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Existence and Uniqueness of Polyhedra with Given Values of the Conditional Curvature

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

The theory of polyhedra and the geometric methods associated with it are interesting not only in their own right but also have a wide outlet in the general theory of surfaces. Certainly, it is only sometimes possible to obtain the corresponding theorem on surfaces from the theorem on polyhedra by passing to the limit. Still, the theorems on polyhedra give directions for searching for the related theorems on surfaces. In the case of polyhedra, the elementary-geometric basis of more general results is revealed. In the present paper, we study polyhedra of a particular class, i.e., without edges and reference planes perpendicular to a given direction. This work is a logical continuation of the author's work, in which an invariant of convex polyhedra isometric on sections was found. The concept of isometry of surfaces and the concept of isometry on sections of surfaces differ from each other. Examples of isometric surfaces that are not isometric on sections and examples of non-isometric surfaces that are isometric on sections. However, they have non-empty intersections, i.e., some surfaces are both isometric and isometric on sections. In this paper, we prove the positive definiteness of the found invariant. Further, conditional external curvature is introduced for "basic" sets, open faces, edges, and vertices. It is proved that the conditional curvature of the polyhedral angle considered is monotonicity and positive definiteness. At the end of the article, the problem of the existence and uniqueness of convex polyhedra with given values of conditional curvatures at the vertices is solved.

Keywords: *Isometry on section, the total conditional angel, invariant, conditional curvature.*
AMS Subject Classification (2020): *Primary: 53C42; Secondary: 53A10.*

1. Introduction

In classical differential geometry, two directions are distinguished. One of them, called geometry "in the small," studies the local properties of geometric objects. The second examines geometric objects along their length and is called geometry "in the large." In 1813, O. Cauchy proved that two closed polyhedral composed of congruent faces are equal. This result is one of the first among the solved geometry problems "in the large." Many problems in geometry "in the large" are related to the isometry of surfaces. If the surfaces are isometric, it is possible to select the coordinate lines so that the surfaces have the same metric. In other words, if isometric surfaces are given, one can choose such parametrizations of surfaces with the same coefficients as the first quadratic forms. Naturally, the question arises of the metric on the surface to what extent determines the surface, i.e. if a linear surface element is given, what can be said about the surface? Proceeding from this, G. Weil posed and outlined the solution to the problem of the existence of a closed convex surface with a given metric. This problem received an exhaustive solution in the most general formulation for metrics of positive curvature in the works of A. D. Aleksandrov, A. V. Pogorelov, and their students [1]. In 1951, A.V. Pogorelov proved that a closed convex surface is uniquely determined by its metric in the class of general closed convex surfaces. That is, closed isometric convex surfaces are equal. A connection is established between the concept of convexity of a surface and a metric of positive curvature, and saddle surfaces represent the metric of negative

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curvature. As D. Hilbert showed [2] in 1901, there is no regular complete surface of constant negative curvature among surfaces in three-dimensional Euclidean space. This result was generalized in the classical work of N.V. Efimov, who proved that the complete metrics of negative curvature separated from zero do not admit regular realization "in the large" in Euclidean space. The problem of incomplete realization of a surface of negative curvature has various applications [3]. Meanwhile, irregular surfaces also deserve attention [4, 5]. For example, polyhedrons, cones, or sharp-edged lens surfaces are not completely regular. The theory of polyhedra and the geometric methods associated with it are interesting not only in and of themselves, but they also have wide access to the general theory of surfaces. Of course, it is not always possible to obtain the corresponding theorem on surfaces from the theorem on polyhedra by passing to the limit. Still, theorems on polyhedra give directions for searching for the corresponding theorem on surfaces. In the case of polyhedra, the elementary geometric basis of more general results is revealed. A. D. Aleksandrov constructed a method that proved [6] that a closed convex polyhedra is uniquely determined by its metric in the class of closed convex polyhedra. The problem of the unique definiteness of closed convex polyhedra was finally solved in the work of S. P. Olovyanishnikov [7]. There are several proofs of this theorem based on entirely different ideas. The first proof is based on the Cauchy method and is due to A. D. Aleksandrov [6]. Other explanations are from E. P. Senkin [8] and A.V. Pogorelov [9].

Many problems of geometry "in the large" [10, 11] are associated with the existence and uniqueness of surfaces with given characteristics. The geometric characteristics can be internal curvature, external or Gaussian curvature, and other functions related to the surface [2]. The existence of a polyhedron with given curvatures of vertices [6-9], or with a given development [12] is also a problem of geometry "in the large." Consequently, the issue of finding the invariants of polyhedra of a certain class and the solution to the problem of the existence and uniqueness of a polyhedron with given values of the invariants are relevant. Problems of the immersions and embeddings of manifolds in Euclidean and other spaces are some of the central issues both in differential geometry as well as in topology [13]. As a result of fundamental research on the problems of embedding and immersion, several topical issues have been solved in recent years through the studies of scientists Azeb Alghanemi, Noura M. Al-Houiti, Siraj Uddin (Saudi Arabia), Bang-Yen Chen (USA) and Mohamed Saleem Lone (India) [14].

The map isometry on sections is a particular case of the isometry of the foliated manifold. In other words, isometry on sections, in which each section of one surface is associated with a section of another surface, some results in this direction were obtained [15-18]. The concept of isometric surfaces on sections is equivalent to the isometry of surfaces in a space with a degenerate metric, particularly the Galilean space [19-21].

In three-dimensional Euclidean R^3 space, consider the surface F and the nonzero vector $\vec{e}$, which lets us the surface F is intersected by all possible planes π^j perpendicular to the vector $\vec{e}$. The set of cross-section points is denoted by $\gamma^j = F \cap \pi^j$. The class of surfaces for which the section γ^j. is homomorphic to a segment, a straight line, or a circle, we denote by $P \in W\left\{\vec{e}\right\}$ [13].

Definition 1.1. Surfaces $F_k \in W\left\{\overrightarrow{e_k}\right\}, k = 1, 2$ are called isometric on sections if there is a homeomorphism of f surfaces satisfying the following conditions [14]:

a) points of the surface F_1 belonging to the same section are compared with points belonging to the same section of the surface F_2. Images of points lying on different sections lie on different sections.

b) the distances between the planes containing sections γ^1 and γ^2 by F_1 and the planes containing curves $f\left(\gamma^1\right)$ and $f\left(\gamma^2\right)$ are equal;

c) the length of the arc between any two points of section $\gamma^j \subset \pi^j$ by F_1 is equal to the length of the arc of the curve $f\left(\gamma\right)$ between the corresponding points.

A polyhedron means a surface made up of a finite number of polygons [6]. Since a polyhedron is also a surface, the definition of isometry on the section also applies to polyhedral. In three-dimensional Euclidean space, we consider a polyhedral surface $P \in W\left\{\vec{e}\right\}$ that does not have edges and support planes perpendicular to vector $\vec{e}$. The planes perpendicular to vector $\vec{e}$ can be referenced only at the edge points of the polyhedra.

2. Invariant of polyhedral isometric on sections and conditional curvature

Let us be given a triangle ABC.

Definition 2.1. [13] The defect of the sides of the triangle ABC relative to the angle A is the number

$$\omega_{ABC} = AB + AC - BC.$$

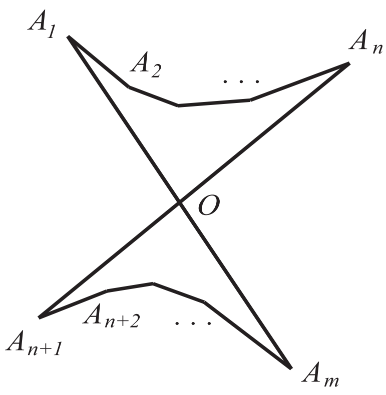

Fig 1. Nonconvex polygon $OA_1A_2...A_nA_{n+1}A_{n+2}...A_m$.

Let us generalize the concept of a side defect for a nonconvex polygon shown in Figure 1. The defect of a polygon relative to angle O is determined by the formula
$\omega = OA_1 + OA_n - P_{A_1A_2....A_n} + OA_{n+1} + OA_m - P_{A_{n+1}A_{n+2}....A_m}$, when $P_{A_1A_2....A_n}$ is the perimeter of the polyline $A_1A_2....A_n$, $P_{A_{n+1}A_{n+2}....A_m}$, is the perimeter of the polyline $A_{n+1}A_{n+2}....A_m$.
Let us consider a trihedron S whose faces lie on the planes

$$a_ix + b_iy + c_iz = 0 \qquad (i = 1, 2, 3)$$

The vertex of the trihedron is at the origin of the coordinates. Assume that the plane $x = const$ is not the reference plane of the trihedron. The S trihedron is intersected by planes $x = \pm 1$. Then one of them intersects one of the edges of the trihedron, the other intersects the rest two. The intersection points of planes $x = \pm 1$ with edges are denoted by A', B, C. The point A' is symmetrically displayed relative to the origin and we get point A. Let us consider triangle ABC.

Definition 2.2. The defect of the sides of the triangle ABC relative to the angle A is called the whole conditional angle of the triangular angle in the direction $\overrightarrow{e}\,(OX)$.

Obviously $\omega_{ABC} > 0$. Using the definition of the whole conditional angle for a triangular angle, we find the whole conditional angle for a tetrahedral angle. Let the vertex of the tetrahedron be at the origin and have no reference planes and edges perpendicular to vector $\vec{e}(OX)$. We intersect the polyhedron by planes $x = \pm 1$.
Let's generalize the concept of a total conditional angle in the direction of $\vec{e}$ for any polyhedral angle from the class $W\,\{\vec{e}\}$.
Let us be given a polyhedral angle, the vertex at the origin, without edges and reference planes perpendicular to the vector $\vec{e}$. We intersect the polyhedron by planes $x = \pm 1$. The points of intersection of the planes with the edges will be denoted by $A'_1, A'_2, ..., A'_{m-1}, A'_m, A'_{m+1}, ..., A'_n$.
Connecting points $A'_1, A'_2, ..., A'_{m-1}, A'_m$, we get a polyline with boundary points A'_1 and A'_m. Polyline $A'_{m+1}, A'_{m+2}, ..., A'_n$ is constructed in the same way. The polyline with boundary points A'_1, A'_m is symmetrically displayed relative to the origin of coordinates and we get a polygon as in case 4 in [13]. Then the total conditional angle corresponds to the defect of the sides of the polygon and is calculated by the formula $\omega = OA_1 + OA_m + OA_{m+1} + OA_n - P_1 - P_2$, when P_1 and P_2 are the length of the broken lines with the beginning A_1 and A_{m+1} and with the ends A_m and A_n, respectively, here O is the intersection point of the segments A_1A_n and A_mA_{m+1}.
It is easy to prove the following theorem.

Theorem 2.1. *[13] The total conditional angle ω is invariant under transformations of the form:*

$$\begin{cases} x' = x + a \\ y' = \alpha x + y\cos\varphi - z\sin\varphi + b \\ z' = \beta x + y\sin\varphi + z\cos\varphi + c \end{cases}$$

To determine the geometric value of the total conditional angle, we will consider a development that preserves the isometry on sections of the trihedral angle. Let the development of the trihedral angle be as in Figure 2.

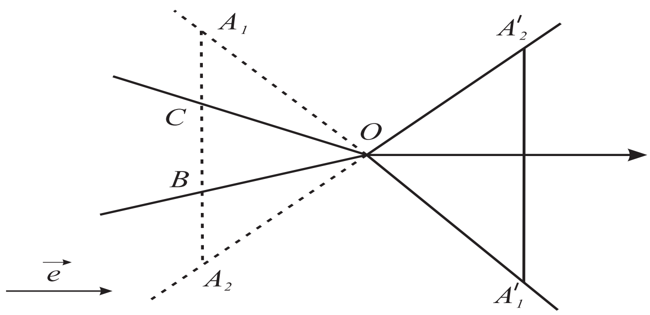

Figure 2. The development of the trihedral angle

Since the total conditional angle is calculated using the formula $\omega = AB + AC - BC$, then considering that $AB = A_1B, AC = A_2C$ when A_1 and A_2 are symmetric points to the glued points A'_1 and A'_2 relative to point O, we get $\omega = A_1B + A_2C - BC = A_1A_2$ from here. Therefore, A_1A_2 is the length of the solution of the edges to be glued, spaced from the top of the trihedron at a unit distance in the direction $\vec{e}$, is the total conditional angle of the trihedron in the direction $\vec{e}$.

3. Existence and uniqueness of polyhedra with given values of the conditional curvature at the vertices

So, if a polyhedron from class $W\{\vec{e}\}$ is given, then the total conditional angles at the vertices of the polyhedron determine the opening of the edges of the development of the polyhedron spaced from the vertices at a unit distance in the direction $\vec{e}$ at the corresponding vertices.

We define curvature as a function of a set, i. e. the set M on the convex polyhedra is associated with some number $\mu(M)$ - the conditional curvature of the set M on the polyhedra P. Let's start by defining conditional curvature for three "basic" sets: open faces, edges, and vertices. An open face is a face of a polyhedron with excluded vertices and sides; an open edge is an edge with excluded ends.

Let Γ be the open face of the polyhedra. The conditional curvature of the open face of the polyhedron is taken to be zero:

$$\mu(M) = 0.$$

The conditional curvature of the open edge L of the polyhedron is taken to be zero:

$$\mu(L) = 0.$$

Let M be the inner vertex of the polyhedron and $\omega(M)$ - the total conditional angle in the direction of $\vec{e}$ of the polyhedral angle with the vertex M, the faces coinciding with the faces of the polyhedron passing through the point M. For the conditional curvature of the vertex M of the polyhedron P we take the number $\mu(M) = \omega(M)$, i.e. conditional full angle of a polyhedral angle with apex M and polyhedron faces passing through point M. If the vertex M belongs to the edge of the polyhedron, the conditional curvature is assumed to be zero.

Let P be a convex polyhedron without edges and support planes perpendicular to the vector $\vec{e}$ and let $P = \sum\limits_{i=1}^{n} B_i$ be its representation as a sum of base sets without common points. For the conditional curvature of the polyhedron P we take the number $\mu(P) = \sum\limits_{i=1}^{n} \mu(B_i)$.

Theorem 3.1. *The conditional curvature of a polyhedral angle is a monotonically increasing function of the argument h.*

First, we prove a lemma about a property of the perimeter of a nonconvex polygon on the plane YOZ, Figure 1. Similar nonconvex polygons arise when considering the conditional curvature of a convex polyhedron. On plane YOZ, consider a non-convex polygon as in case 4 in [14]. That is, the vertex angles are vertical, and the closed polygons $A_1A_2...A_n$ and $A_{n+1}A_{n+2}...A_m$ are convex, while the polygons $A_1A_2...A_n$ and $A_{n+1}A_{n+2}...A_m$ are one-to-one projected onto the axis OY.

Let's make the following amount

$$\omega_1(h) = OA_1\cos^2\varphi_1 + OA_n\cos^2\varphi_{n+1} - \sum_{i=2}^{n} A_{i-1}A_i\cos^2\varphi_i$$

when $\varphi_1, \varphi_2, ..., \varphi_{n+1}$ are the angles between the axis OZ and the sides of the polygon.

Lemma 3.1. *The quantity $\omega_1 = \omega_1(h)$ is a positive function.*

Proof of Lemma 3.1. We use mathematical induction. At $n = 2$, the triangle OA_1A_2 obtained.
Let

$$OA_2\cos^2\varphi_3 = OA_1\cos^2\varphi_1 + A_1A_2\cos^2\varphi_2$$

since

$$OA_1\cos^2\varphi_1 \geq 0$$

it follows that

$$OA_2\cos^2\varphi_3 \geq A_1A_2\cos^2\varphi_2$$

Considering

$$0 \leq \varphi_3 < \varphi_2 \leq \frac{\pi}{2}$$

we get that

$$OA_2\cos^2\varphi_3 > A_1A_2\cos^2\varphi_2$$

hence follows

$$OA_2\cos^2\varphi_3 + OA_1\cos^2\varphi_1 - A_1A_2\cos^2\varphi_2 > 0$$

Let's assume that $n = 3$. Then we get a quadrilateral $OA_1A_2A_3$, quadrilateral $ABDC$. With the help of parallel translation, the origin of coordinates will be transferred to point A_2. The value ω_1 for the quadrangle $OA_1A_2A_3$ is:

$$\omega_1 = OA_1\cos^2\varphi_1 + OA_3\cos^2\varphi_4 - A_1A_2\cos^2\varphi_2 - A_2A_3\cos^2\varphi_3 =$$
$$= \left(OA_1\cos^2\varphi_1 - A_1A_2\cos^2\varphi_2\right) + \left(OA_3\cos^2\varphi_4 - A_2A_3\cos^2\varphi_3\right)$$

Arguing in the same way as in the case of $n = 2$, we obtain the positivity of both expressions in parentheses. Hence $\omega_1 > 0$.

Let $n = 4$. Then the figure consists of an angle A_1OA_4 inside which there is a convex broken line $A_1A_2A_3A_4$ with a convexity $Z < 0$. The axis OY will be transferred in parallel until it touches the broken line $A_1A_2A_3A_4$ and the point of contact will be taken as the origin.

Let us suppose that the point of contact is the vertex A_3. Therefore, we write the expression

$$\omega_1 = OA_1\cos^2\varphi_1 + OA_4\cos^2\varphi_5 - A_1A_2\cos^2\varphi_2 - A_2A_3\cos^2\varphi_3 - A_3A_4\cos^2\varphi_4 =$$
$$= \left(OA_1\cos^2\varphi_1 - A_1A_2\cos^2\varphi_2 - A_2A_3\cos^2\varphi_3\right) + \left(OA_4\cos^2\varphi_5 - A_3A_4\cos^2\varphi_4\right).$$

Let us prove that the expressions in both parentheses are positive.

$$OA_1\cos\varphi_1 > A_1A_2\cos\varphi_2 + A_2A_3\cos\varphi_3$$

Since

$$0 \leq \varphi_1 < \varphi_2 < \varphi_3 \leq \frac{\pi}{2}$$

then we get

$$OA_1\cos^2\varphi_1 > A_1A_2\cos^2\varphi_2 + A_2A_3\cos^2\varphi_3$$

i.e.

$$OA_1\cos^2\varphi_1 > A_1A_2\cos\varphi_2\cos\varphi_1 + A_2A_3\cos\varphi_3\cos\varphi_1 > A_1A_2\cos^2\varphi_2{+}_2A_3\cos^2\varphi_3$$

because

$$\varphi_1 > \cos\varphi_2 > \cos\varphi_3.$$

Similarly, it can be proved that

$$OA_4\cos^2\varphi_5 > A_3A_4\cos^2\varphi_4$$

hence

$$\omega_1 > 0.$$

If the axis OY with a parallel translation touches the broken line $A_1A_2A_3A_4$ at point A_2, then the following expressions can be proved in the same way:

$$OA_1\cos^2\varphi_1 - A_1A_2\cos^2\varphi_2 > 0$$

$$OA_4\cos^2\varphi_5 - A_2A_3\cos^2\varphi_3 - A_3A_4\cos^2\varphi_4 > 0$$

Hence

$$\omega_1 > 0.$$

Let us suppose that inequality $\omega_1 > 0$ is true in case $m = k-1$. In this case, it will also hold for $m = k$. We have

$$\omega_1^{(k-1)} = OA_1\cos^2\varphi_1 + OA_{k-1}\cos^2\varphi_k - A_1A_2\cos^2\varphi_2 -$$
$$-A_2A_3\cos^2\varphi_3 - ... - A_{k-2}A_{k-1}\cos^2\varphi_{k-1} = OA_1\cos^2\varphi_1 - OA_{k-1}\cos^2\varphi_k -$$
$$-\sum_{i=2}^{k-1} A_{i-1}A_i\cos^2\varphi_i > 0.$$

Let all the vertices remain unchanged and add a point A_k to them so that the convexity of the closed polygon $A_1A_2A_3...A_kA_{k-1}$ is preserved. Through points A_{k-2} and A_k we draw straight lines parallel to the axis OY and the points of intersection of lines with the segment OA_{k-1} we denote A'_{k-2}, A'_k.

$$\omega_1^k = \omega_1^{(k-1)} + A_{k-2}A_{k-1}\cos^2\varphi_{k-1} - A_{k-2}A_k\cos^2\varphi_{k_1} - A_kA_{k-1}\cos^2\varphi_{k_2},$$

when $\varphi_{k_1}, \varphi_{k_2}$ are the angles between the OZ axis with links $A_{k-2}A_k$ and A_kA_{k-1}, respectively.
Using

$$A_{k-2}A_k \cos\varphi_{k_1} \leq A'_{k-2}A'_k \cos\varphi_k$$

and

$$\varphi_{k_1} > \varphi_k$$

we get

$$A_{k-2}A_k \cos\varphi_{k_1} < A'_{k-2}A'_k \cos\varphi_k.$$

Similarly, one can prove that

$$A_{k-2}A_k \cos\varphi_{k_2} < A'_{k-2}A'_k \cos\varphi_k$$

when $\varphi_{k_2} > \varphi_k$, adding the last inequalities, we obtain

$$A_{k-2}A_{k-1}\cos^2\varphi_k - A_{k-2}A_k\cos^2\varphi_{k_1} - A_kA_{k-1}\cos^2\varphi_{k_2} > 0$$

Hence $\omega_1^k > 0$. Lemma 3.1 is proved.
Now we prove the stated theorem in the class of pyramids with common boundary γ.
Proof of Theorem 3.1 . Let γ be a convex polygon on the plane XOY that has no vertices on the axis OY and contains the origin. Consider convex pyramids P_1 and P_2 with a common edge γ, convex towards $Z > 0$ in the sense of [18] and whose vertices lie on the axis OZ. Then one of the pyramids is completely contained in the other. Suppose P_1 has a height of 1 and P_2 has a height of h. If $h > 1$ then P_2 contains P_1, and at $h < 1$ then P_2 is contained in P_1.
First, we prove the theorem for triangular pyramids. For a triangular pyramid, the conditional curvature is:

$$\mu(p_2) = \sqrt{y_1^2 + h^2 z_1^2} + \sqrt{y_2^2 + h^2 z_2^2} - \sqrt{y_3^2 + h^2 z_3^2} = OA_1 + OA_2 - A_1A_2$$

Let us find the derivative $\mu(p_2)$ with respect to variable h

$$\mu'(p_2) = \frac{hz_1^2}{\sqrt{y_1^2 + h^2 z_1^2}} + \frac{hz_2^2}{\sqrt{y_2^2 + h^2 z_2^2}} - \frac{hz_3^2}{\sqrt{y_3^2 + h^2 z_3^2}} =$$

$$= \frac{1}{h}\left(OA_1\cos^2\varphi_1 + OA_2\cos^2\varphi_3 - A_1A_2\cos^2\varphi_2\right)$$

By the above lemma, the expression in parentheses is positive and since $h > 0$, it follows that the conditional curvature for a triangular pyramid is a monotonically increasing function. For the case of a quadrangular pyramid:

$$\mu'(p_2) = \frac{1}{h}(OA_1\cos^2\varphi_1 - A_1A_2\cos^2\varphi_2)(OA_3\cos^2\varphi_4 - A_3A_2\cos^2\varphi_3)$$

or

$$\mu'\left(p_2\right) = \frac{1}{h}\left[\left(OA_1\cos^2\varphi_1 + OA_2\cos^2\varphi_3 - A_1A_2\cos^2\varphi_2\right) + \right.$$
$$\left. + \left(OA_3\cos^2\varphi_3 + OA_4\cos^2\varphi_1 - A_3A_4\cos^2\varphi_4\right)\right].$$

Considering the positivity of h and the expressions in brackets, we obtain the theorem for the case of a quadrangular pyramid. In a similar way, let us prove for n - coal pyramid, i.e., calculate the derivative with respect to h conditional curvature $\mu\left(p_2\right)$

$$\mu'(p_2) = \frac{1}{h}[OA_1\cos^2\varphi_1 + OA_n\cos^2\varphi_{n+1} - A_1A_2\cos^2\varphi_2 - ... - A_{n-1}A_n\cos^2\varphi_n].$$

Referring to Lemma 3.1, we obtain the positiveness of the derivative with respect to variable h, which proves the monotonicity of the conditional curvature. Theorem 3.1 is completely proved.

It is also easy to prove the following theorem.

Theorem 3.2. *The conditional curvature of a polyhedral angle is a positive definite function.*

Let us consider a convex polygon G on the plane XOY. We fix points $A_1, A_2, ..., A_n$ inside G. Let γ be a closed polyline in space, which by straight lines parallel to axis OZ is uniquely projected onto plane XOY into a convex polyline $\bar{\gamma}$ bounding polygon G, and vertices of polyline γ correspond to vertices $\bar{\gamma}$. Let $g_1, g_2, ..., g_n$ be any finite system of straight lines parallel to the axis OZ and intersecting polygon G at points $A_1, A_2, ..., A_n$, respectively, $\omega_1, \omega_2, ..., \omega_n$ - any positive numbers, μ - the conditional curvature of the polyhedron, convex towards $Z > 0$ and vertices A'_k on straight lines g_k. Let us denote by Ω_p the set of polyhedra P with edge γ, uniquely projecting onto the plane XOY, convex towards $Z > 0$ and with vertices on straight lines g_k (it is assumed that the polyhedron has no other vertices). Let us consider the following: If positive numbers $\omega_1, \omega_2, ..., \omega_n$ are given, does there exist a convex polyhedron such that $\omega_1, \omega_2, ..., \omega_n$ would be the conditional curvatures of the vertices of the convex polyhedron in the direction $\vec{e}$, lying on the straight lines $g_1, g_2, ..., g_n$, respectively. And to what extent is the polyhedron determined by the conditions that the vertices lie on the given lines and have the given conditional curvatures? The answer to this question is given by the following.

Theorem 3.3. *Let the conditions formulated above in the formulation of the problem in the class of convex polyhedra be satisfied. Then, there is a convex polyhedra $F \in \Omega_p$ with conditional curvatures equal to ω_i in the direction $\vec{e}$ at the vertices A'_i, respectively.*

The proof of the theorem follows from A.V. Pogorelov [10], where this theorem is proved for any function of a vertex of a polyhedron with the property of monotonicity. The monotonicity of the conditional curvature we have defined is proved in the following lemmas:

Lemma 3.2. *Let $P \in \Omega_p$ be a convex polyhedra and $\omega_1, \omega_2, ..., \omega_n$ are the conditional curvatures of the vertices of the polyhedra. If the vertex A'_k with curvature ω_k will be deformed towards $Z > 0$ along the straight line ω_k so that new vertices do not appear, then ω_k increases and the conditional curvatures of other vertices do not increase.*

Lemma 3.3. *Let us consider convex polyhedral angles P_1 and $P_2 \in W\{\vec{e}\}$ with common vertex O. If P_1 is contained in, P_2 then $\mu(P_1) > \mu(P_2)$.*

Proof of Lemma 3.3. We will prove this lemma step by step. Let us suppose that convex polyhedral angles P_1 and P_2 have a common vertex O and P_1 is contained in P_2, and all edges P_1 except OA_k coincide with edges P_2. The edges differing from OA_k are denoted by OA'_k polyhedral angles P_1. The axis OX will be directed along the vector $\vec{e}$ and the vertex O will be taken as the origin. Then the points of intersection of the $x = \pm 1$ planes of the edges of the polyhedral angles P_1 and P_2, except for A_k, coincide. Therefore, the vertices of the polygons obtained by calculating the conditional curvature, except for one, coincide, and the polygon corresponding to the polyhedron P_1 contains the polygon corresponding to the polyhedron P_2. From here

$$\mu\left(m_1\right) = A_1A_n + A_mA_{m+1} - \sum_{i=1}^{n-1} A_iA_{i+1}$$

when $i \neq m$.

Hence
$$\mu\left(m_2\right)=\mu\left(P_1\right)+A_{k-1}A_k+A_kA_{k+1}-A_{k-1}A'_k-A'_kA_{k+1}$$
considering that,
$$A_{k-1}A_k+A_kA_{k+1}-A_{k-1}A'_k-A'_kA_{k+1}<0$$
we get
$$\mu\left(P_1\right)>\mu\left(P_2\right).$$
If polyhedral angle P_1 has one fewer edge than P_2, then
$$\mu\left(P_2\right)=\mu\left(P_1\right)+A_{k-1}A_k-A_{k-1}A'_k-A'_kA_k.$$
From the triangle inequality
$$A_{k-1}A_k-A_{k-1}A'_k-A'_kA_k<0$$
hence it follows that
$$\mu\left(P_1\right)>\mu\left(P_2\right).$$
Let us suppose that two edges P_1 do not coincide with two edges P_2, and the other edges coincide. We construct an intermediate polyhedral angle P'_1, all of whose edges except one coincide with the edges P_1 and P_2 and it contains P_1 and is contained in P_2. By the above statement
$$\mu\left(P_1\right)>\mu\left(P'_1\right)>\mu\left(P_2\right).$$
Hence
$$\mu\left(P_1\right)>\mu\left(P_2\right).$$
We will continue this process until four edges remain, i.e., $OA_1, OA_m, OA_{m+1}, OA_n$.

Now let the edges OA_m and OA_{m+1} of the polyhedral angle do not coincide with the edges of the polyhedral angle P_2, and the remaining edges P_1 coincide with the edges P_2. Considering that P_1 is contained in P_2, we will calculate the conditional curvatures and compare them.

Obviously, the vertices A_m and A_{m+1} of the polygon obtained by cutting the polyhedral angle P_2 do not coincide with the vertices A'_m and A'_{m+1} of the polygon obtained by cutting the polyhedral angle P_1. Since P_1 is contained in P_2, we get the following expression: if the intersection point of the segments A_1A_n and A_mA_{m+1} is denoted by O and this point coincides with the intersection of the segments A_1A_n and $A'_mA'_{m+1}$, then we have the following:
$$A_mA'_m\cos\varphi_0=OA'_m\cos\varphi_1-OA_m\cos\varphi_2,$$
$$0<\varphi_1<\varphi_2<\frac{\pi}{2},$$
$$A_mA'_m\cos\varphi_0=A_{m-1}A'_m\cos\varphi'_1-A_{m-1}A_m\cos\varphi'_2,$$
$$0<\varphi'_1<\varphi'_2<\frac{\pi}{2}$$
when $\varphi_0, \varphi_1, \varphi_2, \varphi'_1, \varphi'_2$ are the angles between the axis OZ and the segments $A_mA'_m, OA'_m$, $OA_m, A_{m-1}A'_m, A_{m-1}A_m$, respectively, and from the condition that P_1 is contained in P_2, we obtain
$$\varphi_1<\varphi'_1, \varphi_2<\varphi'_2.$$
From here
$$OA'_m-OA_m+A_{m-1}A_m-A_{m-1}A'_m\geq OA'_m\cos\varphi_1-OA_m\cos\varphi_1+$$
$$+A_{m-1}A_m\cos\varphi_1-A_{m-1}A'_m\cos\varphi_1>OA'_m\cos\varphi_1-OA_m\cos\varphi_1+$$
$$+A_{m-1}A_m\cos\varphi_1-A_{m-1}A'_m\cos\varphi'_1=0$$
Therefore, we get that
$$OA'_{m-1}-OA_m+A_{m-1}A_m-A_{m-1}A'_m>0.$$
It is proved in a similar way that
$$OA'_{m+1}-OA_{m+1}+A_{m+1}A_{m+2}-A_{m+2}A'_{m+1}>0.$$

Now we calculate the conditional curvature of polyhedral angles and make sure that

$$\mu(P_2) < \mu(P_1).$$

Really,

$$\mu(P_1) = \mu(P_2) + (OA'_m - OA_m + A_{m-1}A_m - A_{m-1}A'_m) +$$
$$+(OA'_{m+1} - OA_{m+1} + A_{m+1}A_{m+2} - A_{m+2}A'_{m+1}) > \mu(P_2).$$

In exactly the same way, it can be proved that if the edges of the polyhedral angle P_2OA_1 and OA_n do not coincide with the edges OA'_1 and OA'_n of the polyhedral angle P_1, respectively, then it is also true $\mu(P_1) > \mu(P_2)$. So we have proved that, step by step, you can achieve the fact that when no edges P_1 and P_2 coincide and when P_1 is contained in P_2, $\mu(P_1) > \mu(P_2)$ is true. Lemma 3.3 is proved.

Proof of Lemma 3.2. Let $P \in \Omega_p$ be a convex polyhedra and $\omega_1, \omega_2, ..., \omega_n$ are the conditional curvatures of the vertices of the polyhedra. Vertex A'_k with curvature ω_k will be deformed towards $Z > 0$ along straight line g_k so that no new vertices appear. The resulting polyhedra is denoted by $P_0 \in \Omega_p$. We move the polyhedra P_0 along the straight line g_k towards $Z > 0$ so that the vertex lying on the straight line g_k coincides with the point A'_k, which is k the vertex of the polyhedron P. Since the polyhedral angle of the polyhedron P_0 with the vertex A'_k is contained in the polyhedral angle with the vertex A'_k of the polyhedron P, it follows that $\mu_P(A'_k) > \mu_{P_0}(A'_k)$. Since the other vertices P_0 and P coincide and the polyhedral angles P_0 contain the polyhedral angles P, then by Lemma 3.3 the conditional curvatures of the neighboring vertices to A'_k decrease, and the conditional curvatures of the remaining vertices remain unchanged. Consequently, upon deformation towards $Z > 0$, we obtain an increase of ω_k, and the conditional curvatures of other vertices do not increase. Lemma 3.2 is proved.

Proof of Theorem 3.3. Let us denote by $T(G, \gamma)$ the set of convex polyhedras $P_G \in \Omega_p$, for which the conditional curvatures of the vertices projecting into points A_i are less than or equal to ω_i. The set $T(G, \gamma)$ is not empty, since the convex hull of the broken line γ belongs to $T(G, \gamma)$. It is a polyhedron with an edge γ without inner vertices. It follows that $\mu_{P_0}(A_i) = 0$. Polyhedron $Q \in T(G, \gamma)$ is bounded by edge γ. Let's denote by h_i the distance between point A_i and vertex A'_i, which is projected to point A_i (even if A'_i is a flat vertex). Let us assign to each polyhedra $P \in T(G, \gamma)$ the function

$$\Omega(P) = h_1 + h_2 + ... + h_n.$$

Function $\Omega(P)$ is continuously dependent on h_i and is limited because all h_i are capped. Therefore, there is a polyhedron $P_0 \in \Omega(G, \gamma)$, for which $\Omega(P)$ reaches the smallest value $T(G, \gamma)$. Let us show that P_0 is a polyhedra, the existence of which is proved in the theorem. If we suppose the opposite, then there is a vertex A_i for which the conditional curvature is less than ω_i. The vertex A_i can be displaced in a straight line towards $Z > 0$ so that the conditional curvature remains less than ω_i.

By Lemma 3.2, the conditional curvatures of other vertices do not increase in this case, that is, the resulting polyhedra P'_0 belongs to $T(G, \gamma)$. But when mixing , the value of will decrease, which is impossible, since

$$\Omega(P_0) = \inf \Omega(P), \quad P \in T(G, \gamma).$$

The contradiction shows that for polyhedron P_0, the conditional curvature of vertex A_i is ω_i. The theorem is completely proved.

The proof of the theorem on the uniqueness of a convex polyhedron with given conditional curvatures is based on the monotonicity of the conditional curvature of the convex polyhedron and on the theorem of A.V. Pogorelov proved for any function at the vertices of a polyhedral angle possessing the monotonicity property. So, the following is true.

Theorem 3.4. *Let P_1 and P_2 be convex polyhedrons with a common edge γ, uniquely projected onto plane XOY, convex towards $Z > 0$, and the corresponding internal vertices are projected to the same point of plane XOY.*

Let the conditional curvatures take the same values at the corresponding vertices of these polyhedra. Then the polyhedras P_1 and P_2 coincide.

4. Conclusions

Many geometry problems "in the large" are connected with the existence and uniqueness of surfaces with given geometric characteristics. The geometric characteristics can comprise intrinsic curvature, extrinsic or

Gaussian curvatures, and other features associated with the surface. We have found such an invariant that has the properties of positive definiteness and monotonicity. It can be taken as a conditional external curvature of a polyhedron. Using A.V. Pogorelov's theorem for any function at the vertices of a polyhedral angle that has the property of monotonicity, we have solved the problem of the existence and uniqueness of a convex polyhedron of a certain class, according to given values of the conditional external curvature at the vertices.

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