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Kendisi ve Tersi Yalınkat Fonksiyonların Balans Polinomları ile Tanımlanan Bazı Yeni Alt Sınıfları Üzerine

On Some New Subclasses of Bi-Univalent Functions Defined by Balancing Polynomials

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Özet. Bu makalede, Balans polinomları kullanılarak kendisi ve tersi yalınkat olan analitik fonksiyonların iki yeni alt sınıfı tanıtılmıştır. Daha sonra, bu yeni sınıflara ait fonksiyonların ilk iki Taylor-Maclaurin katsayıları için katsayı tahminleri belirlenmiştir. Son olarak, tanımlanan sınıflardaki fonksiyonlar için Fekete-Szegö problemi ele alınıp incelenmiştir.

Anahtar Kelimeler: Balans polinomları, Bi-ünivalent fonksiyon, katsayı tahminleri, Fekete-Szegö fonksiyoneli.

Abstract. In this paper, two new subclasses of holomorphic and bi-univalent functions are introduced by using Balancing polynomials. Then, coefficient estmations are determined for the first two Taylor-Maclaurin coefficients of functions belonging to these new classses. Finally, the Fekete-Szegö problem is handled for the functions in subclasses defined.

Keywords: Balancing polynomials, Bi-univalent function, coefficient estimates, Fekete-Szegö functional.
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1. Introduction

Univalent function theory is an important branch of complex analysis in mathematics. It is known that a complex valued function $f : D \subset \mathbb{C} \to \mathbb{C}$ does not take the same value twice, then this function is called univalent or schlicht on D. After coming out the importance of the Riemann Mapping Theorem (see [9]), investigations on the univalent functions has became very intensive. Earlier of 1900, the German mathematician Bieberbach conjectured a coefficient estimate for the n-th coefficient of analytic and univalent function. The well-known Bieberbach conjecture attracted attention of numerious mathematicians in the past century. As a result of this attraction, number of papers were published on the solution of the mentioned problem during the last century. Specially, from the beginning of the 20. century until now, so many function subclasses of analytic and univalent functions were defined and investigated. Also, it is worthy to emphasize here that mentioned function subclasses were defined by using some special polynomial due to catchy of coefficients of their generating functions. In these papers, generally, upper bounds for the coefficients $|a_2|$ and $|a_3|$, Fekete-Szegö and Hankel determinant problems were handled. In the last quartery of the 20. century, De Branges proved Bieberbach's conjecture for the class of analytic and univalent function on the unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by the conditions $f(0) = f'(0) - 1 = 0$. One of the most important subclass of analytic and univalent function class on the unit disk $\mathbb{E}$ is the bi-univalent function class and is denoted

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by Σ. In fact, a function $f \in \mathcal{A}$ is called bi-univalent function in $\mathbb{E}$ if both f and $f^{-1}$are univalent in $\mathbb{E}$. Here, we would like to remind that the problem finding an upper bound for the coefficient $|a_n|$ of the functions belonging to class Σ is still an open problem. A wide range of coefficient estimates for the functions in the class Σ can be found in the literature. For instance, Brannan and Clunie [2], and Lewin[19] gave very important bounds on $|a_2|$, respectively. Also, Brannan and Taha[3] focused on some subclasses of bi-univalent functions and proved certain coefficient estimates. As mentioned above, one of the most attractive open problems in univalent function theory is to find a coefficient estimate on $|a_n|$ $(n \in \mathbb{N}, n \geq 3.)$ for the functions in the class Σ. Since this attraction, motivated by the works [2, 3, 19] and [26], in [4, 5, 6, 7, 10, 11, 15, 16, 21, 27, 28, 29, 30] and references therein, the authors introduced numerous subclasses of bi-univalent functions and obtained non-sharp estimates on the initial coefficients of functions in these subclasses.

In the present paper our main aim is to find upper bounds for the Taylor-Maclaurin coefficients and Fekete-Szegö functional of function subclasses defined. A rich history for the class Σ can be found in the pionnering work [26] published by Srivastava *et al.*

This paper is organized as follows: Section 1 is divided into three subsections. In the first subsection some basic notions of univalent function theory are recalled, while information about the Balancing polynomials is given in the second subsection. In the last subsection, two new function subclasses of analytic and bi-univalent functions are introduced by making use of Balancing polynomials. Section 2 is devoted to determining upper bounds for the first initial Taylor-Maclaurin coefficients of the defined subclasses. In Section 3 we discuss the Fekete-Szegö problem for these new subclasses.

1.1. Some basic notions in univalent function theory

Let $\mathcal{A}$ denote the class of all analytic functions of the form

$$f(z) = z + a_2 z^2 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. It is clear that the functions in $\mathcal{A}$ satisfy the conditions $f(0) = f'(0) - 1 = 0$, known as normalization conditions. We show by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions univalent in $\mathcal{A}$. The Koebe one quarter theorem (see [9]) guarantees that if $f \in \mathcal{S}$, then there exists the inverse function f^{-1} satisfying

$$f^{-1}(f(z)) = z, (z \in \mathbb{E}) \text{ and } f\left(f^{-1}(w)\right) = w, \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

In the univalent function theory, one of the most important notions is subordination principle. Let the functions $f \in \mathcal{A}$ and $F \in \mathcal{A}$. Then, f is called to be subordinate to F if there exists a Schwarz function w such that

$$w(0) = 0, |w(z)| < 1 \text{ and } f(z) = F(w(z)) \qquad (z \in \mathbb{E}).$$

This subordination is shown by

$$f \prec F \quad \text{or} \quad f(z) \prec F(z) \quad (z \in \mathbb{E}).$$

Especially, if the function F is univalent in $\mathbb{E}$, then this subordination is equivalent to

$$f(0) = F(0), \quad f(\mathbb{E}) \subset F(\mathbb{E}).$$

A comprehensive information about the subordination concept can be found in Monographs written by Miller and Mocanu(see [20]).

1.2. Balancing polynomials and its generating function

In the literature, there are many integer number sequences such as Fibonacci, Lucas, Pell and so on. Recently, Behera and Panda [1] introduced a new integer sequence named Balancing numbers. In the last quarter century, on some properties of this new number sequence have been intensively studied and its some generalizations were defined. Interested readers can find comprehensive information regarding Balancing numbers in [8, 13, 14, 17, 18, 22, 23, 25] and references therein. A natural generalization of the Balancing numbers is the Balancing polynomials and its definition and some interesting properties can be found in [24].

Definition 1. *Let $x \in \mathbb{C}$ and $n \geq 2$. Then, Balancing polynomials are defined the following recurrence relation*

$$B_n(x) = 6xB_{n-1}(x) - B_{n-2}(x), \tag{3}$$

where $B_0(x) = 0$ and $B_1(x) = 1$.

Just as other number polynomials, Balancing polynomials may be obtained via some generating functions and one of them is as follows:

Lemma 1. [12] *The ordinary generating function of Balancing polynomials is given by*

$$B(x,z) = \sum_{n=0}^{\infty} B_n(x)z^n = \frac{z}{1-6xz+z^2}.$$

1.3. New subclasses of Bi-univalent functions

In this subsection, we introduce two new function subclasses of holomorphic and bi-univalent function class Σ by using Balancing polynomials. The first of them is bi-convex function class ${}_{\mathcal{B}}\mathcal{C}_{\Sigma}\left(\mathcal{I}(x,z)\right)$ and read as follows:

Definition 2. *A function $f(z) \in \Sigma$ of the form (1) is said to be in the class ${}_{\mathcal{B}}\mathcal{C}_{\Sigma}\left(\mathcal{I}(x,z)\right)$ if the following conditions hold true:*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{B(x,z)}{z} = \frac{1}{1-6xz+z^2} = \mathcal{I}(x,z) \tag{4}$$

and

$$1 + \frac{wg''(w)}{g'(w)} \prec \frac{B(x,w)}{w} = \frac{1}{1-6xw+w^2} = \mathcal{I}(x,w), \tag{5}$$

where $z, w \in \mathbb{E}$, g is inverse of f and it is of the form (2).

Our second function class is bi-starlike function class ${}_{\mathcal{B}}\mathcal{S}^{\star}_{\Sigma}\left(\mathcal{I}(x,z)\right)$ and it is defined as follows:

Definition 3. *A function $f(z) \in \Sigma$ of the form (1) is said to be in the class ${}_{\mathcal{B}}\mathcal{S}^{\star}_{\Sigma}\left(\mathcal{I}(x,z)\right)$ if the following conditions hold true:*

$$\frac{zf'(z)}{f(z)} \prec \frac{B(x,z)}{z} = \frac{1}{1-6xz+z^2} = \mathcal{I}(x,z) \tag{6}$$

and

$$\frac{wg'(w)}{g(w)} \prec \frac{B(x,w)}{w} = \frac{1}{1-6xw+w^2} = \mathcal{I}(x,w), \tag{7}$$

where $z, w \in \mathbb{E}$, g is inverse of f and it is of the form (2).

2. Coefficient Estimates for the classes ${}_{\mathcal{B}}\mathcal{C}_{\Sigma}(\mathcal{I}(x,z))$ and ${}_{\mathcal{B}}\mathcal{S}^{\star}_{\Sigma}(\mathcal{I}(x,z))$

In this section, we present initial coefficients estimates for the functions belonging to the subclasses ${}_{\mathcal{B}}\mathcal{C}_{\Sigma}(\mathcal{I}(x,z))$ and ${}_{\mathcal{B}}\mathcal{S}^{\star}_{\Sigma}(\mathcal{I}(x,z))$, respectively.

Theorem 1. *Suppose that the function $f(z) \in {}_{\mathcal{B}}\mathcal{C}_{\Sigma}(\mathcal{I}(x,z))$ and $x \in \mathbb{C} \setminus \left\{\mp\frac{1}{3\sqrt{2}}\right\}$. Then,*

$$|a_2| \leq \frac{3\sqrt{6}\,|x|\,\sqrt{|x|}}{\sqrt{|1-18x^2|}} \tag{8}$$

and

$$|a_3| \leq |x|\left(1 + 9\,|x|\right). \tag{9}$$

Proof of Theorem 1. Let the function $f(z) \in {}_{\mathcal{B}}\mathcal{C}_{\Sigma}(\mathcal{I}(x,z))$ and $g = f^{-1}$ given by (2). In view of Definition 2, from the relations (4) and (5) we can write that

$$1 + \frac{zf''(z)}{f'(z)} \prec \mathcal{I}(x,z) \tag{10}$$

and

$$1+\frac{wg''(w)}{g'(w)} \prec \mathcal{I}(x,w). \tag{11}$$

The subordinations (10) and (11) imply that

$$1+\frac{zf''(z)}{f'(z)} = \mathcal{I}(x,\kappa(z))$$

and

$$1+\frac{wg''(w)}{g'(w)} = \mathcal{I}(x,\phi(w)).$$

Here, $\kappa(z)=k_1z+k_2z^2+\cdots$ and $\phi(w)=\phi_1w+\phi_2w^2+\cdots$ are Schwarz functions such that $\kappa(0)=\phi(0)=0$, $|\kappa(z)|<1$ and $|\phi(w)|<1$ for all $z,w\in\mathbb{E}$. On the other hand, it is fairly well-known that the conditions $|\kappa(z)|<1$ and $|\phi(w)|<1$ imply

$$|\kappa_j|<1, \tag{12}$$

$$|\phi_j|<1 \tag{13}$$

for all $j\in\mathbb{N}$. Basic computations yield that

$$\begin{aligned}\mathcal{I}(x,\kappa(z)) = {} & B_1(x)+[B_2(x)k_1]\,z+\left[B_2(x)k_2+B_3(x)k_1^2\right]z^2 \\ & +\left[B_2(x)k_3+2B_3(x)k_1k_2+B_4(x)k_1^3\right]z^3+\cdots\end{aligned} \tag{14}$$

and

$$\begin{aligned}\mathcal{I}(x,\phi(w)) = {} & B_1(x)+[B_2(x)\phi_1]\,w+\left[B_2(x)\phi_2+B_3(x)\phi_1^2\right]w^2 \\ & +\left[B_2(x)\phi_3+2B_3(x)\phi_1\phi_2+B_4(x)\phi_1^3\right]w^3+\cdots.\end{aligned} \tag{15}$$

On the other hand, from the straightforward calculations we can write that

$$1+\frac{zf''(z)}{f'(z)} = 1+2a_2z+\left(6a_3-4a_2^2\right)z^2+\left(12a_4-18a_2a_3+8a_2^3\right)z^3+\cdots \tag{16}$$

and

$$1+\frac{wg''(w)}{g'(w)} = 1-2a_2w+\left(8a_2^2-6a_3\right)w^2+\left(42a_2a_3-32a_2^3-12a_4\right)w^3+\cdots. \tag{17}$$

By comparing the coefficients in (16) and (17), with the equations (14) and (15), respectively, we get,

$$2a_2 = B_2(x)\kappa_1, \tag{18}$$

$$-2a_2 = B_2(x)\phi_1, \tag{19}$$

$$6a_3-4a_2^2 = B_2(x)\kappa_2+B_3(x)\kappa_1^2, \tag{20}$$

$$8a_2^2-6a_3 = B_2(x)\phi_2+B_3(x)\phi_1^2, \tag{21}$$

Now, from equations (18) and (19) we have

$$\kappa_1 = -\phi_1, \tag{22}$$

$$\frac{8a_2^2}{(B_2(x))^2} = \kappa_1^2+\phi_1^2. \tag{23}$$

Also, from the summation of the equations (20) and (21), we easily obtain that

$$4a_2^2 = B_2(x)(\kappa_2+\phi_2)+B_3(x)(\kappa_1^2+\phi_1^2). \tag{24}$$

By substituting equation (23) in equation (24) we deduce

$$a_2^2 = \frac{(B_2(x))^3(\kappa_2+\phi_2)}{4(B_2(x))^2-8B_3(x)}. \tag{25}$$

Using recurrence relation given by (3) in Definition 1 we get that

$$B_2(x) = 6x, \tag{26}$$

$$B_3(x) = 36x^2-1. \tag{27}$$

If we replace equations (26) and (27) in (25) we have

$$a_2^2 = \frac{27x^3(\kappa_2+\phi_2)}{1-18x^2}.$$

Now, using triangle inequality with the inequalities (12) and (13), we get

$$\left|a_2^2\right| \leq \frac{54\,|x|^3}{|1-18x^2|}.$$

Taking square root both sides of the last inequality, we deduce (8).

In addition, if we subtract the equation (21) from the equation (20) and consider equation (22), then we obtain

$$a_3 = \frac{B_2(x)(\kappa_2-\phi_2)}{12} + a_2^2. \tag{28}$$

Considering the equation (24) in (28) and a straightforward calculation yield that

$$a_3 = \frac{B_2(x)(\kappa_2-\phi_2)}{12} + \frac{(B_2(x))^2(\kappa_1^2+\phi_1^2)}{8}. \tag{29}$$

By making use of the equation (26), and triangle inequality with the inequalities (12) and (13) in (29) we deduce the inequality (9) which is desired. □

Theorem 2. *Suppose that the function $f(z) \in {}_{\mathcal{B}}\mathcal{S}_\Sigma^\star(\mathcal{I}(x,z))$. Then,*

$$|a_2| \leq 6\sqrt{6}\,|x|\,\sqrt{|x|} \tag{30}$$

and

$$|a_3| \leq 3\,|x|\,(1+12\,|x|)\,. \tag{31}$$

Proof of Theorem 2. Let the function $f(z) \in {}_{\mathcal{B}}\mathcal{S}_\Sigma^\star(\mathcal{I}(x,z))$ and $g = f^{-1}$ given by (2). In view of Definition 3, from the equations (6) and (7) we can write that

$$\frac{zf'(z)}{f(z)} \prec \mathcal{I}(x,z) \tag{32}$$

and

$$\frac{wg'(w)}{g(w)} \prec \mathcal{I}(x,w). \tag{33}$$

By virtue of the relations (32) and (33), there are two Schwarz functions $p(z) = p_1z + p_2z^2 + \cdots$ and $d(w) = d_1w + d_2w^2 + \cdots$ such that

$$\frac{zf'(z)}{f(z)} = \mathcal{I}(x,p(z))$$

and

$$\frac{wg'(w)}{g(w)} = \mathcal{I}(x,d(w)).$$

Since the functions p and d are Schwarz functions, it is known that $p(0) = d(0) = 0$, $|p(z)| < 1$ and $|d(w)| < 1$ for all $z, w \in \mathbb{E}$. Morever,

$$|p_j| < 1, \tag{34}$$

$$|d_j| < 1 \tag{35}$$

for all $j \in \mathbb{N}$. A straightforward calculation yields that

$$\begin{aligned}\mathcal{I}(x,p(z)) = {} & B_1(x) + [B_2(x)p_1]\,z + \left[B_2(x)p_2 + B_3(x)p_1^2\right] z^2 \\ & + \left[B_2(x)p_3 + 2B_3(x)p_1p_2 + B_4(x)p_1^3\right] z^3 + \cdots\end{aligned} \tag{36}$$

and

$$\begin{aligned}\mathcal{I}(x,d(w)) = {} & B_1(x) + [B_2(x)d_1]\,w + \left[B_2(x)d_2 + B_3(x)d_1^2\right] w^2 \\ & + \left[B_2(x)d_3 + 2B_3(x)d_1d_2 + B_4(x)d_1^3\right] w^3 + \cdots.\end{aligned} \tag{37}$$

On the other hand, it can be easily checked that

$$\frac{zf'(z)}{f(z)} = 1 + a_2z + \left(2a_3 - a_2^2\right) z^2 + \left(3a_4 - 3a_2a_3 + a_2^3\right) z^3 + \cdots \tag{38}$$

and

$$\frac{wg'(w)}{g(w)} = 1 - a_2w + \left(3a_2^2 - 2a_3\right) w^2 + \left(12a_2a_3 - 10a_2^3 - 3a_4\right) w^3 + \cdots. \tag{39}$$

By equating the coefficients in (38) and (39), with the equations (36) and (37), respectively, we get,

$$\begin{aligned} a_2 &= B_2(x)p_1, & (40)\\ -a_2 &= B_2(x)d_1, & (41)\\ 2a_3 - a_2^2 &= B_2(x)p_2 + B_3(x)p_1^2, & (42)\\ 3a_2^2 - 2a_3 &= B_2(x)d_2 + B_3(x)d_1^2, & (43) \end{aligned}$$

From equations (40) and (41) we have

$$\begin{aligned} p_1 &= -d_1, & (44)\\ \frac{2a_2^2}{(B_2(x))^2} &= p_1^2 + d_1^2. & (45) \end{aligned}$$

Also, by summing the equations (42) and (43), we easily obtain that

$$2a_2^2 = B_2(x)(p_2 + d_2) + B_3(x)(p_1^2 + d_1^2). \quad (46)$$

Next, by substituting equation (45) in equation (46) we deduce

$$a_2^2 = \frac{(B_2(x))^3 (p_2 + d_2)}{2(B_2(x))^2 - 2B_3(x)}. \quad (47)$$

Plugging equations (26) and (27) into (47), we have

$$a_2^2 = 108x^3(p_2 + d_2). \quad (48)$$

Now, using triangle inequality with the inequalities (34) and (35), we get

$$\left|a_2^2\right| \le 108\,|x|^3\,|(p_2 + d_2)|\,.$$

Taking square root both sides of the last inequality, we deduce inequality (30). In addition, if we subtract the equation (43) from the equation (42) and consider equation (44), then we obtain

$$a_3 = \frac{B_2(x)(p_2 - d_2)}{4} + a_2^2. \quad (49)$$

Considering the equation (45) in (49) and a straightforward calculation yield that

$$a_3 = \frac{B_2(x)(p_2 - d_2)}{4} + \frac{(B_2(x))^2(p_1^2 + d_1^2)}{2}. \quad (50)$$

By making use of the equation (26), and triangle inequality with the inequalities (34) and (35) in (50) we obtain the inequality (31). So, the proof is completed. □

3. Fekete-Szegö inequalities for the classes ${}_{\mathcal{B}}\mathcal{C}_\Sigma(\mathcal{I}(x,z))$ and ${}_{\mathcal{B}}\mathcal{S}^\star_\Sigma(\mathcal{I}(x,z))$

Our result regarding Fekete-Szegö inequality for the function class ${}_{\mathcal{B}}\mathcal{C}_\Sigma(\mathcal{I}(x,z))$ is the following.

Theorem 3. *Let the function $f(z) \in {}_{\mathcal{B}}\mathcal{C}_\Sigma(\mathcal{I}(x,z))$ and $x \in \mathbb{C} \setminus \left\{0, \mp\frac{1}{3\sqrt{2}}\right\}$. Then, for some $\eta \in \mathbb{R}$*

$$\left|a_3 - \eta a_2^2\right| \le \begin{cases} |x|\,, & |1-\eta| \le \frac{\left|1-18x^2\right|}{|54x^2|} \\ \frac{54|x|^3|1-\eta|}{\left|1-18x^2\right|}, & |1-\eta| \ge \frac{\left|1-18x^2\right|}{|54x^2|}. \end{cases}$$

Proof of Theorem 3. Let the function $f(z) \in {}_{\mathcal{B}}\mathcal{C}_\Sigma(\mathcal{I}(x,z))$ and $\eta \in \mathbb{R}$. Then, from the equations (25) and (28), we can write that

$$\begin{aligned} a_3 - \eta a_2^2 &= a_2^2 + \frac{B_2(x)(\kappa_2 - \phi_2)}{12} - \eta a_2^2 \\ &= (1-\eta)a_2^2 + \frac{B_2(x)(\kappa_2 - \phi_2)}{12} \\ &= (1-\eta)\frac{(B_2(x))^3(\kappa_2 + \phi_2)}{4(B_2(x))^2 - 8B_3(x)} + \frac{B_2(x)(\kappa_2 - \phi_2)}{12} \\ &= B_2(x)\left\{\left(h_1(\eta) + \frac{1}{12}\right)\kappa_2 + \left(h_1(\eta) - \frac{1}{12}\right)\phi_2\right\}, \quad (51) \end{aligned}$$

where $h_1(\eta) = \frac{(1-\eta)(B_2(x))^2}{4(B_2(x))^2-8B_3(x)}$. Now, taking modulus and using triangle inequality with the (12), (13), (26) and (27) in (51), we complete the proof. □

Our result regarding Fekete-Szegö inequality for the function class ${}_{\mathcal{B}}\mathcal{S}^{\star}_{\Sigma}(\mathcal{I}(x,z))$ is the following.

Theorem 4. *Let the function $f(z) \in {}_{\mathcal{B}}\mathcal{S}^{\star}_{\Sigma}(\mathcal{I}(x,z))$ and $x \in \mathbb{C} \setminus \{0\}$. Then,*

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} 3\,|x|\,, & |1-\eta| \leq \frac{1}{|72x^2|} \\ 216\,|x|^3\,|1-\eta|\,, & |1-\eta| \geq \frac{1}{|72x^2|} \end{cases}$$

Proof of Theorem 4. Let the function $f(z) \in {}_{\mathcal{B}}\mathcal{S}^{\star}_{\Sigma}(\mathcal{I}(x,z))$. Then, from the equations (48) and (49), it follows that

$$\begin{aligned} a_3 - \eta a_2^2 &= a_2^2 + \frac{B_2(x)(p_2-d_2)}{4} - \eta a_2^2 \\ &= (1-\eta)a_2^2 + \frac{B_2(x)(p_2-d_2)}{4} \\ &= (1-\eta)\frac{(B_2(x))^3\,(p_2+d_2)}{2\,(B_2(x))^2 - 2B_3(x)} + \frac{B_2(x)(p_2-d_2)}{4} \\ &= B_2(x)\left\{\left(h_2(\eta)+\frac{1}{4}\right)p_2 + \left(h_2(\eta)-\frac{1}{4}\right)d_2\right\}, \end{aligned} \tag{52}$$

where $h_2(\eta) = \frac{(1-\eta)(B_2(x))^2}{2(B_2(x))^2-2B_3(x)}$. Now, taking modulus and using triangle inequality with the (26), (27), (34) and (35) in (52), the proof is thus completed. □

4. Conclusion

In this paper, with the help of Balancing polynomials, we have introduced two new subclasses of holomorphic and bi-univalent functions defined in the open unit disk. Then, we have investigated initial Taylor-Maclaurin coefficients estimates and Fekete-Szegö type inequality for these subclasses.

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