\frac{2\omega_0^4\left(\|d_0\|^2-\|c_0\|^4\right)^{1/2}}{4\omega_0^8}$$

$$H(0)=-\frac{\|c_0\|^2+2\omega_0{}^4\left(\|d_0\|^2-\|c_0\|^4\right)^{1/2}}{4\omega_0{}^8}$$

and

$$\mathrm{T}(0)=\sqrt{\kappa(0)^2+\tau(0)^2}$$

$$\mathrm{T}(0)=\sqrt{\left(-\frac{1}{2}\frac{\|c_0\|^2}{\omega_0{}^8}\right)^2+\left(\frac{\left(\|d_0\|^2-\|c_0\|^4\right)^{1/2}}{\omega_0{}^4}\right)^2}$$

$$\mathrm{T}(0)=\sqrt{\frac{1}{4}\frac{\|c_0\|^4}{\omega_0{}^{16}}+\frac{\|d_0\|^2-\|c_0\|^4}{\omega_0{}^8}}$$

$$\mathrm{T}(0)=\sqrt{\frac{1}{4}\frac{\|c_0\|^4+4\omega_0{}^8\|d_0\|^2-4\omega_0{}^8\|c_0\|^4}{\omega_0{}^{16}}}$$

$$\mathrm{T}(0)=\frac{1}{2\omega_0{}^8}\sqrt{\|c_0\|^4+4\omega_0{}^8\|d_0\|^2-4\omega_0{}^8\|c_0\|^4}$$

Corollary 2. The cone Gauss curvature function $K(1)=-\frac{1}{2}\frac{\|c_5\|^2\left(\|d_{12}\|^2-\|c_5\|^4\right)^{1/2}}{\omega_2{}^{12}}$, cone Mean curvature function $H(1)=\frac{-\|c_5\|^2+2\omega_2{}^4\left(\|d_{12}\|^2-\|c_5\|^4\right)^{1/2}}{4\omega_2{}^8}$ and cone total curvature $\mathrm{T}(1)=\frac{1}{2\omega_2{}^8}\sqrt{\|c_5\|^4+4\omega_2{}^8\|d_{12}\|^2-4\omega_2{}^8\|c_5\|^4}$ of the spacelike quadratic rational Bézier curve R(s) parametrized by arclength in 3-dimensional lightlike cone in Minkowski 4-space at $s=0$.

Proof. The cone curvature function $\kappa(1)=-\frac{1}{2}\frac{\|c_5\|^2}{\omega_2{}^8}$ and cone torsion $\tau(1)=\frac{\left(\|d_{12}\|^2-\|c_5\|^4\right)^{1/2}}{\omega_2{}^4}$ at $s=1$ (Özkan Tükel et al., 2021; Yılmaz Ceylan et al., 2021). Thus we can take respectively, Gauss, Mean and Total curvatures are calculated at $s=0$ are defined by

$$K(1)=\kappa(1).\tau(1)$$

$$K(1)=-\frac{1}{2}\frac{\|c_5\|^2}{\omega_2{}^8}.\frac{\left(\|d_{12}\|^2-\|c_5\|^4\right)^{1/2}}{\omega_2{}^4}$$

$$K(1)=-\frac{1}{2}\frac{\|c_5\|^2\left(\|d_{12}\|^2-\|c_5\|^4\right)^{1/2}}{\omega_2{}^{12}}$$

and

$$H(1)=\frac{1}{2}\left(\kappa(1)+\tau(1)\right)$$

$$H(1)=\frac{1}{2}\left[-\frac{1}{2}\frac{\|c_5\|^2}{\omega_2{}^8}+\frac{\left(\|d_{12}\|^2-\|c_5\|^4\right)^{1/2}}{\omega_2{}^4}\right]$$

$$H(1) = \frac{-\|c_5\|^2 + 2\omega_2{}^4\left(\|d_{12}\|^2 - \|c_5\|^4\right)^{1/2}}{4\omega_2{}^8}$$

and

$$\mathrm{T}(1) = \sqrt{\kappa(1)^2 + \tau(1)^2}$$

$$\mathrm{T}(1) = \sqrt{\left(-\frac{1}{2}\frac{\|c_5\|^2}{\omega_2{}^8}\right)^2 + \left(\frac{\left(\|d_{12}\|^2 - \|c_5\|^4\right)^{1/2}}{\omega_2{}^4}\right)^2}$$

$$\mathrm{T}(1) = \frac{1}{2\omega_2{}^8}\sqrt{\|c_5\|^4 + 4\omega_2{}^8\|d_{12}\|^2 - 4\omega_2{}^8\|c_5\|^4}$$

RESULTS AND DISCUSSION

Gauss, Mean and total curvatures of spacelike quadratic rational Bezier curves are calculated in lightlike cone in Minkowski 4-space. In addition we gave special cases calculations for these curvatures of quadratic rational Bezier curves.

CONCLUSION

After making calculations about Gauss, Mean and total curvatures of spacelike quadratic rational Bezier curves, itcan be seen that these curvatures are invariant under Minkowski 4-space

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