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A Note on Fractional Order Derivatives on Periodic Signals According to Fourier Series Expansion

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Abstract: This study presents a discussion on input-output orthogonality property of derivative operators for sinusoidal functions and investigates the effects of fractional order derivative on Fourier series expansion of periodic signals. The findings of this study are useful for the interpretation of fractional order derivative operator for time periodic signals. Fourier series expansion expresses any periodical signals as the sum of sine and cosine functions. Accordingly, it is illustrated that the derivative operator takes effect on the amplitude and phase of Fourier components as follows: The first order derivative of sine and cosine functions leads to a phase shifting of the right angle and an amplitude scaling proportional to angular frequency of sinusoidal component. As a result of the right angle phase shifting of sinusoidal components, the first order derivative generates an orthogonal function for sinusoidal inputs. However, non-integer order derivatives do not conform orthogonality property for sine and cosine functions because it can lead to a phase shifting in the any fraction of right angle. It also results in an amplitude scaling proportional to α-power of angular frequency of sinusoidal components. Moreover, fractional order derivative of periodic signals is expressed on the bases of Fourier series expansion and the interpretation of the operator for signals is discussed on the bases of this formula.

Keywords: Fractional order derivative, orthogonality, Fourier series expansion, periodic signals.

1. INTRODUCTION

There have been many attempts for the interpretation of physical meaning of fractional order derivative operators. However, these interpretations were given regarding to definitions of the derivative, their effects or properties appearing in specific equations or applications. For instance, a probabilistic interpretation of the fractional order differentiation has been introduced with respect to Grünwald-Letnikov definition [1]. Riemman-Lioville definition of fractional derivatives was taken into account arithmetically as the convolution with a kernel [2]. On the other hand, the fractional order derivative was considered as the projection of the area below a function over a nonlinear time scale with a deformation parameter [3]. It was reported that fractional derivative can be identified as a typical superposition integral over the spatial domain of the Levy diffusion process [4]. By considering the basic motion equation, fuzzy velocity and fuzzy acceleration concept has also been considered for fractional order derivative operation [5].

Although a common understanding for the meaning of fractional order derivative operator has not been reached yet, the efforts to explain the role of fractional order derivative in an application deepens our insight on the way of physical interpretation of the fractional order derivative operator. In the current study, we evaluate effects of fractional order derivative on sine and cosine functions and extent this understanding for periodic signals on the bases of Fourier series expansion.

Previously, periodic functions were studied in fractional calculus in many works. It was reported that the fractional derivative of a periodic signal is also periodic if it is defined on the whole real line [6]. Fractional order derivative of a periodic signal was expressed by term by term derivatives [7,8]. Tavazoei presented a discussion on fractional-order derivatives of a periodic functions [9]. For control system problems, step responses of fractional LTI systems can be calculated according to Fourier series of square waves [10].

The conventional first order derivative generates an orthogonal result for sine and cosine functions because of a phase shifting of $\pi/2$ radian. Therefore, this phase shift converts sine waveform to cosine waveform and vice versa. The current study demonstrates that the fractional order derivative may not hold this input-output orthogonality property because the fractional order derivatives of sine and cosine functions allow a phase shift with any fraction of $\pi/2$ radian. In addition, fractional order derivatives also scale the amplitude of sine and cosine functions depending on the angular frequency ω and fractional order α. Since any periodical signal can be expressed the sum of sine and cosine functions by using Fourier series expansion, fractional order derivative of periodical signals is analyzed by considering effects of fractional order derivative on sine and cosine components. Then, fractional order derivative of square wave is calculated by means of Fourier series expansion.

2. FRACTIONAL ORDER DERIVATIVE OF SINE AND COSINE FUNCTIONS AND ORTHOGONALITY PROPERTY

The fractional-order derivatives of sine and cosine functions are expressed as follows [11],

$$D^{\alpha}\sin(\omega t) = \omega^{\alpha}\sin(\omega t + \frac{\pi}{2}\alpha) \tag{1}$$

$$D^{\alpha}\cos(\omega t) = \omega^{\alpha}\cos(\omega t + \frac{\pi}{2}\alpha) \tag{2}$$

Equation (1) and (2) reveal two significant effects of the derivative operator on the sine and cosine functions in time domain [12,13]:

(i) Fractional order derivatives scale the amplitude of sinusoidal function with the factor of ω^{α} (See Figure 1(b)).

(ii) Fractional order derivatives shift the phase of sinusoidal function $\frac{\pi}{2}\alpha$ radian (See Figure 1(b)).

It is obvious that the sine and cosine components of the phasor $v = \cos(\theta) + j\sin(\theta)$ are orthogonal to each others. In order to graphically depict effects of fractional order derivative on the signal of $\cos(\omega t + \theta)$, the phasor representation of this signal, $v = \cos(\theta) + j\sin(\theta)$, are considered in Figure 1. The phasor v' represents the phasor of $D^{\alpha}\cos(\omega t + \theta)$ in the figure.

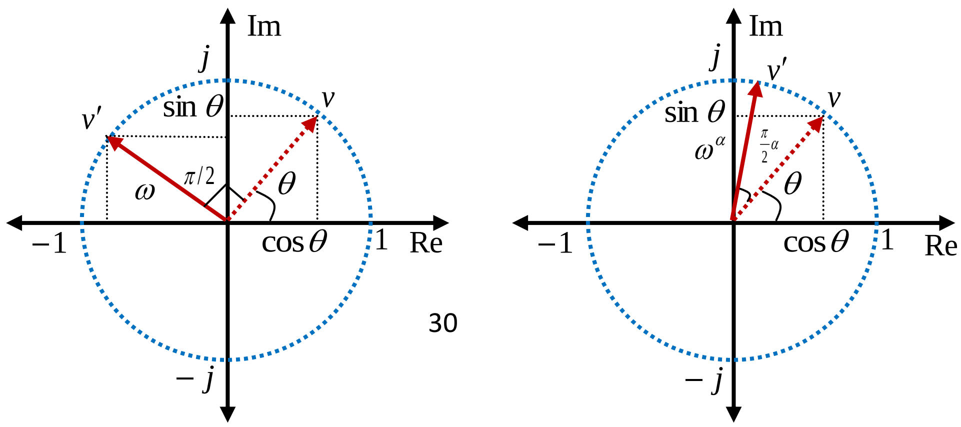

Figure 1. Phasor representation of (a) the first order derivative and (b) fractional order derivative of the signal $\cos(\omega t+\theta)$.

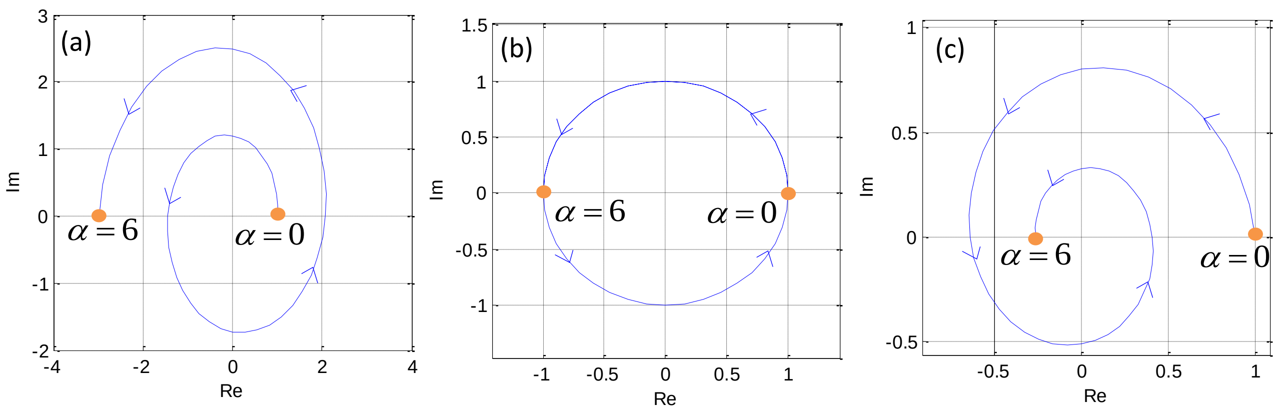

Figure 2. Change of phasor of $D^{\alpha}\cos(\omega t+\theta)$ for (a) $\omega=1.2$, (b) $\omega=1.0$ and (c) $\omega=0.8$ in the range of $\alpha\in(0,6)$ and $\theta=0$.

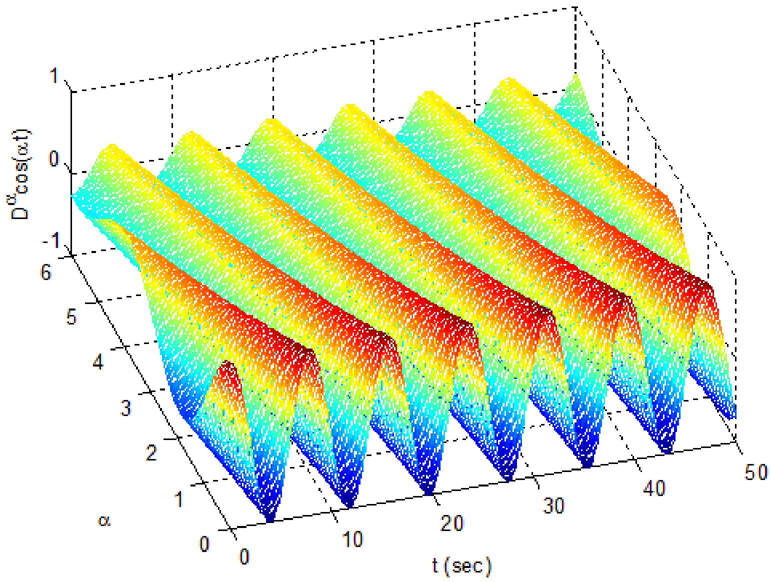

Figure 2 shows the trajectory of phasors of fractional order derivatives of the signal $\cos(\omega t+\theta)$.

In the figure, phasors of fractional order derivative of $\cos(\omega t+\theta)$, $v'(\omega,\alpha)=\omega^{\alpha}\cos(\theta+\frac{\pi}{2}\alpha)+j\omega^{\alpha}\sin(\theta+\frac{\pi}{2}\alpha)$, were drawn for $\omega=1.2$, $\omega=1.0$ and $\omega=0.8$ rad/sec from $\alpha=0$ to $\alpha=6$. Due to ω^{α} factoring of the amplitude, it traces three different circular trajectories: a growing one for $\omega>1$, a steady one for $\omega=1$ and a decaying one for $\omega<1$.

Figure 3 shows the temporal evolution of $D^{\alpha}\cos(\omega t)$ for $\omega = 0.8$ for various $\alpha > 0$. Figure clearly illustrates effects of fractional order derivative on amplitude and phase of a sinusoidal function. The phase shifting of $D^{\alpha}\cos(\omega t)$ is apparent in the figure. Due to $\omega < 0.8$, as fractional order α increases, the amplitude of $D^{\alpha}\cos(\omega t)$ decreases.

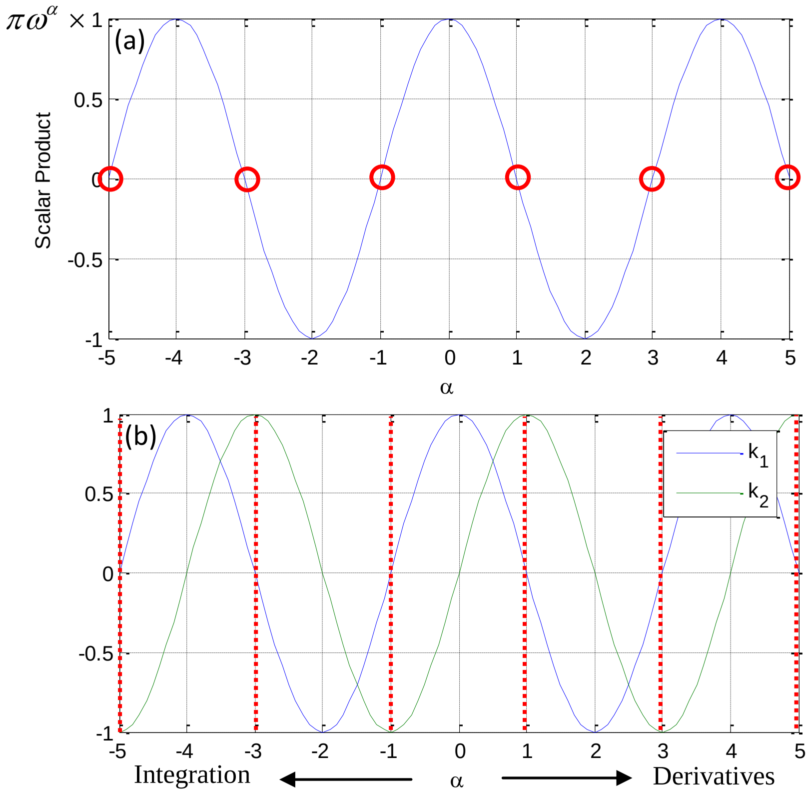

Figure 3. Values of $D^{\alpha}\cos(\omega t)$ for $t \in (0,50)$, $\omega = 0.8$ and $\alpha \in (0,6)$.

Let us show that the first order derivative operator generates an orthogonal output signal to sinusoidal input signals.

To verify this effect, one can show that the scalar product of $\cos(\omega t)$ and $D^{(1)}\cos(\omega t)$ functions is zero.

$$< \cos(\omega t), D^{(1)}\cos(\omega t) > = < \cos(\omega t), -\omega\sin(\omega t) > = -\omega\int_0^{2\pi}\cos(\omega t)\sin(\omega t)dt = 0 \qquad (3)$$

This result is valid for integer angular frequencies ($\omega \in Z$) and it confirms that the first derivative operator generates orthogonal output signal for sinusoidal input signals. Let us show that the none-integer derivative can generate a non-orthogonal output signal for cosine input signals for $\omega \in Z^{+}$. For this purpose, one can demonstrate that the scalar product of $\cos(\omega t)$ and $D^{\alpha}\cos(\omega t)$ is not zero for none-integer order derivatives as follows:

$$< \cos(\omega t), D^{\alpha}\cos(\omega t) >=< \cos(\omega t), \omega^{\alpha}\cos(\omega t+\frac{\pi}{2}\alpha) >$$
$$= \omega^{\alpha}\int_{0}^{2\pi}\cos(\omega t)\cos(\omega t+\frac{\pi}{2}\alpha)dt = \pi\omega^{\alpha}\cos(\frac{\pi}{2}\alpha) \tag{4}$$

Scalar product of $\cos(\omega t)$ and $D^{\alpha}\cos(\omega t)$ is not zero for non-interger values of α and $\omega \neq 0$. This result indicates that fractional order derivative operator for non-integer α values does not yield an orthogonal output signal for cosine input signals for $\omega \in Z^{+}$.

3. PHASE FACTOR DECOMPOSITION OF FRACTIONAL ORDER DERIVATIVE OF SINE AND COSINE FUNCTIONS

In this section , we decompose the sine and cosine components from phase shifting part as,

$$\sin(\omega t+\frac{\pi}{2}\alpha) = \cos(\frac{\pi}{2}\alpha)\sin(\omega t)+\sin(\frac{\pi}{2}\alpha)\cos(\omega t) = k_1\sin(\omega t)+k_2\cos(\omega t) \tag{5}$$

$$\cos(\omega t+\frac{\pi}{2}\alpha) = \cos(\frac{\pi}{2}\alpha)\cos(\omega t)-\sin(\frac{\pi}{2}\alpha)\sin(\omega t) = k_1\cos(\omega t)-k_2\sin(\omega t) \tag{6}$$

where, $k_1 = \cos(\frac{\pi}{2}\alpha)$ and $k_2 = \sin(\frac{\pi}{2}\alpha)$ are called as the phase factors of the derivative.

Fractional order derivative of sine and cosine functions can be written in the following forms,

$$D^{\alpha}\sin(\omega t) = \omega^{\alpha}\sin(\omega t+\frac{\pi}{2}\alpha) = \omega^{\alpha}k_1\sin(\omega t)+\omega^{\alpha}k_2\cos(\omega t) \tag{7}$$

$$D^{\alpha}\cos(\omega t) = \omega^{\alpha}\cos(\omega t+\frac{\pi}{2}\alpha) = \omega^{\alpha}k_1\cos(\omega t)-\omega^{\alpha}k_2\sin(\omega t) \tag{8}$$

Figure 3(a) shows scalar product of input and output of fractional order derivative operator for the order range of $\alpha \in [-5,5]$. The circular marks in the figure indicate the α values that make scalar products of input and output of fractional order derivative zero ($\pi\omega^{\alpha}\cos(\frac{\pi}{2}\alpha) = 0$). As seen from the figure, for the odd integer values of $\alpha = 2n-1,\ n \in Z$, the scalar product becomes zero and verifies the input-output orthogonality of the derivative operator. For other α values, fractional order derivative does not generate orthogonal results. When the fractional order derivative exhibits orthogonal input-output relation, the phase factors takes the values of $\{k_1 = 0,\ k_2 = 1\}$ as in Figure 3(b). In other cases of $\{k_1, k_2\}$, fractional order derivative does not exhibit input-output orthogonality for cosine signal. One can see that k_1 and k_2 are also orthogonal each other. Figure 4 shows the drawing of k_2 versus k_1, forming an unit circle, and the corresponding fractional operations. For $\alpha = 0$, it does not perform any operation on signals and yields signal itself (null operation).

3. INVESTIGATION OF EFFECTS OF FRACTIONAL-ORDER DERIVATIVE ON FOURIER SERIES COMPONENTS

Fractional order derivative of Fourier series of a periodic signal was derived in [7,8]. In this section, we discuss effects of fractional order derivative on the components of Fourier series. Let us consider a periodical signal defined as $x(t) = x(t + T_0)$. It is obvious that one can express the rest of the signal by periodic extensions with T_0 period. Accordingly, the fundamental frequency of the signal is defined as $\omega_0 = 2\pi / T_0$. As known, Fourier series expansion of the time series signal is written in open form as,

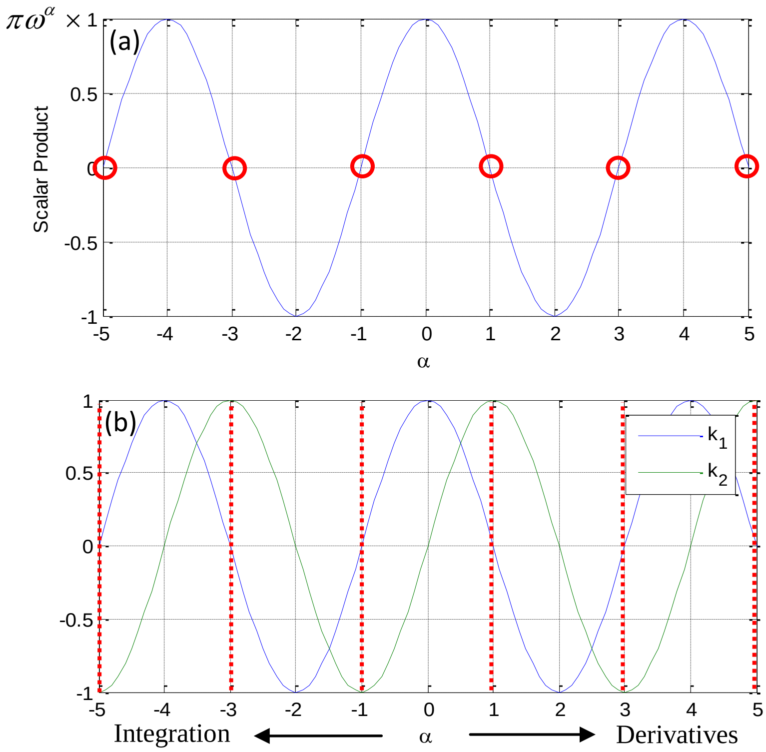

Figure 3. (a) Scalar product $< V(t), D^{(1)}V(t) >= \omega^{\alpha}\cos(\frac{\pi}{2}\alpha)$ versus α; (b) phase factors

$\{\ k_1 = \cos(\frac{\pi}{2}\alpha)\ ,\ k_2 = \sin(\frac{\pi}{2}\alpha)\ \}$ versus α

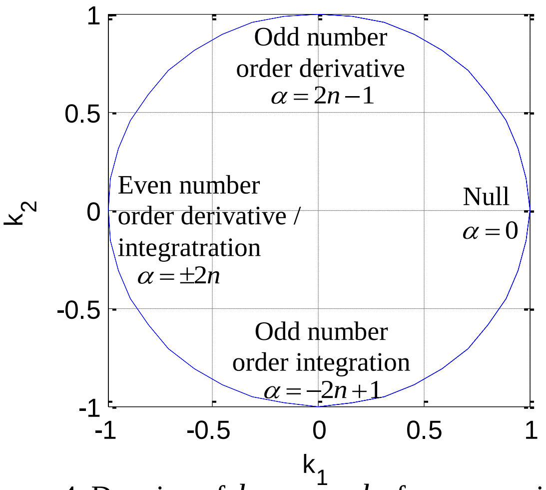

Figure 4. Drawing of k_2 versus k_1 forms an unit circle

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \tag{9}$$

Where, a_n and b_n are constants which are calculated as,

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt \tag{10}$$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos(n\omega_0 t)\, dt \tag{11}$$

$$b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin(n\omega_0 t)\, dt \tag{12}$$

Let us write the fractional order derivative of periodic signal $x(t)$ by considering equation (1) and (2) as,

$$D^{\alpha} x(t) = \frac{a_0}{\Gamma(1-\alpha)} t^{-\alpha} + \sum_{n=1}^{\infty} a_n (n\omega_0)^{\alpha} \cos(n\omega_0 t + \frac{\pi}{2}\alpha) + \sum_{n=1}^{\infty} b_n (n\omega_0)^{\alpha} \sin(\omega_0 t + \frac{\pi}{2}\alpha) \tag{13}$$

Here, fractional order derivative of constant a_0 can be written as $\frac{a_0}{\Gamma(1-\alpha)} t^{-\alpha}$ [14]. By applying the phase factor decomposition of fractional order derivative of sine and cosine functions, fractional order derivative of $x(t)$ can be reorganized as follows,

$$D^{\alpha} x(t) = \frac{a_0}{\Gamma(1-\alpha)} t^{-\alpha} + \sum_{n=1}^{\infty} (n\omega_0)^{\alpha} (k_1 a_n + k_2 b_n) \cos(n\omega_0 t) + \sum_{n=1}^{\infty} (n\omega_0)^{\alpha} (k_1 b_n - k_2 a_n) \sin(n\omega_0 t) \tag{14}$$

Equation (14) transfers the phase shifting effect on the amplitude of Fourier components so that one can compare it with the Fourier transform of the signal $x(t)$ given in Equation (9). Under consideration of equations (13) and (14), one can state that the fractional order derivative takes effect on $x(t)$ in the following manner, (i) Fractional order derivative turns the constant term (DC term) of $x(t)$, which is a_0, into a time-depended component. For the derivative operation ($\alpha > 0$), it decays in time and approximates to zero and then sinusoidal components become more effective in results. For the integration operations ($\alpha < 0$), it grows in time. (ii) Phases of sinusoidal components are shifted $\frac{\pi}{2}\alpha$ radian. For the derivative operation ($\alpha > 0$), phase of sinusoidal components are shifted forwards $\frac{\pi}{2}\alpha$ radian. For the integrator operation ($\alpha < 0$), the phase of sinusoidal components are shifted backwards $\frac{\pi}{2}\alpha$ radian. (iii) Amplitude of sinusoidal components are scaled by $(n\omega_0)^{\alpha}$ factors. For the derivative operation ($\alpha > 0$), $(n\omega_0)^{\alpha}$ factor scales down the components at low frequencies, and it scales up the sinusoidal components at higher frequencies. This means that the scaling factor is frequency-depended. The fractional order derivative operator amplifies higher frequency components and suppresses very lower frequency components. So, it sharpens the signal

$x(t)$ and the degree of sharpening also depends on the order α. For the integrator operation ($\alpha < 0$), $(n\omega_0)^\alpha$ factor takes effect in reverse direction to the derivative operator. Fractional order integration amplifies lower frequency components and suppresses higher frequency components. So, it smoothes the signal $x(t)$ and the degree of smoothing of $x(t)$ also depends on the order α. (iv) Equation (14) reveals that effect of fractional order derivative on Fourier components can be transferred to amplitudes of cosine and sine components according to series of $s_n = (n\omega_0)^\alpha (k_1 a_n + k_2 b_n)$ and $p_n = (n\omega_0)^\alpha (k_1 b_n - k_2 a_n)$. Indeed, s_n and p_n are a linear combination of Fourier coefficients a_n and b_n. In the case of odd integer values of α, the integer order derivative exhibit input-output orthogonality relation and the amplitude of cosine and sine components of Fourier series become $s_n = (n\omega_0)^\alpha k_2 b_n$ and $p_n = -(n\omega_0)^\alpha k_2 a_n$.

4. FRACTIONAL-ORDER DERIVATIVE OF A PERIODIC SQUARE WAVE

This section illustrates the calculation of fractional order derivative of square waves by using Fourier series method. One of applications of this approach was effectively used for the calculation of step response of LTI systems by Tan et al. [10]. They showed that Fourier transform method can provide useful analytical solutions to obtain time response of fractional order transfer function [10,15].

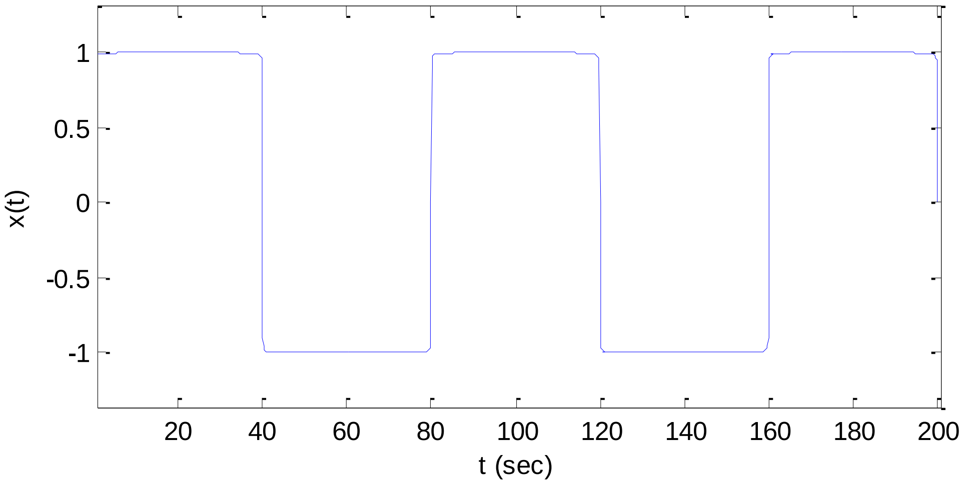

Figure 4. Square waveform

Fourier series expansion of the square waveform shown in Figure 4 can be written as,

$$x(t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k-1)\omega_0 t)}{(2k-1)} \tag{15}$$

One can express the fractional order derivative of this function as,

$$D^\alpha x(t) = \frac{4}{\pi} \omega^\alpha \sum_{k=1}^{\infty} (2k-1)^{\alpha-1} \sin((2k-1)\omega_0 t + \frac{\pi}{2}\alpha) \tag{16}$$

Figure 5 shows the fractional order derivation and integration of square wave, which is calculated by equation (16). For $\alpha > 0$, equation (16) yields the fractional order derivatives, and for $\alpha < 0$, it yields the fractional order integration of Fourier series of square waves in Figure 5.

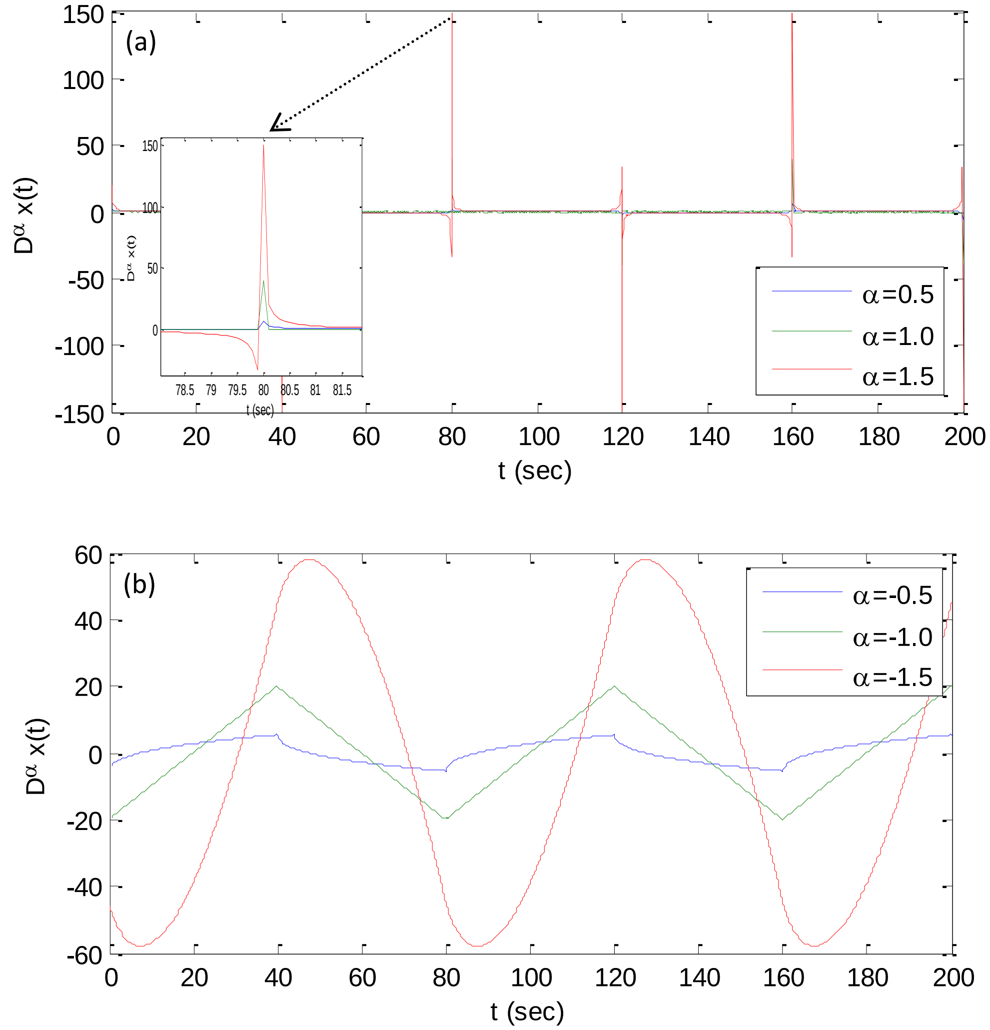

Figure 5. (a) Fractional order derivative of square waveform for $\alpha = \{0.5, 1.0, 1.5\}$, (b) Fractional order integration of square waveform for $\alpha = \{-0.5, -1.0, -1.5\}$

5. CONCLUSION AND DISCUSSIONS

This study presents a theoretical investigation on input-output orthogonality relation of fractional order derivative operator and analyzes effects of fractional order derivative on periodical signals on the bases of Fourier series expansion. The results of analyses can be useful for the interpretation of impacts of fractional order derivative operator on periodic signals.

We observed that the conventional the first order derivative results in a $\frac{\pi}{2}$ radian phase shifting. For non-integer derivative orders, the operator does not exhibit input-output orthogonality property for cosine components. This result suggests us that it may not be effective to interpret physical meaning of the non-integer order derivatives by considering the conventional first order derivative. Therefore, efforts to match some properties of non-integer order derivative with the first derivatives may not be a necessity to understand true nature of the fractional order derivative operator. Perhaps, the power of fractional order tools may come from divergence of fractional order derivative from the characteristic properties of the conventional derivative concept.

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