Turk. J. Math. Comput. Sci.
15(2)(2023) 407–413
© MatDer
DOI : 10.47000/tjmcs.1249554

A Modified Quadratic Lorenz Attractor in Geometric Multiplicative Calculus

Hatice Aktöre[1,*], Buğçe Eminağa Tatlicioglu[2]

[1] *Department of Computer Engineering, Faculty of Engineering, Girne American University, Kyrenia, North Cyprus, via Mersin 10, Turkey.*

[2] *Department of Computer Engineering, Faculty of Engineering, Final International University, Kyrenia, North Cyprus, via Mersin 10, Turkey.*

Received: 09-02-2023 • Accepted: 20-09-2023

Abstract. In this study, the modified quadratic Lorenz attractor is introduced in geometric multiplicative calculus. The new system is analyzed and discussed for the chaotic behaviour in detail. The equilibria points, the eigenvalues of the multiplicative Jacobian, and the Lyapunov exponents are determined. The numerical simulations are conducted using the Runge-Kutta method in the framework of geometric multiplicative calculus highlighting the chaotic behaviour.

2020 AMS Classification: 37N30, 37M05, 37M10

Keywords: Chaos, dynamical system, multiplicative calculus.

1. Introduction

Many researchers in the field of dynamic systems consider Henry Poincaré as the founding father of dynamic systems [5]. Poincaré published the fundamental ideas of the theory of dynamic systems at the end of the 19th and the beginning of the 20th century in [12, 13]. In parallel, Aleksandr Lyapunov defined the stability of sets of ordinary differential equations [9]. The Lyapunov exponent has been proven to be one of the most important tools in the analysis of the stability of differential equations to decide if the system is chaotic or not. The Lorenz attractor [6], proposed in 1962, and the Rössler attractor [15], proposed in 1976, are the most popular systems of ordinary differential equations with chaotic behaviour. Many publications based on these two fundamental attractors analysing these attractors to the utmost detail have been published.

Numerous research endeavors have explored the realm of attractors, seeking to unveil new ordinary differential equations capable of generating chaotic behavior. In one such instance, the article by Pehlivan et al. [11] introduces a distinct chaotic system, underlining its unique mathematical characteristics and its capacity to yield Lorenz-like attractors. These findings are substantiated through both computational simulations and experimental validations. Similarly, Chen's work in [3] brings to light a fresh chaotic attractor within a simple three-dimensional autonomous system, featuring qualities reminiscent of both Lorenz and Rossler attractors. In a related vein, Lu [7] identifies a novel chaotic attractor in a basic three-dimensional autonomous system, which acts as a bridge connecting the Lorenz attractor and

*Corresponding Author
Email addresses: haticeaktore@gau.edu.tr (H. Aktöre), bugce.tatlicioglu@final.edu.tr (B.E.Tatlicioglu)

Chen's attractor, effectively capturing the transitional dynamics between the two. The authors in [8] perform a numerical investigation to explore the chaotic behaviors exhibited by the fractional-order Lu system. Building upon these concepts, the study presented in this article follows the idea of exploring attractors to discover new ordinary differential equations that exhibit chaotic behavior. This new chaotic attractor, the modified quadratic Lorenz attractor, was proposed by the authors in the framework of standard calculus in [4]. The modified quadratic Lorenz attractor as proposed in [4] is:

$$\frac{dx}{dt} = s(yz - x), \tag{1.1}$$

$$\frac{dy}{dt} = rx - xz, \tag{1.2}$$

$$\frac{dz}{dt} = (xy)^2 - bz, \tag{1.3}$$

where s, r, and b are real constants. The chaotic properties of this new attractor are discussed using the standard techniques of dynamic systems, as discussed in any textbook like [10].

In this paper, we will discuss the modified quadratic Lorenz attractor in the framework of geometric multiplicative calculus. Aniszewska applied a version of multiplicative calculus to analyse dynamic systems [1] by inventing the so-called Multiplicative Runge-Kutta method. This method was also used later by Aniszewska and Rybaczuk [2] in the analysis of dynamic systems. Later, the Runge-Kutta method, building the foundation of the numerical simulations of these systems, was developed by Riza and Aktöre [14] in the framework of geometric multiplicative calculus, which is a different multiplicative calculus than the one used by Aniszewska.

In this article, we delve into the exploration of the modified quadratic Lorenz attractor within the context of geometric multiplicative calculus. Section 2 of this paper delves into an in-depth examination of equilibrium points, the multiplicative Jacobian, and the computation of eigenvalues, all within the framework of geometric multiplicative calculus. In contrast, Section 3 shifts our focus to the Newtonian interpretation of the modified quadratic Lorenz attractor, where we employ conventional textbook methods for our analysis.

Interestingly, our numerical simulations yield consistent results for both approaches, regardless of whether we analyze the system through the lens of geometric multiplicative calculus or conventional methods. Consequently, we employ the Runge-Kutta method within the framework of geometric multiplicative calculus to generate our numerical findings. Our assessment of the Lyapunov exponents unmistakably reveals the chaotic nature inherent in the proposed system.

2. Multiplicative Chaotic Attractor Defined By Multiplicative Derivatives

Remembering that addition, subtraction and multiplication of functions in Newtonian calculus correspond to multiplication, division and powers of functions in multiplicative calculus, the equation (1.1) can be written as

$$\frac{dx}{dt} = s(yz - x) \quad \rightarrow \quad \frac{d^*x}{dt} = \left(\frac{y^{\ln z}}{x}\right)^{\ln s}.$$

Rewriting the equations (1.2) and (1.3) in the same way results in the modified multiplicative quadratic Lorenz attractor defined by the multiplicative derivatives as

$$\begin{aligned} \frac{d^*x}{dt} &= \left(\frac{y^{\ln z}}{x}\right)^{s}, \\ \frac{d^*y}{dt} &= \frac{x^r}{x^{\ln z}}, \\ \frac{d^*z}{dt} &= \frac{x^{\ln x (\ln y)^2}}{z^b}. \end{aligned} \tag{2.1}$$

The powers of the functions are chosen according to the rule $x^{\ln y} = y^{\ln x}$ and the constants $\ln s, \ln r$ and $\ln b$ are replaced by s, r and b.

2.1. **Analysis of the System.** The first step of analyzing a chaotic system is to find the equilibrium points of the system. The equilibrium points are obtained from the solution of the system

$$\begin{aligned} \frac{d^*x}{dt} &= \left(\frac{y^{\ln z}}{x}\right)^s = 1, \\ \frac{d^*y}{dt} &= \frac{x^r}{x^{\ln z}} = 1, \\ \frac{d^*z}{dt} &= \frac{x^{\ln x (\ln y)^2}}{z^b} = 1. \end{aligned} \tag{2.2}$$

Solution of the system (2.2) gives the following points of equilibria:

$$\begin{aligned} E_1 &= (1,1,1), \\ E_2 &= \left(\exp\left(\sqrt[4]{b\,r^3}\right), \exp\left(\sqrt[4]{\frac{b}{r}}\right), \exp(r)\right), \\ E_3 &= \left(\exp\left(-\sqrt[4]{b\,r^3}\right), \exp\left(-\sqrt[4]{\frac{b}{r}}\right), \exp(r)\right). \end{aligned}$$

The 3×3 multiplicative Jacobian matrix can be defined as following:

$$J = \begin{bmatrix} \ln\left(\frac{\partial^* f_1}{\partial x}\right) & \ln\left(\frac{\partial^* f_1}{\partial y}\right) & \ln\left(\frac{\partial^* f_1}{\partial z}\right) \\ \ln\left(\frac{\partial^* f_2}{\partial x}\right) & \ln\left(\frac{\partial^* f_2}{\partial y}\right) & \ln\left(\frac{\partial^* f_2}{\partial z}\right) \\ \ln\left(\frac{\partial^* f_3}{\partial x}\right) & \ln\left(\frac{\partial^* f_3}{\partial y}\right) & \ln\left(\frac{\partial^* f_3}{\partial z}\right) \end{bmatrix}.$$

Therefore, we get for the multiplicative Jacobian of the multiplicative modified quadratic Lorenz system (2.1)

$$\begin{aligned} J &= \begin{bmatrix} \ln\left(\exp\{-\frac{s}{x}\}\right) & \ln\left(\exp\{\frac{s\ln z}{y}\}\right) & \ln\left(\exp\{\frac{s\ln y}{z}\}\right) \\ \ln\left(\exp\{\frac{r-\ln z}{x}\}\right) & \ln\left(\exp\{0\}\right) & \ln\left(\exp\{-\frac{\ln x}{z}\}\right) \\ \ln\left(\exp\{\frac{2\ln x(\ln y)^2}{x}\}\right) & \ln\left(\exp\{\frac{2(\ln x)^2\ln y}{y}\}\right) & \ln\left(\exp\{-\frac{b}{z}\}\right) \end{bmatrix} \\ &= \begin{bmatrix} -\frac{s}{x} & \frac{s\ln z}{y} & \frac{s\ln y}{z} \\ \frac{r-\ln z}{x} & 0 & -\frac{\ln x}{z} \\ \frac{2\ln x(\ln y)^2}{x} & \frac{2(\ln x)^2\ln y}{y} & -\frac{b}{z} \end{bmatrix}. \end{aligned}$$

Thus, for the equilibrium points E_1, E_2 and E_3 the corresponding Jacobian matrices has the form

$$J(E_1) = \begin{bmatrix} -s & 0 & 0 \\ r & 0 & 0 \\ 0 & 0 & -b \end{bmatrix}, \text{ and } J(E_{2,3}) = \begin{bmatrix} -se^{cr} & sre^c & -cse^{-r} \\ 0 & 0 & -cre^{-r} \\ 4c^2re^{cr} & 4c^2re^c & -be^{-r} \end{bmatrix}, \tag{2.3}$$

with $c = \pm\sqrt[4]{\frac{b}{r}}$. In order to get the eigenvalues of the Jacobian matrices, we will substitute the corresponding equilibrium points in (2.3). Thus, the eigenvalues can be summarized in the following table.

Equilibrium Point	λ_1	λ_2	λ_3
E_1	−12	−4	0
E_2	−0.0286149	$0.00644902 + 0.272747i$	$0.00644902 - 0.272747i$
E_3	−10017.4	$0.000690801 + 0.892422i$	$0.000690801 - 0.892422i$

Table 1. Eigenvalues of the Jacobian matrices of the multiplicative chaotic system at the equilibrium points

Since the two eigenvalues of the equilibrium point E_1 are negative real numbers, we can say that the system is unstable at this equilibrium point. On the other hand, each of the other two equilibrium points, E_2 and E_3, have two complex conjugate numbers with positive real parts and a negative real number, which also proves that the system is again unstable at these equilibrium points.

3. Multiplicative Chaotic Attractor Defined By Additive Derivatives

The modified quadratic Lorenz attractor can also be defined by the additive derivatives. Keeping in mind that the relation between the ordinary and the multiplicative derivative of a function $f(x)$ is

$$f^{*}(x) = e^{\frac{f'(x)}{f(x)}}.$$

The modified multiplicative quadratic Lorenz attractor can be expressed in terms of the additive derivatives as

$$\begin{aligned}
\frac{dx}{dt} &= xs\left(\ln(y)\ln(z) - \ln(x)\right), \\
\frac{dy}{dt} &= y(r\ln(x) - \ln(x)\ln(z)), \\
\frac{dz}{dt} &= z((\ln(x)\ln(y))^2 - b\ln(z)).
\end{aligned} \tag{3.1}$$

3.1. **Analysis of the System.** As we have discussed in Section 2.1, in order to analyze the system (3.1) the first step is to find the equilibria. Since the system is defined by the additive derivatives, we will solve the system

$$\begin{aligned}
\frac{dx}{dt} &= xs\left(\ln(y)\ln(z) - \ln(x)\right) = 0, \\
\frac{dy}{dt} &= y(r\ln(x) - \ln(x)\ln(z)) = 0, \\
\frac{dz}{dt} &= z((\ln(x)\ln(y))^2 - b\ln(z)) = 0.
\end{aligned} \tag{3.2}$$

The solution of the system shows that the equilibrium points of the system (3.2) are the same with the ones that we have obtained from the solution of the system (2.2), which are given in the Table 2. Defining the system as

$$\begin{aligned}
f_1 &= \frac{dx}{dt} = xs\left(\ln(y)\ln(z) - \ln(x)\right), \\
f_2 &= \frac{dy}{dt} = y(r\ln(x) - \ln(x)\ln(z)), \\
f_3 &= \frac{dz}{dt} = z((\ln(x)\ln(y))^2 - b\ln(z))
\end{aligned} \tag{3.3}$$

and keeping in mind that the Jacobian matrix of a 3×3 system defined by the additive derivatives can be written as

$$J = \begin{bmatrix} \frac{df_1}{dx} & \frac{df_1}{dy} & \frac{df_1}{dz} \\ \frac{df_2}{dx} & \frac{df_2}{dy} & \frac{df_2}{dz} \\ \frac{df_3}{dx} & \frac{df_3}{dy} & \frac{df_3}{dz} \end{bmatrix}.$$

The Jacobian matrix of the system (3.3) can be written as

$$J = \begin{bmatrix} s\ln y\ln z - s\ln x - s & \frac{sx\ln z}{y} & \frac{sx\ln y}{z} \\ \frac{ry}{x} - \frac{y\ln z}{x} & r\ln(x) - \ln(x)\ln(z) & -\frac{y\ln x}{z} \\ \frac{2z\ln x(\ln y)^2}{x} & \frac{2z(\ln x)^2\ln y}{y} & (\ln(x)\ln(y))^2 - b\ln z - b \end{bmatrix}.$$

Substituting the equilibrium points E_1, E_2 and E_3 in the Jacobian matrix, results in the corresponding Jacobian matrices as follows

$$J(E_1) = \begin{bmatrix} -s & 0 & 0 \\ r & 0 & 0 \\ 0 & 0 & -b \end{bmatrix} \text{ and } J(E_{2,3}) = \begin{bmatrix} -s & e^{-c+cr}sr & e^{cr-r}sc \\ 0 & 0 & -e^{c-r}cr \\ 4c^2re^{-cr+r} & 4c^2re^{-c+r} & -b+4c^2r-br \end{bmatrix},$$

where $c = \pm\sqrt[4]{\frac{b}{r}}$. In order to get the eigenvalues of the system we will solve the jacobian matrices $J(E_1)$ and $J(E_{2,3})$ for the constants s, b and r.

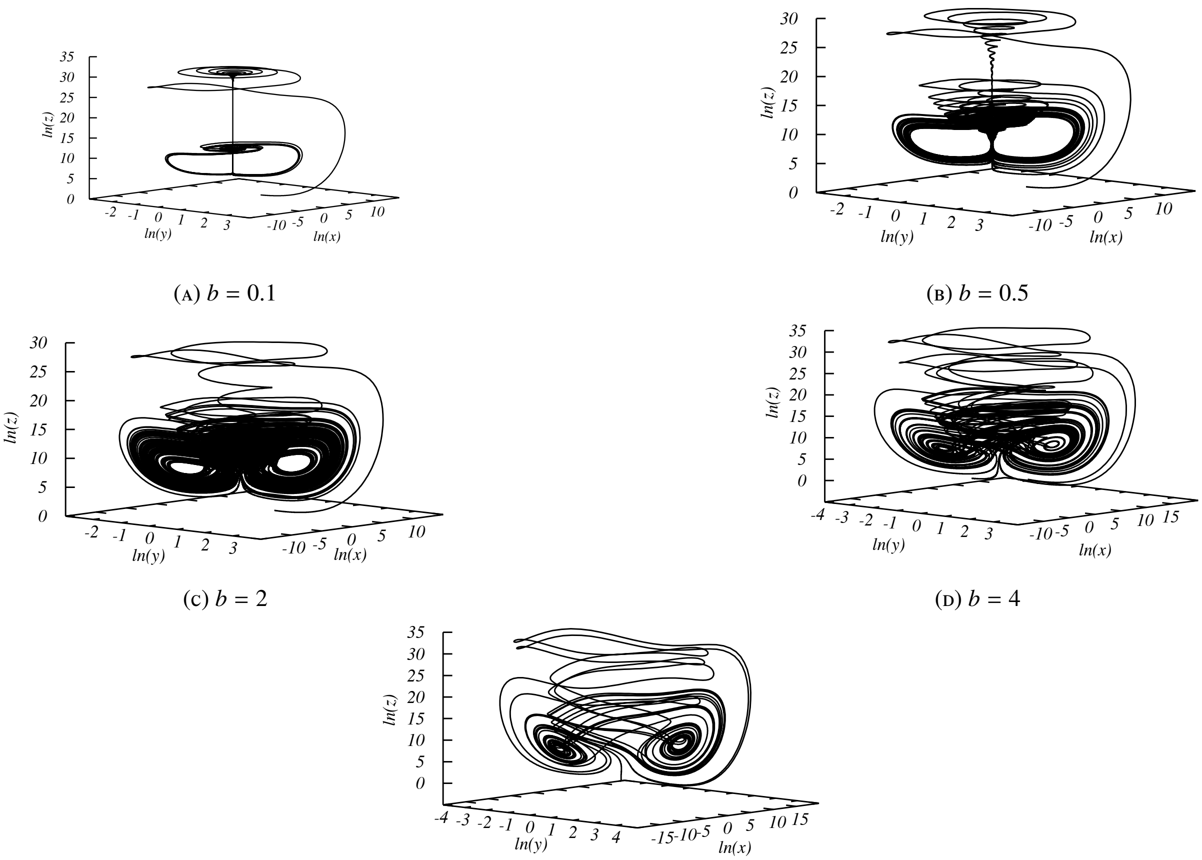
Thus, the eigenvalues of the system (3.1) can be listed in Table 2.

Equilibrium Point	λ_1	λ_2	λ_3
E_1	-12	-4	0
E_2	-21.3002	$2.65011 + 23.872i$	$2.65011 - 23.872i$
E_3	-21.3002	$2.65011 + 23.872i$	$2.65011 - 23.872i$

Table 2. Eigenvalues of the Jacobian matrices of the multiplicative chaotic system, defined by additive derivatives, at the equilibrium points

The results summarised in the Table 2 reveal the instability of the system at the equilibrium point E_1, because two of the three eigenvalues of the Jacobian at E_1 are negative real numbers. Moreover, the eigenvalues of the Jacobian at the 2$^{\text{nd}}$ and the 3$^{\text{rd}}$ equilibrium point also show the instability of the system as two of the eigenvalues are complex with positive real parts being conjugate to each other.

The modified multiplicative chaotic system (3.1) is solved by the 4$^{\text{th}}$ order multiplicative Runge-Kutta method. The chaotic behaviour changes as the value of b changes. The graphs for fixed values of s and r and varying values of b of the solution of the system are shown in the following figure.



(A) $b = 0.1$ (B) $b = 0.5$

(C) $b = 2$ (D) $b = 4$

(E) $b = 10$

Figure 1. Simulation of the multiplicative chaotic system

In Figure 1 , we present a visual representation of our defined chaotic system with specific parameter values: $s = 12$, $r = 8$, and various values of b. The figure illustrates the system's behavior under different b values. Notably, we observe that the system exhibits chaotic behavior across a range of b values between 2 to 10. However, it becomes evident that the system consistently displays more chaotic dynamics when $s = 12$, $r = 8$, and $b = 4$.

3.2. **Lyapunov Exponents of the System.** The chaotic behaviour of a system can also be tested by the Lyapunov exponents. A system is considered chaotic if a system has at least one positive Lyapunov exponent. In Newtonian calculus the Lyapunov exponents of a chaotic system can be evaluated by the formula

$$l = \lim_{N\to\infty} \sum_{n=1}^{N} \ln \frac{d_1}{d_0}.$$

The Lyapunov exponents of a multiplicative chaotic system can also be evaluated in analogy to the ordinary case. Then, the formula used to evaluate the Lyapunov exponents for the multiplicative chaotic systems can be written as

$$l = \lim_{N\to\infty} \sum_{n=1}^{N} \ln \frac{d_1^{(\ln)}}{d_0^{(\ln)}}.$$

It can be seen that, for the multiplicative chaotic systems the distances are evaluated in logarithmic scale.

The maximal Lyapunov exponents of the system (3.1) are obtained as $l_1 = 8.1806, l_2 = 1.0684$ and $l_3 = -15.1893$. Moreover, the Lyapunov dimension of the system can also be calculated as

$$D_L = j + \frac{\sum_{i=1}^{j} l_i}{\left|l_{j+1}\right|} = 2 + \frac{l_1 + l_2}{|l_3|} = 2.6089.$$

As the system has a positive Lyapunov exponent and the Lyapunov dimension is in the range 2-3 the multiplicative system can be considered as a chaotic system. The following graph shows all of the Lyapunov exponents of the system (3.1).

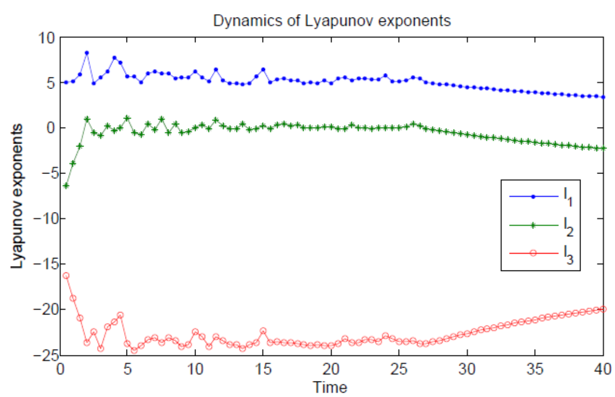


FIGURE 2. Plot of Lyapunov exponents of the Multiplicative Chaotic System

In Figure 2, which displays the Lyapunov exponents of the Multiplicative Chaotic System. Each point on the graph represents a Lyapunov exponent, and their distribution and values are indicative of the system's chaotic nature. The presence of positive exponents and the overall pattern observed in the graph provide compelling evidence for the chaotic dynamics of our system

4. Conclusion

In this research, we introduced a novel attractor called the "modified quadratic Lorenz attractor" and conducted a comprehensive examination of its characteristics. Our investigation encompassed several aspects, including the equilibrium points, the Jacobian matrix, the eigenvalues of the Jacobian, and the Lyapunov exponents. We performed this analysis using both geometric multiplicative calculus and conventional calculus. To obtain numerical results, we applied the geometric multiplicative Runge-Kutta method. The findings from our study consistently demonstrate the presence of chaotic behavior in this newly proposed system, aligning with the outcomes of our numerical simulations. This system will used several areas in engineering.

5. Authors Contribution Statement

The study was a collaborative effort in which all authors actively participated in the planning, execution, and analysis phases. Each author's contribution was substantial enough to warrant their inclusion as authors. Furthermore, all authors have thoroughly reviewed and consented to the final version of the manuscript prior to publication. Specifically, the authors collaborated on the results section, demonstrating a collective engagement in the research process. This statement affirms that each author has played a significant role in the study and is in concurrence with the content presented in the published manuscript.

6. Conflicts of Interest

The authors affirm that there are no conflicts of interest associated with the publication of this manuscript. Each author declares that they have no financial or personal relationships that could inappropriately influence the work presented. The research was conducted with the utmost integrity and objectivity, and any potential conflicts that arose were transparently addressed during the planning and execution of the study.

References

[1] Aniszewska, D., *Multiplicative runge–kutta methods,* Nonlinear Dynamics, **50**(1)(2007), 265–272.

[2] Aniszewska, D., Rybaczuk, M., *Lyapunov type stability and Lyapunov exponent for exemplary multiplicative dynamical systems,* Nonlinear Dynamics, **54**(4)(2008), 345–354.

[3] Chen, G., Ueta, T., *Yet another chaotic attractor*, International Journal of Bifurcation and Chaos, **9**(07)(1999), 1465–1466.

[4] Eminağa, B., Aktöre, H., Riza, M. *A modified quadratic Lorenz attractor*, ArXiv e-prints 1508.06840 (Aug. 2015).

[5] Holmes, P., *Poincaré, celestial mechanics, dynamical-systems theory and chaos,* Physics Reports, **193**(3)(1990), 137–163.

[6] Lorenz, E.N., *Deterministic nonperiodic flow,* Journal of the Atmospheric Sciences, **20**(2)(1963), 130–141.

[7] Lu, J., Chen, G., *A new chaotic attractor coined,* International Journal of Bifurcation and Chaos, **12**(03)(2002), 659–661.

[8] Lu, J.G., *Chaotic dynamics of the fractional-order Lu system and its synchronization,* Physics Letters A, **354**(4)(2006), 305–311.

[9] Lyapunov, A.M., *The general problem of the stability of motion,* International Journal of Control, **55**(3)(1992), 531–534.

[10] Ott, E., Chaos in Dynamical Systems, Cambridge University Press, 2002.

[11] Pehlivan, I., Uyaroglu, Y., *A new chaotic attractor from general lorenz system family and its electronic experimental implementation,* Turkish Journal of Electrical Engineering and Computer Sciences, **18**(2)(2010), 171–184.

[12] Poincare, H., Lectures on Celestial Mechanics, 1965.

[13] Poincare, H., New Methods of Celestial Mechanics, History of Modern Physics and Astronomy, New York: American Institute of Physics 156 (AIP),— c1993 1, 1993.

[14] Riza, M., Aktöre, H., *The runge–kutta method in geometric multiplicative calculus,* LMS Journal of Computation and Mathematics, **15818**(2015), 539–554.

[15] Rössler, O., *An equation for continuous chaos,* Physics Letters A, **57**(5)(1976), 397–398.