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Research Article

On τ-Discrete Modules

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Abstract
An R-module M is said to be *(quasi)* τ*-discrete* if M is τ-lifting and has the property (D_2) (respectively, has the property (D_3)), where τ is a preradical in $R-mod$. It is shown that: (1) direct summands of a (quasi) τ-discrete module are (quasi) τ-discrete; (2) a projective module M is τ-discrete if and only if $\frac{M}{\tau(M)}$ is semisimple and $\tau(M)$ is QSL; (3) if a projective module M is *Soc-lifting,* then $\frac{M}{Soc(M)}$ is *Soc-discrete* and $Rad(\frac{M}{Soc(M)})$ is semisimple.

Keywords: preradical, τ-lifting module, (quasi) τ-discrete module.

τ-Ayrık Modüller Üzerine

Öz
τ tüm sol R-modüllerin kategorisinde öncül radikal olmak üzere τ-yükseltilebilir ve (D_2) özelliğini sağlayan (sırasıyla, (D_3) özelliğini sağlayan) bir R-modülü M'e *(ayrık)* τ*-ayrık* denir. Şu gösterilmiştir: (1) Bir (quasi) τ-ayrık modülün her direkt toplam terimi (quasi) τ-ayrıktır; (2) bir projektif M modülünün τ-ayrık olması için gerek ve yeter koşul $\frac{M}{\tau(M)}$ nin yarıbasit ve $\tau(M)$ nin QSL olmasıdır; (3) bir projektif M modülü Soc-yükseltilebilirse, $\frac{M}{Soc(M)}$ Soc-ayrıktır ve $Rad(\frac{M}{Soc(M)})$ yarıbasittir.

Anahtar Kelimeler: öncül radikal, τ-yükseltilebilir modül, (yarı) τ-ayrık modül.

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1. Introduction

In our article, all rings are associative with identity and all modules are unity left modules over these rings. For a ring R, $R-mod$ denotes the category of all left R-modules. A submodule N of a module M will be denoted by $N \leq M$. A nonzero $E \leq M$ is called *essential* in M and written by $E \trianglelefteq M$ if $E \cap F \neq 0$ for every nonzero submodule F of M. We call a module M *extending* if it satisfies (C_1), that is, its submodules are essential in a direct summand of M as in [5].

We call an extending module M *continuous* if it satisfies (C_2), that is, every submodule isomorphic to a direct summand of M is a direct summand as in [5].

We call an extending module M *quasi continuous* if it satisfies (C_3), that is, whenever $M = A \oplus B = C \oplus D$ and $A \cap C = 0$, M has a decomposition $M = (A \oplus C) \oplus E$ as in [5] Since a module M with (C_2) has the property (C_3) every continuous module is quasi continuous. Injective modules are an example of a continuous module.

As a dual notation of an essential submodule of A, one call a proper submodule S of A *small* in M and denoted by $S \ll M$ if $S+X$ is not M for every proper submodule $X<M$. With the notation of immediately extending modules, lifting modules are defined as: M is *lifting* if it satisfies

(D_1) For any $A \leq M$, we can write $M = A_1 \oplus L$, $A_1 \leq A$ and $A \cap L \ll L$ for submodules A_1, L of M.

We call a lifting module M *quasi-discrete* if it satisfies

(D_2) If $A \leq M$ with $\frac{M}{A} \cong B$ and $M = B \oplus C$, we can write $M = A \oplus A'$.

We call a lifting module M *discrete* if it satisfies

(D_3) Whenever $M = A \oplus B$, $M = C \oplus D$ and $M = A + C$, M has a decomposition $M = (A \cap C) \oplus E$.

The modules that provide quasi-projective and the property (D_2) are coincide. Since a module M with (D_2) provides (D_3), quasi-discrete modules are a generalization of discrete modules. It is obvious that (quasi) discrete modules are a dual notion of (quasi) continuous modules. Although injective modules are continuous, a projective module usually does not have to be discrete. Hollow modules (that is, its proper submodules are small) are quasi-discrete. The family of (quasi-) discrete modules are extensively studied by researchers. A module M has the property P^* if for every submodule A of M M has the decomposition $M = A' \oplus B$ such that $A' \leq A$ and $\frac{A}{A'} \leq Rad(\frac{M}{A'})$ for some submodules A' and B of M. Every lifting module has the property P^*. Also, a finitely generated module with the property P^* is lifting. In general, a module with the property P^* need not be lifting. For example, consider the left $\mathbb{Z}$-module $M = {}_{\mathbb{Z}}\mathbb{Q}$. Since radical modules have the property P^*, M has the property P^*. On the other hand, M is not lifting.

In recent years, types of lifting modules have been defined and studied in $R-mod$ with the help of preradicals. A functor τ from the category $R-mod$ to itself is said to be *preradical* if it provides the following properties:

(1) $\tau(M) \leq M$, where $M \in R-mod;$
(2) If $f: M \longrightarrow M'$ is homomorphism, then $f(\tau(M)) \subseteq \tau(M)$ and $\tau(f)$ is the restriction of to $\tau(M')$.

A preradical τ for $R-mod$ is called *exact* if for $N \leq M$ $\tau(N) = N \cap \tau(M)$, and it is called *radical* if $\tau\left(\frac{M}{\tau(M)}\right) = 0$.

Rad(M) and *Soc(M)* denote the radical, socle of a module *M*, respectively. *Rad* and δ are radical in $R-mod$, and *Soc* is an exact preradical in $R-mod$.

Let τ be a preradical in $R-mod$. Following [1, 2.8 and 2.9], we call *M* τ*-lifting* if for any $N \leq M$, we can write $M = A \oplus B$ with $A \subseteq N$ and $N \cap B \leq \tau(B)$ for $A, B \leq M$. In [1], for $\tau = Rad$, *M* is Rad-lifting if and only if *M* has the property P^*. Lifting modules are an example of *Rad*-lifting modules. It is shown in [1, 2.10 (2)] that whenever $M = A \oplus B$ is a τ-lifting module, so does *A*.

2. Preliminaries

Let R be a ring and τ be a preradical in $R-mod$. In our study, we introduce the concept of (quasi) τ-discrete modules. We obtain some properties of such modules. In particular, we show that direct summands of a (quasi) τ-discrete module are (quasi) τ-discrete. Moreover, we prove that a projective module *M* is τ-discrete if and only if $\frac{M}{\tau(M)}$ is semisimple and $\tau(M)$ is QSL. Also, we show that if a projective module *M* is *Soc-lifting,* $\frac{M}{Soc(M)}$ is *Soc-discrete* and $Rad(\frac{M}{Soc(M)})$ is semisimple.

3. Main Theorem and Proof

In this section, we study on (quasi) τ-discrete modules.

Definition 3.1 A module M is called τ*-discrete* (respectively, *quasi* τ*-discrete*) if M is τ-lifting with (D_2) (respectively, (D_3)).

Theorem 3.2 Given a (quasi) τ-discrete module $M = N \oplus N'$. Then *N* is (quasi) discrete.

Proof. By [9, 2.10.(2)], we obtain that *N* is τ-lifting. Hence *N* is (quasi) τ-discrete by [5, Lemma 4.6].

Given modules $U \leq X$. In [6], U is said to be *strongly lifting* in *X* provided whenever $\frac{X}{U} = \frac{A+U}{U} \oplus \frac{B+U}{U}$, we can write $M = Z \oplus T$ where $Z \subseteq A$, $\frac{A+U}{U} = \frac{Z+U}{U}$ and $\frac{B+U}{U} = \frac{T+U}{U}$. Alkan [3] generalizes the definition; *U* is called *quasi strongly lifting (QSL)* in *X* if whenever $\frac{X}{U} = \frac{A+U}{U} \oplus \frac{C}{U}$, we can write $X = Z \oplus T$, $Z \subseteq A$ and $Z + U = A + U$. Observe from [3, Lemma 3.5] that if a module M is τ-lifting, then $\tau(M)$ is *QSL*. Using this fact we obtain that a characterization of (quasi) τ-discrete modules.

Proposition 3.3 Let M be a module with (D_2) (respectively, (D_3)). Then the following statements are equivalent:

(1) it is (quasi) τ-discrete,
(2) it is τ-supplemented and $\tau(M)$ is QSL.
(3) $\frac{M}{\tau(M)}$ is semisimple with $QSL\ \tau(M)$.

Proof. By Lemma 3.5 and Proposition 3.6 in [3].

Corollary 3.4 A projective module *M* is τ-discrete if and only if $\frac{M}{\tau(M)}$ is semisimple and τ(M) is QSL.

Proof. Since projective modules are (D_2), it follows from Proposition 3.3.

Given a module E. We call E *(quasi) Rad-discrete* if E has the property P^* and (D_2) (respectively, has the property P^* and (D_3)) as in [7].

Lemma 3.5 A projective M is Rad-discrete if and only if M is semilocal and $Rad(M)$ is QSL.

Proof. The proof follows from Corollary 3.4.

Theorem 3.6 The following statements are equivalent for a ring *R*:

(1) R is semiperfect;
(2) R is Rad-discrete;
(3) R has the property (P^*);
(4) R is Rad-⊕-supplemented;
(5) R is semilocal and $Rad(R)$ is QSL.

Proof. (1)⇒(2)⇒(3)⇒(4)⇒(1) By [7, Corollary 2.10].

(1)⇔(5) It follows from Corollary 3.4.

Follows from [6, Theorem 10], the socle Soc$({}_RR)$ of a ring R is strongly lifting. Using this fact we characterize Soc-discrete rings in the following.

Proposition 3.7 A ring R is Soc-discrete if and only if $\frac{R}{Soc({}_RR)}$ is semisimple.

Proof. By Corollary 3.4 and [6, Theorem 10].

Given a module *E*. We call E *τ-torsion free* if $\tau(E) = 0$.

Proposition 3.8 Let M be a τ-torsion free module. If it is quasi τ-discrete, it is semisimple.

Proof. Let $N \leq M$. By assumption, we can write $M = A \oplus B$ with $A \leq N$ and $N \cap B \subseteq \tau(B)$. Since M is τ-torsion free, we can write $N \cap B \subseteq \tau(B) \subseteq \tau(M) = 0$ and so $N = N \cap B = A \oplus (N \cap B) = A$, as required.

Recall from [2] that a submodule Z of a module E is *a τ–supplement* of some submodule $T \leq M$ provided $Z+T$ is M and $Z \cap T \subseteq \tau(Z)$.

Theorem 3.9 Let τ be an exact preradical and let M be a τ-lifting module and V be τ-supplement in M. Then V is τ-lifting.

Proof. Let $N \leq V$. Since M is τ-lifting, we can write $M = A \oplus B$, $A \leq N$ and $N \cap B \subseteq \tau(B)$. By the modularity, we can write V *is* $A \oplus (V \cap B)$, and clearly, $N \cap (V \cap B) = N \cap B \subseteq \tau(B)$. Since τ is an exact preradical in $R\text{-}Mod$, we can write $\tau(V \cap B)$ is $V \cap \tau(B)$. Now $N \cap B \subseteq V \cap \tau(B)$ is $\tau(V \cap B)$. It means that V is τ-lifting.

Corollary 3.10 Let τ be an exact preradical in $R - Mod$ and M be a uniform R-module. If M is τ-lifting, then every τ-supplement submodule V of M is quasi τ-discrete.

Proof. By Theorem 3.9, we obtain that V is τ-lifting. Since uniform modules have the property (D_3), we get that V is quasi τ-discrete.

Proposition 3.10 Let τ be a radical in $R - Mod$ and M be a (quasi) τ-discrete module with small $\tau(M)$. Then $\tau(M) = Rad(M)$ and it is (quasi) discrete.

Proof. By [2, 2.10 (1)], we obtain that $Rad(M) \subseteq \tau(M)$. Since $\tau(M) \ll M$, $\tau(M) = Rad(M)$ is small in M. So M is lifting. Hence it is (quasi) discrete.

A module E is called *τ-torsion* if $E=\tau(E)$. For example, semisimple modules are Soc-torsion, radical modules are Rad-torsion, and projective semisimple modules are δ-torsion.

Lemma 3.11 Suppose that M is a τ-lifting module. If $N \leq M$ is τ-torsion, $\frac{M}{N}$ is τ-lifting.

Proof. Let $N \leq A \leq M$. Then we can write $M = A' \oplus B$, $A' \leq A$ and $A \cap B \subseteq \tau(B)$ for submodules $A', B \leq M$. It follows that $\frac{M}{N} = \frac{A'+N}{N} + \frac{B+N}{N}$ and $\frac{A \cap B+N}{N} \subseteq \tau(\frac{B+N}{N})$. Since N is τ-torsion, we can write $\left(\frac{A'+N}{N}\right) \cap \left(\frac{B+N}{N}\right) = 0$. Thus $\frac{M}{N}$ is τ-lifting.

Theorem 3.12 Suppose that N is a τ-torsion submodule of a projective module M. If M is τ-lifting, $\frac{M}{N}$ is τ-discrete.

Proof. Since M is a projective module and N is τ-torsion, $\frac{M}{N}$ has the property (D_2). Applying Lemma 3.11, we deduce that $\frac{M}{N}$ is τ-discrete.

Corollary 3.13 If M is a projective and Soc-lifting module, then $\frac{M}{Soc(M)}$ is Soc-discrete and its radical is semisimple.

Proof. Following Theorem 3.12, we get that $\frac{M}{Soc(M)}$ is Soc-discrete. Also, applying [2, 2.10 (1)], $Rad(\frac{M}{Soc(M)})$ is semisimple. This completes the proof.

4. Conclusion

In this article, we introduce the concept of (quasi) τ-discrete modules and investigate the basic properties of these modules by preradicals in $R-mod$, where R is an associative ring with identity. We characterize projective τ-discrete modules. We show that if a module is τ-lifting, then its factor modules by τ-torsion submodules are τ-lifting. We prove that if a projective module M is *Soc*-lifting, then $\frac{M}{Soc(M)}$ is Soc-discrete and its radical is semisimple

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

Author Contributions

All authors have investigated and studied no the published version of the manuscript.

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