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The distortion of tetrads under quasimeromorphic mappings of Riemann sphere

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Abstract

On the Riemann sphere, we consider the ptolemaic characteristic of a four of non-empty pairwise non-intersecting compact subsets (generalized tetrad, or generalized angle). We obtain an estimate for distortion of this characteristic under the inverse to a K-quasimeromorphic mapping of the Riemann sphere which takes each of its values at no more then N different points. The distortion function in this estimate depends only on K and N. In the case K=1, it is an essentially new property of complex rational functions.

1. Introduction

For a mapping $f : D \to \mathbb{C}$ of a domain $D \subset \mathbb{C}$ the following concepts are equivalent: K-quasiregular mapping [6, 2.20, Definition], K-quasiconformal function [4, 5.2], and the mapping with bounded distortion $\leq K$ [8, Ch.1, 4.2]. Moreover, each of these mappings has a representation $f = g \circ h$ where $h : D \to \mathbb{C}$ is a K-quasiconformal mapping, and $g : h(D) \to \mathbb{C}$ is a holomorphic function.

The more general concept of K-*quasimeromorphic mapping* $f : D \to \overline{\mathbb{C}}$ of a domain $D \subset \overline{\mathbb{C}}$ was introduced in [7, Section 2] (see also the definition for mappings with bounded distortion in Riemann manifolds [8, Ch. I, 5.2]), and it is as well equivalent to a notion of K-quasiconformal function in $\overline{\mathbb{C}}$. In this case the representation $f = g \circ h$ is also true, where $h : D \to \overline{\mathbb{C}}$ is a K-quasiconformal mapping, and $g : h(D) \to \overline{\mathbb{C}}$ is a meromorphic function (see [5, Ch. VI, Definition, Satz 1.1, Satz 2.2]).

The ptolemaic characteristic $\beta(\Psi)$ of a tetrad $\Psi = (z_1, z_2; z_3, z_4)$ in $\overline{\mathbb{C}}$ had been employed in [9] and was modified in [1] for the case of generalized tetrads $\Psi = (A_1, A - 2; A_3, A_4)$ in $2^{\overline{\mathbb{C}}}$ (the definitions see below in section 2). The main result is the following Theorem

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Theorem 1.1. *Given a positive integer N and $K \geq 1$, let $\mathbf{F}(N,K)$ denote the family of all non-constant K-quasimeromorhpic mappings $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which take each value at no more then N different points. Then there exists a homeomorphism $\omega_{N,K} : [0,+\infty) \to [0,+\infty)$ such that for each generalized tetrad Ψ in $\overline{\mathbb{C}}$ and for each mapping $f \in \mathbf{F}(N,K)$ the following inequality holds*

$$\beta(f^{-1}(\Psi)) \leq \omega_{N,K}(\beta(\Psi)). \tag{1.1.1}$$

The proof of this theorem will be given in Section 3.

It is worth to be noticed that this proof is based on the estimate (1.1.1) with $K = 1$ which will be established in the Lemma 2.2 and seems to present an essentially new global property of complex rational functions in $\overline{\mathbb{C}}$.

The inverse theorem is also true. It has been established in [2, Corollary] as follows.

Theorem 1.2. *Let N be a positive integer, $\omega : [0,+\infty) \to [0,+\infty)$ be a homeomorphism, and $\mathbf{G}(N,\omega)$ denote the family of all multivalued mappings $F : \overline{\mathbb{C}} \to 2^{\overline{\mathbb{C}}}$ such that $\#F(w) \leq N$ for every $w \in \overline{\mathbb{C}}$ and $F(w_1) \cap F(w_2) = \emptyset$ provided $w_1 \neq w_2$. If $F \in \mathbf{G}(N,\omega)$ and the inequality*

$$\beta(F(\Psi)) \leq \omega(\beta(\Psi)). \tag{1.2.1}$$

holds for each tetrad Ψ in $\overline{\mathbb{C}}$ then $F(\overline{\mathbb{C}}) = \overline{\mathbb{C}}$ and the left inverse mapping $f = F^{-1} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ belongs to $\mathbf{F}(N,K)$ where K depends only on N and ω.

2. Definitions and the Main Lemma

The chordal (spherical) metric $q(\cdot,\cdot)$ on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is defined by

$$q(z_1,z_2) = \frac{|z_1 - z_2|}{\sqrt{(1+|z_1|^2)(1+|z_2|^2)}} \text{ for } z_1,z_2 \in \mathbb{C}; \quad q(z,\infty) = \frac{1}{\sqrt{1+|z|^2}} \,.$$

By a *tetrad* $\Psi = (z_1,z_2;z_3,z_4)$ we mean a four of distinct points in $\overline{\mathbb{C}}$ divided into two pairs. Its *ptolemaic characteristic* is defined by

$$\beta(\Psi) = \frac{q(z_1,z_3)\cdot q(z_2,z_4) + q(z_1,z_4)\cdot q(z_2,z_3)}{q(z_1,z_2)\cdot q(z_3,z_4)} \,.$$

We also consider a *generalized tetrad* as a four of non-empty pairwise non-intersecting compact subsets in $\overline{\mathbb{C}}$ divided into two pairs. Then the ptolemaic characteristic of a generalized tetrad $\Psi = (A_1,A_2;A_3,A_4)$ is defined by

$$\beta(\Psi) = \max_{a_1\in A_1;\ a_2\in A_2} \left\{ \min_{a_3\in A_3;\ a_4\in A_4} \beta(a_1,a_2;a_3,a_4) \right\} \,.$$

Given a positive integer N, let $\mathbf{R}(N)$ denote the family of all rational functions $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $\#f^{-1}(w) \leq N$ for each $w \in \overline{\mathbb{C}}$.

Generalized tetrads are a special kind of so called *generalized angles.* This notion was introduced in [1, Section 3] as a four of arbitrary subsets $\Psi = (A_1,A_2;A_3,A_4)$ in a ptolemaic metric space (X,ρ) under the conditions $A_3 \neq \emptyset \neq A_4$ and $\#(A_3 \cup A_4) \geq 2$. The *angular characteristic* (or the *value*) $\alpha(\Psi)$ of a generalized angle Ψ was defined as

$$\alpha(\Psi) = \inf_{a_1\in A_1;\ a_2\in A_2} \left\{ \sup_{a_3\in A_3;\ a_4\in A_4} \frac{\rho(a_1,a_2)\cdot\rho(a_3,a_4)}{\rho(a_1,a_3)\cdot\rho(a_2,a_4) + \rho(a_1,a_4)\cdot\rho(a_2,a_3)} \right\}$$

under the agreement $\alpha(\Psi) = 1$ if $A_1 = \emptyset$ or $A_2 = \emptyset$. It is clear that $\beta(\Psi) = 1/\alpha(\Psi)$ for general tetrads in $\overline{\mathbb{C}}$. So the result [1, Lemma 4.2] may be reformulated as follows

Proposition 2.1. *Let X and Y be ptolemaic metric spaces, $F : X \to 2^Y$ be a multivalued mapping such that $F(x_1) \cap F(x_2) = \emptyset$ for all $x_1 \neq x_2$, and $\omega : [0, +\infty) \to [0, +\infty)$ be a homeomorphism. If the inequality $\beta(F(\Psi)) \leq \omega(\beta(\Psi))$ holds for any tetrad Ψ in X then it is also true for any generalized tetrad Ψ in X.*

Now we can prove the main lemma.

Lemma 2.2. *Given a positive integer N, there exists a homeomorphism $\omega_N : [0, +\infty) \to [0, +\infty)$ such that for each generalized tetrad $\Psi = (A_1, A_2; A_3, A_4)$ in $\overline{\mathbb{C}}$ and for all rational functions $f \in \mathbf{R}(N)$ the following inequality holds*

$$\beta(f^{-1}(A_1), f^{-1}(A_2); f^{-1}(A_3), f^{-1}(A_4)) \leq \omega_N(\beta(A_1, A_2; A_3, A_4)),$$

that is

$$\beta(f^{-1}(\Psi)) \leq \omega_N(\beta(\Psi)) \ . \tag{2.2.1}$$

Proof. Regarding the Proposition 2.1 we reduce the proof of the estimate (2.2.1) to the case where Ψ is an arbitrary tetrad in $\overline{\mathbb{C}}$. Let us prove that in this case it suffices to show that for all $t \geq 1$

$$\eta_N(t) := \sup \beta(f^{-1}(\Psi)) < \infty \tag{2.2.2}$$

where the supremum is taken over all tetrads Ψ in $\overline{\mathbb{C}}$ with $\beta(\Psi) \leq t$ and all finctions $f \in \mathbf{R}(N)$.

Indeed, the function $\eta_N(t)$ is non-decreasing in $[1, +\infty)$. Letting $\eta_N^*(t) \equiv \sup\{\eta_N(t) : n \leq t < n+1\}$ as $t \in [n, n+1)$, $n \geq 1$, and $\eta_N^* \equiv 0$ as $t \in [0, 1)$ we obtain a non-decreasing majorant for η_n. Letting $\omega_N^* = \eta_N^*(n) + (\eta_N^*(n+1) - \eta_N^*(n)) \cdot (t - n)$ for $t \in [n, n+1]$, $n = 0, 1, ...$, we obtain a continuous majorant for η_n. Finally, the function $\omega_N(t) = \omega_N^*(t) + t$ is a homeomorphism which is desired in (2.2.1).

Now we shall prove (2.2.2). Suppose, on contrary, that (2.2.2) fails for some fixed $t \geq 1$. It means that there exist a sequence of tetrads $\{\Psi_n = (a_n, b_n; c_n, d_n)\}$ with $\beta(\Psi_n) \leq t$, and a sequence of rational functions $\{f_n \in \mathbf{R}(N)\}$ such that

$$\beta(f^{-1}(\Psi_n)) \to \infty \ \text{ as } \ n \to \infty \ . \tag{2.2.3}$$

For each given n we consider the Möbius transformation $\mu_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $\mu_n(a_n) = 0$, $\mu_n(b_n) = 1$, $\mu_n(d_n) = \infty$, and mark the point $Z_n = \mu_n(c_n)$. Since the ptolemaic characteristic β of generalized tetrads is invariant under Möbius transformations, we obtain the estimate $\beta(\mu_n(\Psi_n)) = \beta(\Psi_n) \leq t$. Therefore, replacing in (2.2.3) functions f_n by $\mu_n \circ f_n \in \mathbf{R}(N)$ and tetrads Ψ_n by $\mu_n(\Psi_n)$ we pass to the following situation: there exist a sequence of points $Z_n \in \mathbb{C} \setminus \{0, 1\}$ such that

$$\beta(\Psi_n) = \beta(0, 1; Z_n, \infty) = |Z_n| + |1 - Z_n| \leq t \tag{2.2.4}$$

and a sequence of functions $f_n \in \mathbf{R}(N)$ such that

$$\beta(f_n^{-1}(\Psi_n)) = \beta(f_n^{-1}(0), f_n^{-1}(1); f_n^{-1}(Z_n), f_n^{-1}(\infty)) \to \infty \ \text{ as } \ n \to \infty \ . \tag{2.2.5}$$

Since

$$\beta(f_n^{-1}(\Psi_n)) = \max_{x_n \in f_n^{-1}(0);\ y_n \in f_n^{-1}(1)} \left\{ \min_{z_n \in f_n^{-1}(Z_n);\ w_n \in f_n^{-1}(\infty)} \beta(x_n, y_n;\ z_n, w_n) \right\}$$

there exist points $x_n^o \in f_n^{-1}(0)$ and $y_n^o \in f_n^{-1}(1)$ such that

$$\beta(f_n^{-1}(\Psi)) = \min_{z_n \in f_n^{-1}(Z_n);\ w_n \in f_n^{-1}(\infty)} \beta(x_n^o, y_n^o;\ z_n, w_n) \ .$$

Chose some point $w_n^o \in f_n^{-1}(\infty)$ and consider the Möbius transformation $\eta_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that

$$\eta_n(0) = x_n^o, \ \eta_n(1) = y_n^o, \ \eta_n(\infty) = w_n^o \ .$$

Because of möbius-invariance of ptolemaic characteristic β we have the equality

$$\beta(f_n^{-1}(\Psi_n)) = \beta(\eta_n^{-1}(f_n^{-1}(\Psi_n))) \ .$$

Now we can replace functions f_n in (2.2.5) by functions $f_n \circ \eta_n \in \mathbf{R}(N)$ and pass to the following situation.

Situation. Given a positive integer N, there exist a number $t \geq 1$, a sequence of points $\{Z_n\}$ in $\mathbb{C}\backslash\{0,1\}$, and a sequence of functions $\{f_n\}$ in $\mathbf{R}(N)$ such that

$$|Z_n| + |1 - Z_n| \leq t \ , \tag{2.2.6}$$

$$f_n(0) = 0, \ f_n(1) = 1, \ f_n(\infty) = \infty \ , \tag{2.2.7}$$

and

$$\beta(0,1;z_n,w_n) = \frac{q(0,z_n)\cdot q(1,w_n) + q(0,w_n)\cdot q(1,z_n)}{q(0,1)\cdot q(z_n,w_n)} \to \infty \ \text{ as } n \to \infty \tag{2.2.8}$$

under an arbitrary choice of points $z_n \in f_n^{-1}(Z_n)$, $w_n \in f_n^{-1}(\infty)$.

In particular, letting $w_n \equiv \infty$ we obtain from (2.2.8) the convergence

$$|z_n| + |1 - z_n| \to \infty \ \text{ as } \ n \to \infty \tag{2.2.9}$$

under an arbitrary coice of points $z_n \in f_n^{-1}(Z_n)$.

Suppose there exists a bounded subsequence $\{w_n \in f_n^{-1}(\infty)\}$. Then we would have for this subsequence the convergences

$$\frac{q(0,z_n)}{q(w_n,z_n)} = \frac{|z_n|\cdot\sqrt{1+|w_n|^2}}{|w_n - z_n|} \to 1 \ ,$$

$$\frac{q(1,z_n)}{q(w_n,z_n)} = \frac{|1-z_n|\cdot\sqrt{1+|w_n|^2}}{\sqrt{2}\cdot|w_n - z_n|} \to 1 \ .$$

Then we obtain boundedness of subsequences

$$\frac{q(0,z_n)\cdot q(1,w_n)}{q(0,1)\cdot q(w_n,z_n)} \sim \frac{|1-w_n|}{2\sqrt{1+|w_n|^2}} \ ,$$

$$\frac{q(0,w_n)\cdot q(1,z_n)}{q(0,1)\cdot q(w_n,z_n)} \sim \frac{|w_n|}{\sqrt{2}\cdot\sqrt{1+|w_n|^2}}$$

which contradicts (2.2.8). Therefore, the assumption above is impossible, so $|w_n| \to \infty$ as $n \to \infty$ under an arbitrary choice of $w_n \in f_n^{-1}(\infty)$.

Thus we have the convergence of the sets

$$f_n^{-1}(\infty)\} \to \{\infty \ , \ f_n(_1)(Z_n) \to \{\infty\} \ . \tag{2.2.10}$$

Moreover, passing to a suitable subsequences we may assume without loss of generality that

$$Z_n \to Z_0 \neq \infty \ \text{ as } n \to \infty \ . \tag{2.2.11}$$

For more convenience of notation let us denote f_n^{-1} by F_n. Consider an increasing sequence of open disks $B_j = \{z : |z| < R^{(j)}\}$ where $1 < R^{(1)} < ... < R^{(j)} < ...$ and $R^{(j)} \to +\infty$ as $j \to \infty$.

Let $\{f_n^{(j)}\}$ be a given subsequence in $\{f_n\}$, and $\{Z_n^{(j)}\}$ be the corresponding subsequence of points. Because of (2.2.10) there exists a number $n^{(j)}$ such that

$$F_n^{(j)}(\infty) \cap \overline{B}_{j+1} = \emptyset \ ; \ F_n^{(j)}(Z_n^{(j)}) \cap \overline{B}_{j+1} = \emptyset$$

for all $n \geq n^{(j)}$.

It means that for all $n \geq n^{(j)}$ the functions $f_n^{(j)}(z)$ do not take the values ∞ and $Z_n^{(j)}$ in the disk $\overline{B}_{j+1}$. Thus we have two sequences

$$\{\varphi_n^{(j)}(z) = f_n^{(j)}(z) - Z_n^{(j)}\} \ ; \ \left\{\psi_n^{(j)}(z) = \frac{1}{f_n^{(j)}(z) - Z_n^{(j)}}\right\}, \ n \geq n^{(j)} \tag{2.2.12}$$

of holomorphic fnctions in the disk B_{j+1} which do not take the value 0.

Since each of these functions takes any its value at no more then N distinct points, the conditions of generalized Montel's Theorem ([3, Ch. II, Section 7, Theorem 2]) are fulfilled. According to this theorem both families (2.2.12) of holomorphic functions in B_{j+1} are normal families. It means that there exist a subsequence $\{f_n^{(j+1)}\} \subset \{f_n^{(j)}\}$ and the corresponding subsequence of points $\{Z_n^{(j+1)}\} \subset \{Z_n^{(j)}\}$ such that each of two sequences

$$\{\varphi_n^{(j+1)}(z) = f_n^{(j+1)}(z) - Z_n^{(j+1)}\}\ ,\ \left\{\psi_n^{(j+1)}(z) = \frac{1}{f_n^{(j+1)}(z) - Z_n^{(j+1)}}\right\}$$

converges uniformly on compact subsets of B_{j+1} either to ∞ or to a holomorphic function in B_{j+1}.

It follows from (2.2.11) that $f_n^{(j+1)}(0) - Z_n^{(j+1)} = -Z_n^{(j+1)} \to -Z_0 \neq \infty$, and therefore the sequence $\{\varphi_n^{(j+1)}\}$ converges to a holomotphic function $\varphi^{(j+1)}$ uniformly on compacts in B_{j+1}.

In the case where the initial sequences $\{\varphi_n^{(j)}\}$ and $\{\psi_n^{(j)}\}$ were converging to holomorphic functions $\varphi^{(j)}$ and $\psi^{(j)}$ in B_j, we have $\varphi^{(j+1)}(z) = \varphi^{(j)}(z)$ and $\psi^{(j+1)}(z) = \psi^{(j)}(z)$ for all $z \in B_j$.

Thus the chains $\{\varphi_n^{(1)}\} \supset \{\varphi_n^{(2)}\} \supset ...\{\varphi_n^{(j)}\} \supset ...$ and $\{\psi_n^{(1)}\} \supset \{\psi_n^{(2)}\} \supset ... \supset \{\psi_n^{(j)}\} \supset ...$ of subsequences are obtained. Then the diagonal sequences $\{\varphi_j^{(j)}\}$ and $\{\psi_j^{(j)}\}$ converge uniformly on compacts in $\mathbb{C}$ to holomorphic functions $\varphi(z)$ and $\psi(z)$. Since $\varphi(0) = -Z_0 \neq \varphi(1) = 1 - Z_0$ and $\psi(0) = -1/Z_0 \neq \psi(1) = 1/(1 - Z_0)$, each of the fuctions $\varphi(z)$ and $\psi(z)$ is non-constant.

Each function in a sequence $\{\varphi_j^{(j)}\}$ takes any its value at no more then N points. So according to [3, Ch. I, Section 1, Theorem 2] the entire function $\varphi(z)$ is a polynomial of degree $\leq N$. But the equality $\varphi_j^{(j)}(z) \cdot \psi_j^{(j)}(z) \equiv 1$ for all j and $z \in B_j$ implies the equality $\varphi(z) \cdot \psi(z) \equiv 1$ at every point $z \in \mathbb{C}$. It is impossible because the polynomial $\varphi(z)$ has a point z_0 where $\varphi(z) = 0$.

This contradiction being obtained completes the proof of the lemma. □

3. Proof of the Main Theorem

The *absolute ratio* (or *cross-ratio*) $|x_1, x_2, x_3, x_4|$ of distinct points in $\overline{\mathbb{R}^d}$ is defined by

$$|x_1, x_2, x_3, x_4| = \frac{q(x_1, x_3) \cdot q(x_2, x_4)}{q(x_1, x_2) \cdot q(x_3, x_4)}$$

where $q(\cdot, \cdot)$ denotes the chordal distance between points in $\overline{\mathbb{R}^d}$. The distortion of absolute ratio under K-quasiconformal mappings $f : \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ had been thoroughly studied by M. Vuorinen in his paper [10]. He had obtained the estimate [10, Theorem 3.5]

$$|f(x_1), f(x_2), f(x_3), f(x_4)| \leq \eta_{K,d}(|x_1, x_2, x_3, x_4|)$$

where the distortion function $\eta_{K,d}$ depends only on K and d. The distortion function $\eta_{K,2}$ has the explicit expression [10, (1.9), (1.10)].

Since

$$\beta(x_1, x_2; x_3, x_4) = |x_1, x_2, x_3, x_4| + |x_1, x_2, x_4, x_3|$$

we can just obtain the estimate for the distortion of ptolemaic characteristic of a tetrad $\Psi = (x_1, x_2; x_3, x_4)$ under a K-quasiconfornal automorphism of $\overline{\mathbb{R}^d}$

$$\beta(f(\Psi)) = |f(x_1), f(x_2), f(x_3), f(x_4)| + |f(x_1), f(x_2), f(x_4), f(x_3)| \leq$$

$$2\eta_{K,d}(|x_1, x_2, x_3, x_4| + |x_1, x_2, x_4, x_3|) = 2\eta_{K,d}(\beta(\Psi))\ .$$

Moreover, it follows from the Proposition 2.1 that the estimate

$$\beta(f(\Psi)) \leq 2\eta_{K,d}(\beta(\Psi)) \tag{3.1.1}$$

is also valid for every general tetrad in $\overline{\mathbb{R}^d}$.

Now let us consider $f \in \mathbf{F}(N, K)$ and a tetrad Ψ in $\overline{\mathbb{C}}$. It has been mentioned in section 1 that $f = g \circ h$ where $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a K-quasiconformal mapping and $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational function $g \in \mathbf{R}(N)$. Applying the Lemma 2.2 to the rational function $g \in \mathbf{R}(N)$ we get the estimate

$$\beta(g^{-1}(\Psi)) \le \omega_N(\beta(\Psi)) \ .$$

Applying the estimate (3.1.1) to the K-quasiconformal mapping h^{-1} and the generalized tetrad $g^{-1}(\Psi)$ we get the estimate

$$\beta(h^{-1}(g^{-1}(\Psi))) \le 2\eta_{K,2}(\beta(g^{-1}(\Psi))) \le 2\eta_{K,2}(\omega_N(\beta(\Psi))) \ .$$

Thus we obtain the desired estimate

$$\beta(f^{-1}(\Psi)) \le \omega_{N,K}(\beta(\Psi))$$

with the distortion function $\omega_{N,K} = 2\eta_{K,2} \circ \omega_N$ depending only on K and N. Now the Theorem 1.1 is proved.

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