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Some Notes on the Extendibility of an Especial Family of Diophantine P_2 Pairs

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Abstract

Although it is known that there are an infinite number of Diophantine P_1 triples, there is no complete characterization for these triples.

This paper is a continuation and a generalization of one of the recent papers (see [ref. 35]) in which several numerical results are demonstrated and some properties are given for special Diophantine P_2 pairs and triples. Here, the expansion of the single-element set {2} into a Diophantine P_2 binary special family as {2, s} (with s values expressed as a recurrence/iteration of natural numbers) is obtained firstly. Then, binary special family {2, s} is extended as {2, s, a_s} Diophantine P_2 triples (a_s is determined in the terms of s) using solutions of Diophantine equations. Lastly, it is proved that {2, s, a_s} can not be extended Diophantine P_2 quadruples using elementary and algebraic methods different from other works in the literaure.

Keywords: Diophantine P_2 sets, System of Equations, Integral Solutions, Non- extendable Diophantine P_2 triples, Elementary Number Theory, Natural Numbers.

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1. Introduction, Materials and Method

Until today, many different studies have been brought to the literature in Diophantine set theory with different methods and approximations especially for Diophantine P_t pairs, Diophantine P_t triples, Diophantine P_t quadruples and so on…

Dujella, A. ([14 - 24]) , who is a very famous in the Diophantine set theory, has proved many significant results on this subject and his contributions in this field have a very important place in the literature.

Some of the results obtained on Diophantine P_t pairs and their structures with extensions can be found in the work of mathematicians such as Cipu, Filipino, Fujita, Özer and Park ([9, 26-27, 35-36]).

In addition, studies with varies approximations/perspections on Diophantine P_t triples and their extensions, characterization with classification and studies with different methods that are guiding can be discovered in the works of mathematicians such as Adzaga, Baker, ..., Özer, Thamotherampillai, Zhang etc ([1, 3, 5-7, 11-12, 29, 33-34, 38-39]).

Bashmakova, Mollin, Dickson and many valuable mathematicians ([4, 8, 10, 13, 25, 28, 30, 32]) who have very valuable studies and books in the field of number theory such as Diophantine set theory, Pell and Diophantine equation solutions, elementary number theory, algebraic number theory, lead the way in the creation of today's studies.

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Even very famous mathematicians such as Euler, Fermat, Dickson, Davenport obtained amny significant results on the problem of Diophantus, there are still characterization/classification and algorithm problems on this subject nowadays.

So, followings can be given from the literature as briefly:

Definition 1.1 (Diophantine P_t Pairs) Let t be a positive integer and Ш ={£, ¥} be a set containing two positive integers. It is named by a Diophantine P_t pair if (£.¥ + t) is a perfect square integer.

In ([27]), it is proved that {k − 1, k + 1} set is a Diophantine P_1 pair for natural number k and it can be extended to Diophantine P_1 triples with some positive integers.

Definition 1.2 (Diophantine P_t Triples) Let t be a positive integer and Ш ={£, ¥, Ʀ} be a set containing three positive integers. It is named by a Diophantine P_t triple if all three (i) £.¥ + t and (ii) £.Ʀ + t with (iii) ¥.Ʀ + t are equal to a perfect square integer.

In ([29]), it is investigated that {k + 1, 4k, 9k + 3} set is the family of Diophantine P_1 triples for natural number k and whether this set could be quadruple or not.

Definition 1.3 (Diophantine P_t r - Tuples) Let t be a positive integer and Ш ={Ƞ$_1$, Ƞ$_2$, …, Ƞ$_r$} be a set containing r different positive integers. It is named by a Diophantine P_t r- tuple if (Ƞ$_i$. Ƞ$_j$ + t) is a square integer for $1 \leq i \neq j \leq r$.

Definition 1.4 (Regularity Condition for Diophantine P_t Triples) A Diophantine P_t triple Ш = {£, ¥, Ʀ} set of the positive integers is called as a regular if the condition (Ʀ − ¥ − £)^2 = 4. (£. ¥ + t) is satisfied.

As a different way of the other works, congruences, definitions of Diophantine P_t sets fundamental solutions of the Diophantine equations, elementary and algebraic operations with logical basic and simple approximations are used to obtain our results in this work as method of the paper.

2. Results

In [35] the author proved that a type of Diophantine D(2) (with different represention it can be used as P_2) that pairs started with number two {2} and, continues with positive integers such as {2,7}, { 2,17}, {2,31} can be extended to regular Diophantine D(2) triples but can not be extended to Diophantine D(2) quadruples using the techniques with notations such as quadratic congruence's solutions, Stolt fundamental solutions method and fundamental solutions of binary quadratic form equations.

Now, the expansion of the single-element set {2} into a Diophantine P_2 pair special family as {2, s} (with s values expressed as a recurrence/iteration of natural numbers) will be obtained and special family {2, s} will be expanded as {2, s, a_s} regular Diophantine P_2 triples (a_s is determined in the terms of s by natural numbers) using solutions of Diophantine equations. Also, it will be demonstarated that that {2, s, a_s} can not be extended Diophantine P_2 quadruples by the helping of elementary and algebraic methods as follows:

Theorem 2.1. The single-element set {2} is expanded into a Diophantine P_2 pair special family as {2, s} with followings:

(i) $s = 2n^2 - 1$ in the terms of natural numbers for $2 \leq n \in \mathbb{N}$.
(ii) $s = 2n^2 + 4n + 1$ in the terms of natural numbers for $2 \leq n \in \mathbb{N}$.

Proof. It needs to be proved whether or not {2, s} is a special family of Diophantine P_2 pairs for $s = 2n^2 - 1$ or $s = 2n^2 + 4n + 1$ in the terms of natural numbers for $n \geq 2$.

(i) For $n \geq 2$ natural numbers, Is $\{2, 2n^2 - 1\}$ a special family of Diophantine P_2 pairs? Using definition of the Diophantine P_2 pair, following equation is obtained and need to solvable for $x \in \mathbb{Z}$

$$2.(2n^2 - 1) + 2 = x^2$$

From the left side of equation, we have

$$2(2n^2 - 1) + 2 = 4n^2 - 2 + 2 = 4n^2$$

It implies $x = \pm 2n \in \mathbb{Z}$.

So, $\{2,2n^2 - 1\}$ is a special family of Diophantine P_2 pairs natural numbers for $n \geq 2$.

(ii) In a similar way, $\{2,2n^2 + 4n + 1\}$ is a special family of Diophantine P_2 pairs for $n \geq 2$ since

$$2.(2n^2 + 4n + 1) + 2 = 4n^2 + 8n + 2 + 2 = 4n^2 + 8n + 4 = 4(n^2 + 2n + 1)$$
$$= 4(n + 1)^2 = [\mp 2(n + 1)]^2$$

Hence, $\{2,2n^2 + 4n + 1\}$ a special family of Diophantine P_2 pairs natural numbers for $n \geq 2$.

Theorem 2.2. Let $\{2,\ 2n^2 - 1\}$ be a special family of Diophantine P_2 pairs for $2 \leq n \in \mathbb{N}$. Special family of Diophantine P_2 pairs $\{2,\ 2n^2 - 1\}$ ($2 \leq n \in \mathbb{N}$) generates a regular special family of Diophantine P_2 triples as $\{2, s, a_s\} = \{2,2n^2 - 1,\ 2n^2 + 4n + 1\}$ for $2 \leq n \in \mathbb{N}$.

Proof. Suppose that t be a positive integer such that $\{2,\ 2n^2 - 1, t\}$ be a special family of Diophantine P_2 triples for $2 \leq n \in \mathbb{N}$. Then, following equations are obtained for some x, y integers.

$$2t + 2 = x^2 \qquad \text{and} \qquad (2n^2 - 1)t + 2 = y^2.$$

Dropping t between above equations, we have following Diophantine equation

$$(2n^2 - 1)x^2 - 2y^2 = (4n^2 - 6)$$

for some $x, y \in \mathbb{Z}$.

Considering $2 \leq n$ natural numbers and using techniques to solve Diophantine equation mentioned above, one of the family solutions x is obtained as $x = \mp 2(n + 1)$ and also it is got that t equals to $2n^2 + 4n + 1$ in the terms of natural numbers.

Additionally, special family of Diophantine P_2 pairs $\{2,\ 2n^2 - 1\}$ ($2 \leq n \in \mathbb{N}$) generates a special family of Diophantine P_2 triples as $\{2, s, a_s\} = \{2,2n^2 - 1,\ 2n^2 + 4n + 1\}$ for $2 \leq n \in \mathbb{N}$.

Regularity condition need to be satisfied for this special family of Diophantine P_2 triples determined as $\{2, s, a_s\} = \{2,2n^2 - 1,\ 2n^2 + 4n + 1\}$ for $2 \leq n \in \mathbb{N}$.

From the definition of regularity condition which is defined by "Diophantine P_2 triple Ш $= \{£, ¥, Ɽ\}$ is a set of the positive integers called as regular if the condition $(Ɽ - ¥ - £)^2 = 4.(£.¥ + 2)$ is satisfied", it is easily seen that $\{2,2n^2 - 1,\ 2n^2 + 4n + 1\}$ is regular for $2 \leq n \in \mathbb{N}$.

Theorem 2.3. The regular special family of Diophantine P_2 triples $\{2, s, a_s\}=\{2, 2n^2-1,\ 2n^2+4n+1\}$ for $2 \leq n\ \in\ \mathbb{N}$ can not be extended to Diophantine P_2 quadruples.

Proof. Let assume that the special family of Diophantine P_2 triples $\{2, s, a_s\}=\{2, 2n^2-1,\ 2n^2+4n+1\}$ can be extended for any positive integer ¥ while $n \geq 2$ natural numbers such that $\{2,\ 2n^2-1,\ 2n^2+4n+1,\ ¥\}$ is a special family of Diophantine P_2 quadruples. Therefore, we have equations as follows for x, y, z integers;

$$2¥+2=x^2$$

$$(2n^2-1)¥+2=y^2$$

$$(2n^2+4n+1)¥+2=z^2$$

for $n\ \geq 2$ natural numbers.

Eliminating ¥ from the above equations, we obtain following Diophantine equations to solve while $n\ \geq 2$ natural numbers.

$$(2n^2-1)x^2-2y^2\ =\ 4n^2-6$$

$$(2n^2+4n+1)\,x^2-2z^2\ =\ 4n^2+8n-2$$

$$(2n^2+4n+1)\,y^2\ -\ (2n^2-1)\,z^2\ =\ 8n+4.$$

Firstly let us consider first Diophantine equation $(2n^2-1)x^2-2y^2\ =\ 4n^2-6$. Since the right side of this equation is an even integer and the left side has to be an even integer too. Hence, x has to be an even integer.

Let $x=2A$ be an even integer for $A\in\ \mathbb{Z}$. If we put this value into the $(2n^2-1)x^2-2y^2\ =\ 4n^2-6$, then we get following results and obtain that y has to be odd integer.

$$(2n^2-1).\,4A^2-2y^2=4n^2-6$$

$$\Rightarrow 2y^2=(2n^2-1).\,4A^2-4n^2+6$$

$$\Rightarrow y^2=4A^2n^2-2A^2-2n^2+3$$

$$\Rightarrow y=2B+1$$

such that B is an integer. If we put both $y=2B+1,\ x=2A$ integers for $A, B\in\ \mathbb{Z}$ into the first Diophantine equation $(2n^2-1)x^2-2y^2\ =\ 4n^2-6$, then followings are obtained.

$$4B^2+4B+1=4A^2n^2-2A^2-2n^2+3$$

$$\Rightarrow 2B^2+2B=A^2(2n^2-1)-(n^2-1).$$

If $n\ \geq 2$ natural numbers are odd numbers then it is determined that $A\in\ \mathbb{Z}$ is an even integer, otherwise $A\in\ \mathbb{Z}$ is an odd integer if $n\ \geq 2$ natural numbers are even numbers.

If we consider the second Diophantine equation $(2n^2+4n+1)\,x^2-2z^2\ =\ 4n^2+8n-2$ and put $x=2A$ value into this one, then we get followings;

$$4A^2.\,(2n^2+4n+1)-2z^2=4n^2+8n-2$$

$$\Rightarrow z^2=4n^2A^2+8A^2n+2A^2-2n^2-4n+1$$

This result implies that $z=2C+1$ is an odd integer for $C\in\ \mathbb{Z}$.

Considering last Diophantine equation $(2n^2 + 4n + 1)\, y^2 - (2n^2 - 1)\, z^2 = 8n + 4$ with $y = 2B + 1$ and $z = 2C + 1$ for $B, C \in \mathbb{Z}$ then following equations are obtained.

$$(2n^2 + 4n + 1).(2B + 1)^2 - (2n^2 - 1)(2C + 1)^2 = 8n + 4$$

$$\Rightarrow (4B^2 + 4B + 1)(2n^2 + 4n + 1) - (4C^2 + 4C + 1)(2n^2 - 1) = 8n + 4$$

$$\Rightarrow 8B^2n^2 + 16B^2n + 4B^2 + 8Bn^2 + 16Bn + 4B + 2n^2 + 4n + 1 - 8C^2n^2 + 4C^2 - 8Cn^2 + 4C - 2n^2 + 1 = 8n + 4$$

$$\Rightarrow 8n^2(B^2 + B - C^2 - C) + 4n(4B^2 + 4B) + 4(B^2 + B + C^2 + C + n) + 2 = 8n + 4$$

If we apply $(mod\ 4)$ in the equation mentioned as above

$$8n^2(B^2 + B - C^2 - C) + 4n(4B^2 + 4B) + 4(B^2 + B + C^2 + C + n) + 2 = 8n + 4$$

Then, we obtain $2 \equiv 0 (mod 4)$. This is a contradiction.

Thus, there is no solution of the system of the Diophantine equations mentioned as above. So, the regular special family of Diophantine P_2 triples $\{2, s, a_s\} = \{2, 2n^2 - 1,\ 2n^2 + 4n + 1\}$ for $2 \leq n \in \mathbb{N}$ can not be extended to Diophantine P_2 quadruples.

3. Discussion and Conclusion

Classifying and characterization problem of P_t sets has been studied extensively at the present time even this issue comes from ancient times of Diophantus of Alexandria as you see in the references. This problem is tried to be solved by establishing connections with many different methods and other issues.

Therefore, in this study, some results were obtained by using algebraic and elementary methods for a special family Diophantine P_2 sets and a contribution to the literature was made In this study, the mathematical steps to obtain our results are given in detail. New results can be obtained by performing similar studies within different Diophantine P_2 or P_k families.

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Declaration

This study does not require ethics committee approval.

Conflict of Interest

There is no conflict of authors in this work.

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