Black Sea Journal of Engineering and Science
doi: 10.34248/bsengineering.1253548

Open Access Journal
e-ISSN: 2619 – 8991

Research Article
Volume 6 - Issue 2: 98-101 / April 2023

VAJDA'S IDENTITIES FOR DUAL FIBONACCI AND DUAL LUCAS SEDENIONS

Zafer ÜNAL[1]*

[1]*Kastamonu University, Faculty of Sciences, Department of Mathematics, 37100, Kastamonu, Türkiye*

Abstract: Fibonacci and Lucas numbers have been the most popular integer sequences since they were defined. These integer sequences have many uses, from nature to computer science, from art to financial analysis. Many researchers have worked on this subject. Sedenions form a 16-dimensional algebra on the field of real numbers. Various systems can be constructed by using the terms of special integer sequences instead of terms in sedenions. In this study, we define dual Fibonacci (DFS) and dual Lucas sedenions (DLS) with the help of Fibonacci and Lucas termed sedenions. Then we calculate some special identities for DFS and DLS such as Vajda's, Catalan's, d'Ocagne's, Cassini's.

Keywords: Fibonacci and Lucas numbers, Dual numbers, Sedenions, Vajda's identity, Binet-like formula

***Corresponding author:** Kastamonu University, Faculty of Sciences, Department of Mathematics, 37100, Kastamonu, Türkiye
E mail: zunal@kastamonu.edu.tr (Z. ÜNAL)
Zafer ÜNAL https://orcid.org/0000-0003-2445-1028

Received: February 20, 2023
Accepted: March 11, 2023
Published: April 01, 2023
Cite as: Ünal Z. 2023. Vajda's identities for dual fibonacci and dual lucas sedenions. BSJ Eng Sci, 6(2): 98-101.

1. Introduction

Fibonacci numbers are the most famous integer sequence in mathematics. For $n > 1$, Fibonacci numbers have $F_n = F_{n-1} + F_{n-2}$ recurrence relation with the initial conditions, $F_0 = 0, F_1 = 1$. Lucas numbers have same recurrence relation $L_n = L_{n-1} + L_{n-2}$, but initial conditions are $L_0 = 2, L_1 = 1$. Binet-like formulas of Fibonacci and Lucas numbers are $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$, respectively. $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are roots of the second order equation $x^2 - x - 1 = 0$ (Koshy, 2001). For more details about Fibonacci and Lucas numbers, one can see following studies (Horadam, 1961; Wilcox, 1986; Muskat, 1993; Yayenie, 2011; Bilgici, 2014).

Clifford (1871), extended real numbers to dual numbers with a structure similar to that of complex numbers. For any real numbers, a *and* a^*, a dual number d can be expressed as $d = a + \varepsilon a^*$, where, ε is the dual unit that satisfies. $\varepsilon \neq 0, \varepsilon^2 = 0$ The set of all dual numbers

$$\mathrm{D} = \left\{ d = a + \varepsilon a^* : a, a^* \in \mathrm{R}, \varepsilon \neq 0, \varepsilon^2 = 0 \right\}$$

is a commutative ring with unity with respect to the following binary operations: For any dual numbers

$$d_1 = a_1 + \varepsilon a_1^*, d_2 = a_2 + \varepsilon a_2^* \in \mathrm{D}$$
$$d_1 + d_2 = a_1 + a_2 + \varepsilon(a_1^* + a_2^*)$$
$$d_1 d_2 = a_1 a_2 + \varepsilon(a_1 a_2^* + a_1^* a_2).$$

But since has zero divisors, dual numbers ring is not a field.

Ünal et al. (2017), gave some properties of dual Fibonacci and Lucas octonions. Tokeşer et al. (2022), studied on split dual Fibonacci and Lucas octonions, recently.

Sedenions, defined by Imaeda and Imaeda (2000) and denoted by S, is a 16-dimensional algebra over real numbers. Since sedenions have zero divisors, it is not a division algebra. Also, sedenions form a non-commutative and non-associative algebra. Any sedenion s is $s = \sum_{i=0}^{15} a_i e_i$, where, coefficients a_i are reals and $\{e_0, \ldots, e_{15}\}$ is the basis elements of S. The multiplication table of basis elements of sedenions is given by Cawagas (2004) as follows (Table 1):

Bilgici et al. (2017), defined Fibonacci and Lucas sedenions as

$$FS_n = \sum_{i=0}^{15} F_{n+i} e_i \text{ and } LS_n = \sum_{i=0}^{15} L_{n+i}$$

Binet's formulas for Fibonacci and Lucas sedenions are given in Equation 1;

$$FS_n = \frac{\alpha^n \alpha^* - \beta^n \beta^*}{\alpha - \beta} \text{ and } LS_n = \alpha^n \alpha^* + \beta^n \beta^* \qquad (1)$$

where, $\alpha^* = \sum_{i=0}^{15} \alpha^i e_i$ and $\beta^* = \sum_{i=0}^{15} \alpha \beta^i e_i$. In the same study, they gave some special identities such as Catalan, Cassini, etc.

Table 1. The multiplication table for basis of sedenions by setting $i = e_i$, $(0 \leq i \leq 15)$.

.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	-0	3	-2	5	-4	-7	6	9	-8	-11	10	-13	12	15	-14
2	2	-3	-0	1	6	7	-4	-5	10	11	-8	-9	-14	-15	12	13
3	3	2	-1	-0	7	-6	5	-4	11	-10	9	-8	-15	14	-13	12
4	4	-5	-6	-7	-0	1	2	3	12	13	14	15	-8	-9	-10	-11
5	5	4	-7	6	-1	-0	-3	2	13	-12	15	-14	9	-8	11	-10
6	6	7	4	-5	-2	3	-0	-1	14	-15	-12	13	10	-11	-8	9
7	7	-6	5	4	-3	-2	1	-0	15	14	-13	-12	11	10	-9	-8
8	8	-9	-10	-11	-12	-13	-14	-15	-0	1	2	3	4	5	6	7
9	9	8	-11	10	-13	12	15	-14	-1	-0	-3	2	-5	4	7	-6
10	10	11	8	-9	-14	-15	12	13	-2	3	-0	-1	-6	-7	4	5
11	11	-10	9	8	-15	14	-13	12	-3	-2	1	-0	-7	6	-5	4
12	12	13	14	15	8	-9	-10	-11	-4	5	6	7	-0	-1	-2	-3
13	13	-12	15	-14	9	8	11	-10	-5	-4	7	-6	1	-0	3	-2
14	14	-15	-12	13	10	-11	8	9	-6	-7	-4	5	2	-3	-0	1
15	15	14	-13	-12	11	10	-9	8	-7	6	-5	-4	3	2	-1	-0

2. Material and Methods

In this part, we will define dual Fibonacci and Lucas sedenions and give the Binet-like formulas.

Definition 1 For $n > 1$, n-th dual Fibonacci and dual Lucas sedenions are

$$DFS_n = FS_n + \varepsilon FS_{n+1},$$
$$DLS_n = LS_n + \varepsilon LS_{n+1}.$$

Instead of these recursive definitions we can use Binet-like formulas.

Theorem 2 (Binet-like Formula) For $n > 1$, Binet formulas of n-th dual Fibonacci and dual Lucas sedenions are

$$DFS_n = \frac{\alpha^n \alpha' - \beta^n \beta'}{\alpha - \beta} \text{ and } DLS_n = \alpha^n \alpha' + \beta^n \beta'$$

where,

$$\alpha' = (1+\varepsilon\alpha)\sum_{i=0}^{15} \alpha^i e_i = (1+\varepsilon\alpha)\alpha^*$$

and

$$\beta' = (1+\varepsilon\beta)\sum_{i=0}^{15} \beta^i e_i = (1+\varepsilon\beta)\beta^*.$$

Proof Using the Equation (1), the proof is completed. ■

3. Results and Discussion

In this section, we will give the Vajda's identities for dual Fibonacci and dual Lucas sedenions. First of all, let us consider the following lemma that has useful results for calculations.

Lemma 3 For α^* and β^* we have

(i) $\alpha^* \beta^* = LS_0 - \sqrt{5}(FS_0 - K)$,

(ii) $\beta^* \alpha^* = LS_0 + \sqrt{5}(FS_0 - K)$,

(iii) $(\alpha^*)^2 = -\xi_1 + LS_0 + \sqrt{5}(FS_0 - \xi_2)$,

(iv) $(\beta^*)^2 = -\xi_1 + LS_0 - \sqrt{5}(FS_0 - \xi_2)$,

where, FS_0 and LS_0 are 0-th Fibonacci and Lucas sedenions, respectively, and, $\xi_1 = 1505175$, $\xi_2 = 673134$ and $K = 94e_9 + 94e_{10} + 188e_{11} + 282e_{12} - 188e_{13} + 94e_{14} + 893e_{15}$ (Bilgici et al., 2017).

Theorem 4 (Vajda's Identity) For any integers n, r, and, s, followings are hold in Equations 2 and 3:

$$DFS_{n+r}DFS_{n+s} - DFS_n DFS_{n+r+s} = (-1)^n F_r \left[F_s LS_0 + (FS_0 - K)L_s \right](1+\varepsilon) \quad (2)$$

$$DLS_{n+r}DLS_{n+s} - DLS_n DLS_{n+r+s} = 5(-1)^{n+1} F_r \left[F_s LS_0 + (FS_0 - K)L_s \right](1+\varepsilon) \quad (3)$$

where, F_r, F_s, L_s, FS_0 and LS_0 are r-th, s-th Fibonacci, s-th Lucas numbers, 0-th Fibonacci and Lucas sedenions, respectively, and

$$K = 94e_9 + 94e_{10} + 188e_{11} + 282e_{12} - 188e_{13} + 94e_{14} + 893e_{15}.$$

Proof We will prove the first equation. From the definition of dual Fibonacci sedenions, we get

$$\begin{aligned}DFS_{n+r}DFS_{n+s}-DFS_{n}DFS_{n+r+s}&=(FS_{n+r}+\varepsilon FS_{n+r+1})(FS_{n+s}+\varepsilon FS_{n+s+1})\\&-(FS_{n}+\varepsilon FS_{n+1})(FS_{n+r+s}+\varepsilon FS_{n+r+s+1})\\&=FS_{n+r}FS_{n+s}-FS_{n}FS_{n+r+s}+\varepsilon(FS_{n+r}FS_{n+s+1}\\&-FS_{n}FS_{n+r+s+1}+FS_{n+r+1}FS_{n+s}-FS_{n+1}FS_{n+r+s})\end{aligned}\tag{4}$$

Let us consider the real part of the Equation (4). From Equation (1)

$$\begin{aligned}&FS_{n+r}FS_{n+s}-FS_{n}FS_{n+r+s}\\&=\frac{\alpha^{n+r}\alpha^{*}-\beta^{n+r}\beta^{*}}{\alpha-\beta}\frac{\alpha^{n+s}\alpha^{*}-\beta^{n+s}\beta^{*}}{\alpha-\beta}-\frac{\alpha^{n}\alpha^{*}-\beta^{n}\beta^{*}}{\alpha-\beta}\frac{\alpha^{n+r+s}\alpha^{*}-\beta^{n+r+s}\beta^{*}}{\alpha-\beta}\\&=\frac{1}{(\alpha-\beta)^{2}}[-\alpha^{n+r}\beta^{n+s}\alpha^{*}\beta^{*}-\alpha^{n+s}\beta^{n+r}\beta^{*}\alpha^{*}+\alpha^{n}\beta^{n+r+s}\alpha^{*}\beta^{*}+\alpha^{n+r+s}\beta^{n}\beta^{*}\alpha^{*}]\\&=\frac{(-1)^{n}}{(\alpha-\beta)^{2}}[(-\alpha^{r}+\beta^{r})\beta^{s}\alpha^{*}\beta^{*}+(\alpha^{r}-\beta^{r})\alpha^{s}\beta^{*}\alpha^{*}]\\&=\frac{(-1)^{n}F_{r}}{\alpha-\beta}[\alpha^{s}(LS_{0}+\sqrt{5}(FS_{0}-K))-\beta^{s}(LS_{0}-\sqrt{5}(FS_{0}-K))]\\&=(-1)^{n}F_{r}\left\{F_{s}LS_{0}+(FS_{0}-K)L_{s}\right\}\end{aligned}\tag{5}$$

Now we will calculate the dual part in two steps: In Equation (5), replacing s to s+1, we get

$$FS_{n+r}FS_{n+s+1}-FS_{n}FS_{n+r+s+1}=(-1)^{n}F_{r}\left\{F_{s+1}LS_{0}+(FS_{0}-K)L_{s+1}\right\}\tag{6}$$

and replacing n to n+1 and s to s−1, we find

$$FS_{n+r+1}FS_{n+s}-FS_{n+1}FS_{n+r+s}=(-1)^{n+1}F_{r}\left\{F_{s-1}LS_{0}+(FS_{0}-K)L_{s-1}\right\}.\tag{7}$$

From Equations (6) and (7) and using the relations $F_s = F_{s+1} - F_{s-1}$ and $L_s = L_{s+1} - L_{s-}$ we have

$$FS_{n+r}FS_{n+s+1}-FS_{n}FS_{n+r+s+1}+FS_{n+r+1}FS_{n+s}-FS_{n+1}FS_{n+r+s}=(-1)^{n}F_{r}\left\{F_{s}LS_{0}+(FS_{0}-K)L_{s}\right\}\tag{8}$$

Finally, from Equations (4) and (8) we get

$$\begin{aligned}&DFS_{n+r}DFS_{n+s}-DFS_{n}DFS_{n+r+s}\\&\qquad=(-1)^{n}F_{r}[F_{s}LS_{0}+(FS_{0}-K)L_{s}](1+\varepsilon)\end{aligned}$$

In a similar way, the second part of the proof can be done easily. ■

Now, we will give following corollaries without proof as a consequence of the Theorem 4.

Corollary 5 (Catalan-like Identity) For any integers n and r, Catalan's identities of dual Fibonacci and dual Lucas sedenions are as follows:

$$\begin{aligned}DFS_{n+r}DFS_{n-r}-DFS_{n}^{\,2}&=(-1)^{n+r+1}\left[F_{r}^{\,2}LS_{0}-(FS_{0}-K)F_{2r}\right](1+\varepsilon)\\DLS_{n+r}DLS_{n-r}-DLS_{n}^{\,2}&=5(-1)^{n+r}\left[F_{r}^{\,2}LS_{0}-(FS_{0}-K)F_{2r}\right](1+\varepsilon)\end{aligned}$$

Where F_r, F_{2r}, FS_0 and LS_0 are r-th, (2r)-th Fibonacci numbers, 0-th Fibonacci and Lucas sedenions, respectively, and $K=94e_{9}+94e_{10}+188e_{11}+282e_{12}-188e_{13}+94e_{14}+893e_{15}$.

Proof In Equations (2) and (3), if we write $-r$ instead of s and use the identities $F_{-r}=(-1)^{r+1}F_r$, $L_{-r}=(-1)^{r+1}L_r$ and $F_{2r}=F_rL_r$, the proof is completed.■

Corollary 6 (Cassini-like Identity) Let n be any integer. Then Cassini's identities of dual Fibonacci and dual Lucas sedenions are

$$\begin{aligned}DFS_{n+1}DFS_{n-1}-DFS_{n}^{\,2}&=(-1)^{n}\left[2FS_{-1}+K\right](1+\varepsilon)\\DLS_{n+1}DLS_{n-1}-DLS_{n}^{\,2}&=-5(-1)^{n}\left[2FS_{-1}+K\right](1+\varepsilon)\end{aligned}$$

where, FS_{-1} (−1)-th Fibonacci sedenion and $K=94e_{9}+94e_{10}+188e_{11}+282e_{12}-188e_{13}+94e_{14}+893e_{15}$.

Proof In Equations (2) and (3), if we write $s = -r = -1$ and use the relation $LS_0 - FS_0 = 2FS_{-1}$; it is completed.■

Corollary 7 (d'Ocagne-like Identity) Let m and n are any integer numbers. Then d'Ocagne identities of dual Fibonacci and dual Lucas sedenions are as follows:

$$\begin{aligned}DFS_{n+1}DFS_{m}-DFS_{n}DFS_{m+1}&=(-1)^{n}\left[F_{m-n}LS_{0}+(FS_{0}-K)L_{m-n}\right](1+\varepsilon)\\DLS_{n+1}DLS_{m}-DLS_{n}DLS_{m+1}&=-5(-1)^{n}\left[F_{m-n}LS_{0}+(FS_{0}-K)L_{m-n}\right](1+\varepsilon)\end{aligned}$$

where, F_{m-n}, L_{m-n}, LS_0 and FS_0 are (m-n)-th Fibonacci and Lucas numbers, 0-th Fibonacci and Lucas sedenions, respectively, and

$$K = 94e_9 + 94e_{10} + 188e_{11} + 282e_{12} - 188e_{13} + 94e_{14} + 893e_{15}.$$

Proof If we write *s* = m – n and *r* = 1 in Equations (2) and (3), we get the result. ■

Some identities of DFS and DLS are given without proof in the next theorem.

Theorem 8 Following identities for dual Fibonacci and Lucas sedenions are valid.

$$DLS_{n+r}DFS_{n+s} - DLS_{n+s}DFS_{n+r} = 2(-1)^{n+r} LS_0 FS_{s-r}(1+\varepsilon),$$

$$DFS_{m+n} + (-1)^n DFS_{m-n} = DFS_m L_n,$$

$$DFS_m DLS_n - DLS_n DFS_m = 2(-1)^{m+1} L_0 F_{n-m}(1+\varepsilon).$$

4. Conclusion

In this study, dual Fibonacci and dual Lucas sedenions are defined. Then we calculated Vajda's identities for DFS and DLS. By using this identity some classical identities are given for example, Catalan, Cassini and d'Ocagne etc.

Author Contributions

The percentage of the author contributions is present below. The author reviewed and approved final version of the manuscript.

	Z.Ü.
C	100
D	100
S	100
DCP	100
DAI	100
L	100
W	100
CR	100
SR	100
PM	100
FA	100

C=Concept, D= design, S= supervision, DCP= data collection and/or processing, DAI= data analysis and/or interpretation, L= literature search, W= writing, CR= critical review, SR= submission and revision, PM= project management, FA= funding acquisition.

Conflict of Interest

The author declared that there is no conflict of interest.

References

Bilgici G. 2014. New generalizations of Fibonacci and Lucas sequences. Appl Math Sci, 8(29): 1429-1437.

Bilgici G, Tokeşer Ü, Ünal Z. 2017. Fibonacci and Lucas Sedenions. J Integer Seq, 20: 17.1.8.

Cawagas RE. 2004. On the structure and zero divisors of the Cayley-Dickson sedenion algebra. Discuss Math Gen Algebra Appl, 24: 251-265.

Clifford WK. 1871. Preliminary sketch of bi-quaternions. Proc Lond Math Soc, 4(1): 381-395.

Horadam AF. 1961. A generalized Fibonacci sequence. The American Math Monthly, 68(5): 455-459.

Imaeda K, Imaeda M, 2000. Sedenions: algebra and analysis. Appl Math Comput, 115: 77-88.

Koshy T. 2001. Fibonacci and lucas numbers with applications. Wiley-Interscience Publication, Quebec, Canada, pp: 77-78.

Muskat JB. 1993. Generalized Fibonacci and Lucas sequences and rootfinding methods. Math Comput, 61(203): 365-372.

Tokeşer Ü, Mert T, Dündar Y. 2022. Some properties and Vajda theorems of split dual Fibonacci and split dual Lucas octonions. AIMS Math, 7(5): 8645-8653.

Ünal Z, Tokeşer Ü, Bilgici G. 2017. Some properties of dual Fibonacci and dual Lucas octonions. Adv Appl Clifford Algebras, 27: 1907-1916.

Wilcox HJ. 1986. Fibonacci sequences of period n in groups. Fibonacci Quart, 24(4): 356-361.

Yayenie O. 2011. A note on generalized Fibonacci sequences. Appl Math Comput, 217(12): 5603-5611.