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Research Article

Regarding equitable colorability defect of hypergraphs

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Abstract

After Lovász's break-through in determining the chromatic number of Kneser graphs (1978), and after extending this result to the chromatic number of r-uniform Kneser hypergraphs by Alon, Frankl, and Lovász (1986), some important parameters such as colorability defect and equitable colorability defect were introduced in order to provide sharp lower bounds for the chromatic number of general r-uniform Kneser hypergraphs. As a generalization of many earlier results in this area, Azarpendar and Jafari (2023) introduced the s-th equitable r-colorability defect $\mathrm{ecd}^r(\mathcal{F}, s)$; a parameter which provides a lower bound for the chromatic number of generalized Kneser hypergraphs $\mathrm{KG}^r(\mathcal{F}, s)$. They proved the following nice inequality

$$\chi\left(\mathrm{KG}^r(\mathcal{F}, s)\right) \geq \left\lceil \frac{\mathrm{ecd}^r\left(\mathcal{F}, \lfloor \frac{s}{2} \rfloor\right)}{r-1} \right\rceil,$$

and noted that it is plausible that the above inequality remains true if one replaces $\lfloor \frac{s}{2} \rfloor$ with s.

In this paper, considering the relation $\mathrm{ecd}^r\left(\mathcal{F}, x\right) \geq \mathrm{cd}^r\left(\mathcal{F}, x\right)$ which always holds, we show that even in the weaker inequality

$$\chi\left(\mathrm{KG}^r(\mathcal{F}, s)\right) \geq \left\lceil \frac{\mathrm{cd}^r\left(\mathcal{F}, \lfloor \frac{s}{2} \rfloor\right)}{r-1} \right\rceil,$$

no number x greater than $\lfloor \frac{s}{2} \rfloor$ could be replaced by $\lfloor \frac{s}{2} \rfloor$.

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1. Introduction

A *hypergraph* $\mathcal{F}$ consists of a finite set $V(\mathcal{F})$, together with a subset of $2^{V(\mathcal{F})} \setminus \{\varnothing\}$ which is denoted by $E(\mathcal{F})$. Any member of $V(\mathcal{F})$ is called a *vertex* of $\mathcal{F}$, and members of $E(\mathcal{F})$ are called *hyperedges* of $\mathcal{F}$. The hypergraph $\mathcal{F}$ is called *r-uniform* whenever $|e| = r$ for each hyperedge e of $\mathcal{F}$.

Let $\mathcal{F}$ be an arbitrary (uniform or nonuniform) hypergraph and $r \in \{2, 3, 4, \ldots\}$. If s is a nonnegative integer such that $s < |e|$ for each hyperedge e of $\mathcal{F}$, then the *generalized Kneser hypergraph* $\mathrm{KG}^r(\mathcal{F}, s)$ is defined as an r-uniform hypergraph with vertex set $V\left(\mathrm{KG}^r(\mathcal{F}, s)\right) := E(\mathcal{F})$ in such a way that r hyperedges $e_1, e_2, \ldots, e_r$ of $\mathcal{F}$ form a hyperedge of $\mathrm{KG}^r(\mathcal{F}, s)$ whenever $|e_i \cap e_j| \leq s$ for all distinct indices i and j in $\{1, 2, \ldots, r\}$. As a definition, the *chromatic number* of $\mathrm{KG}^r(\mathcal{F}, s)$, denoted by $\chi\left(\mathrm{KG}^r(\mathcal{F}, s)\right)$, is the minimum cardinality of a set C for which a function $f : V\left(\mathrm{KG}^r(\mathcal{F}, s)\right) \longrightarrow C$ exists in such a way that $|\{f(e_1), f(e_2), \ldots, f(e_r)\}| \geq 2$ for each hyperedge $\{e_1, e_2, \ldots, e_r\}$ of $\mathrm{KG}^r(\mathcal{F}, s)$.

For nonnegative integers n and k, let the symbols $[n]$ and $\binom{[n]}{k}$ denote the following sets

$$[n] := \{1, 2, \ldots, n\}; \quad \text{and} \quad \binom{[n]}{k} := \{A : A \subseteq [n] \text{ and } |A| = k\}.$$

If n is a positive integer and k is a nonnegative integer, then the *complete k-uniform hypergraph* K_n^k, is a hypergraph with $V\left(K_n^k\right) := [n]$ and $E\left(K_n^k\right) := \binom{[n]}{k}$.

One can easily observe that $\chi\left(\mathrm{KG}^2\left(K_{2n+k}^n, 0\right)\right) \leq k + 2$ and $\chi\left(\mathrm{KG}^r\left(K_n^k, 0\right)\right) \leq \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$. In 1955, Kneser [7] conjectured that $\chi\left(\mathrm{KG}^2\left(K_{2n+k}^n, 0\right)\right) = k+2$. This conjecture was settled by Lovász [10] in 1978. Later, in 1986, Alon, Frankl, and Lovász [2] proved a conjecture of Erdős [6] which asserts that if $n \geq r(k-1)+1$ then $\chi\left(\mathrm{KG}^r\left(K_n^k, 0\right)\right) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$.

If A and B are two sets and s is a nonnegative integer, then as a notation, we write $A \subseteq_s B$ whenever $|A \setminus B| \leq s$.

Let $\mathcal{F}$ be a hypergraph, and r and s be nonnegative integers such that $r \geq 2$ and $s < |e|$ for each hyperedge e of $\mathcal{F}$. As a definition, the *s-th r-colorability defect* of $\mathcal{F}$, denoted by $\mathrm{cd}^r(\mathcal{F}, s)$, is equal to the minimum size of a subset X_0 of $V(\mathcal{F})$ for which a partition $\{X_1, X_2, \ldots, X_r\}$ of $V(\mathcal{F}) \setminus X_0$ exists such that :

$$\text{If } e \in E(\mathcal{F}) \text{ and } X_i \in \{X_1, X_2, \ldots, X_r\}, \text{ then } e \nsubseteq_s X_i.$$

We note that in this definition, some of $X_1, X_2, \ldots, X_r$ may be equal to the empty set.

Dolnikov [5] and Kříž [8,9] proved that for any hypergraph $\mathcal{F}$ and each integer $r \geq 2$ we have

$$\chi\left(\mathrm{KG}^r(\mathcal{F}, 0)\right) \geq \left\lceil \frac{\mathrm{cd}^r(\mathcal{F}, 0)}{r-1} \right\rceil.$$

It is easily seen that $\mathrm{cd}^2\left(K_{2n+k}^n, 0\right) = k+2$. Also, it is evident that if $n \geq r(k-1)+1$ then $\mathrm{cd}^r\left(K_n^k, 0\right) = n - r(k-1)$. Therefore, the theorem of Dolnikov and Kříž is a generalization of theorems of Lovász [10] and Alon, Frankl, and Lovász [2].

If $\mathcal{F}$ is a hypergraph and r and s are nonnegative integers such that $r \geq 2$ and $s < |e|$ for each hyperedge e of $\mathcal{F}$, then the *s-th equitable r-colorability defect* of $\mathcal{F}$, denoted by $\mathrm{ecd}^r(\mathcal{F}, s)$, is defined as the minimum cardinality of a subset X_0 of $V(\mathcal{F})$ in such a way that a partition $\{X_1, X_2, \ldots, X_r\}$ of $V(\mathcal{F}) \setminus X_0$ with the two following properties exists :

- If $1 \leq i < j \leq r$ then $||X_i| - |X_j|| \leq 1$;
- If $e \in E(\mathcal{F})$ and $X_i \in \{X_1, X_2, \ldots, X_r\}$, then $e \nsubseteq_s X_i$.

We note that in this definition, some of $X_1, X_2, \ldots, X_r$ could be equal to the empty set. Obviously, $\mathrm{ecd}^r(\mathcal{F}, s) \geq \mathrm{cd}^r(\mathcal{F}, s)$. As a generalization of the theorem of Dolnikov and

Kříž, it was proved by Abyazi Sani and Alishahi [1] that the relation

$$\chi\left(\mathrm{KG}^r(\mathcal{F},0)\right) \geq \left\lceil \frac{\mathrm{ecd}^r(\mathcal{F},0)}{r-1} \right\rceil$$

always holds.

Azarpendar and Jafari [4], as a generalization of many earlier results [1–3,5,8–10] proved the following theorem.

Theorem 1.1 ([4])**.** *Let $\mathcal{F}$ be a hypergraph. If r and s are nonnegative integers such that $r \geq 2$ and $s < |e|$ for each hyperedge e of $\mathcal{F}$, then*

$$\chi\left(\mathrm{KG}^r(\mathcal{F},s)\right) \geq \left\lceil \frac{\mathrm{ecd}^r\left(\mathcal{F},\lfloor\frac{s}{2}\rfloor\right)}{r-1} \right\rceil.$$

Azarpendar and Jafari in [4] noted that it is plausible that the above theorem remains true if one replaces $\lfloor\frac{s}{2}\rfloor$ with s.
In this paper, as our first result, we show that the inequality

$$\chi\left(\mathrm{KG}^r(\mathcal{F},s)\right) \geq \left\lceil \frac{\mathrm{ecd}^r(\mathcal{F},s)}{r-1} \right\rceil$$

does not hold in general.

We know that

$$\mathrm{ecd}^r(\mathcal{F},0) \leq \mathrm{ecd}^r(\mathcal{F},1) \leq \cdots \leq \mathrm{ecd}^r\left(\mathcal{F},\left\lfloor\frac{s}{2}\right\rfloor\right) \leq \cdots \leq \mathrm{ecd}^r(\mathcal{F},s).$$

One may ask whether the inequality

$$\chi\left(\mathrm{KG}^r(\mathcal{F},s)\right) \geq \left\lceil \frac{\mathrm{ecd}^r\left(\mathcal{F},\lfloor\frac{s}{2}\rfloor\right)}{r-1} \right\rceil$$

is still true if we put some other values larger than $\lfloor\frac{s}{2}\rfloor$ instead of $\lfloor\frac{s}{2}\rfloor$. In order to answer this natural question, we consider the relation

$$\mathrm{cd}^r(\mathcal{F},0) \leq \mathrm{cd}^r(\mathcal{F},1) \leq \cdots \leq \mathrm{cd}^r\left(\mathcal{F},\left\lfloor\frac{s}{2}\right\rfloor\right) \leq \cdots \leq \mathrm{cd}^r(\mathcal{F},s),$$

and also the relation $\mathrm{ecd}^r(\mathcal{F},x) \geq \mathrm{cd}^r(\mathcal{F},x)$ which always holds. As our second result, we show that even in the weaker inequality

$$\chi\left(\mathrm{KG}^r(\mathcal{F},s)\right) \geq \left\lceil \frac{\mathrm{cd}^r\left(\mathcal{F},\lfloor\frac{s}{2}\rfloor\right)}{r-1} \right\rceil,$$

no number x greater than $\lfloor\frac{s}{2}\rfloor$ could be replaced by $\lfloor\frac{s}{2}\rfloor$.

2. We cannot replace s instead of $\left\lfloor\frac{s}{2}\right\rfloor$.

In this section, our aim is showing that the inequality

$$\chi\left(\mathrm{KG}^r(\mathcal{F},s)\right) \geq \left\lceil \frac{\mathrm{ecd}^r(\mathcal{F},s)}{r-1} \right\rceil$$

is not correct. If we put $r = 2$, then the expression $\left\lceil\frac{\mathrm{ecd}^r(\mathcal{F},s)}{r-1}\right\rceil$ will be equal to $\mathrm{ecd}^2(\mathcal{F},s)$. Hence, in order to disprove $\chi\left(\mathrm{KG}^r(\mathcal{F},s)\right) \geq \left\lceil\frac{\mathrm{ecd}^r(\mathcal{F},s)}{r-1}\right\rceil$, it is enough to find a hypergraph $\mathcal{F}$ for which

$$\chi\left(\mathrm{KG}^2(\mathcal{F},s)\right) < \mathrm{ecd}^2(\mathcal{F},s).$$

In this regard, we state and prove the following theorem, which is the first result of this paper.

Theorem 2.1. *For any two positive integers l and s with $l \geq 2$, there exists a hypergraph $\mathcal{F}$ such that*

$$\chi\left(\mathrm{KG}^2(\mathcal{F}, s)\right) = l \quad \textit{and} \quad \mathrm{ecd}^2(\mathcal{F}, s) = l + s.$$

Proof. Put $k := l-2$. Let K^n_{2n+k} be the hypergraph with vertex set $[2n+k]$ and hyperedge set $\binom{[2n+k]}{n}$. So,

$$\chi\left(\mathrm{KG}^2\left(K^n_{2n+k}, 0\right)\right) = k + 2 = l.$$

Now, let $S := \{a_1, a_2, \ldots, a_s\}$ be a set such that $S \cap [2n+k] = \varnothing$; and then define a new hypergraph $\mathcal{F}$ with vertex set $V(\mathcal{F}) := S \cup [2n+k]$ whose hyperedge set equals

$$E(\mathcal{F}) := \{e \cup S : e \in E\left(K^n_{2n+k}\right)\} = \left\{e \cup S : e \in \binom{[2n+k]}{n}\right\}.$$

Any two vertices e_1 and e_2 from $\mathrm{KG}^2\left(K^n_{2n+k}, 0\right)$ are adjacent iff their corresponding vertices in $\mathrm{KG}^2(\mathcal{F}, s)$ are adjacent; that is, $\{e_1, e_2\} \in E\left(\mathrm{KG}^2\left(K^n_{2n+k}, 0\right)\right)$ if and only if $\{e_1 \cup S, e_2 \cup S\} \in E\left(\mathrm{KG}^2(\mathcal{F}, s)\right)$. So, we observe that

$$\chi\left(\mathrm{KG}^2(\mathcal{F}, s)\right) = \chi\left(\mathrm{KG}^2\left(K^n_{2n+k}, 0\right)\right) = k + 2 = l.$$

Now, we show that $\mathrm{ecd}^2(\mathcal{F}, s) = k + 2 + s = l + s$. In this regard, our first objective is showing that $\mathrm{ecd}^2(\mathcal{F}, s) \leq k + 2 + s$. Put

- $Y_1 := [n-1] = \{1, 2, \ldots, n-1\}$;
- $Y_2 := [2n-2] \setminus [n-1] = \{n, n+1, \ldots, 2n-2\}$;
- $Y_0 := \{2n-1, 2n, 2n+1, \ldots, 2n+k\} \cup S$.

Obviously, $\{Y_0, Y_1, Y_2\}$ is a partition of $V(\mathcal{F})$ such that $|Y_1| = |Y_2| = n-1$. Also, if $e \in E(\mathcal{F})$ and $i \in \{1, 2\}$ then :

$$|e \setminus Y_i| = |e \setminus (e \cap Y_i)| = |e| - |e \cap Y_i| \geq |e| - |Y_i| = (s+n) - (n-1) = s + 1 > s;$$

and therefore, $e \not\subseteq_s Y_i$. We conclude that $\mathrm{ecd}^2(\mathcal{F}, s) \leq |Y_0| = k + 2 + s$.

As our next task, we aim to prove that $\mathrm{ecd}^2(\mathcal{F}, s) \geq k+2+s$. Suppose, on the contrary, that $\mathrm{ecd}^2(\mathcal{F}, s) \leq k + 1 + s$. So, one can regard a partition $\{X_0, X_1, X_2\}$ of $V(\mathcal{F})$ with $|X_0| = \mathrm{ecd}^2(\mathcal{F}, s) \leq k + 1 + s$ and $|X_1| \geq |X_2|$ in such a way that $e \not\subseteq_s X_1$ and $e \not\subseteq_s X_2$ for each hyperedge e in $E(\mathcal{F})$. Hence,

$$2|X_1| \geq |X_1| + |X_2| = |V(\mathcal{F})| - |X_0| \geq (2n + k + s) - (k + 1 + s) = 2n - 1;$$

and therefore,

$$|X_1| \geq \left\lceil \frac{2n-1}{2} \right\rceil = n.$$

Choose a subset X'_1 of X_1 such that $|X'_1| = n$. Define

$$A := X'_1 \cap [2n+k] \quad \text{and} \quad B := X'_1 \cap S.$$

So, $|A| + |B| = |X'_1| = n$. Also, suppose that

$$[2n+k] \setminus X'_1 = \left\{i_1, i_2, \ldots, i_{2n+k-|A|}\right\} \quad \text{and} \quad i_1 < i_2 < \cdots < i_{2n+k-|A|}.$$

Now, $e := X_1' \cup \left\{i_1, i_2, \ldots, i_{n-|A|}\right\} \cup (S \setminus X_1')$ is a hyperedge of $\mathcal{F}$ that satisfies

$$\begin{aligned}|e \setminus X_1| \leq |e \setminus X_1'| &= \left|\left\{i_1, i_2, \ldots, i_{n-|A|}\right\} \cup (S \setminus X_1')\right| \\ &= \left|\left\{i_1, i_2, \ldots, i_{n-|A|}\right\}\right| + |S \setminus X_1'| \\ &= n - |A| + |S \setminus (S \cap X_1')| \\ &= n - |A| + |S| - |S \cap X_1'| \\ &= n - |A| + |S| - |B| \\ &= |S| + n - (|A| + |B|) = |S| = s.\end{aligned}$$

We conclude that $|e \setminus X_1| \leq s$, a contradiction to the fact that $e \not\subseteq_s X_1$. It follows that $\mathrm{ecd}^2(\mathcal{F}, s) \geq k+2+s$. Therefore, $\mathrm{ecd}^2(\mathcal{F}, s) \geq k+2+s$ and $\mathrm{ecd}^2(\mathcal{F}, s) \leq k+2+s$ imply $\mathrm{ecd}^2(\mathcal{F}, s) = k+2+s = l+s$; as desired. □

3. A stronger result

This section concerns with determining the set of values which can be replaced by $\lfloor \frac{s}{2} \rfloor$ in the general inequality $\chi(\mathrm{KG}^r(\mathcal{F}, s)) \geq \left\lceil \frac{\mathrm{ecd}^r(\mathcal{F}, \lfloor \frac{s}{2} \rfloor)}{r-1} \right\rceil$. Since

$$\mathrm{ecd}^r(\mathcal{F}, 0) \leq \mathrm{ecd}^r(\mathcal{F}, 1) \leq \cdots \leq \mathrm{ecd}^r\left(\mathcal{F}, \left\lfloor \frac{s}{2} \right\rfloor\right) \leq \cdots \leq \mathrm{ecd}^r(\mathcal{F}, s),$$

one observes that each nonnegative integer which is less than or equal to $\lfloor \frac{s}{2} \rfloor$, could be replaced by $\lfloor \frac{s}{2} \rfloor$. The aim of this section is showing that no number x greater than $\lfloor \frac{s}{2} \rfloor$ could be replaced by $\lfloor \frac{s}{2} \rfloor$. In this regard, we consider the relation $\mathrm{ecd}^r(\mathcal{F}, x) \geq \mathrm{cd}^r(\mathcal{F}, x)$ which always holds; and we show that even in the weaker inequality

$$\chi(\mathrm{KG}^r(\mathcal{F}, s)) \geq \left\lceil \frac{\mathrm{cd}^r(\mathcal{F}, \lfloor \frac{s}{2} \rfloor)}{r-1} \right\rceil,$$

no number x greater than $\lfloor \frac{s}{2} \rfloor$ could be replaced by $\lfloor \frac{s}{2} \rfloor$. We restrict our attention just to the case where $r = 2$ and s is an even positive integer.

Theorem 3.1. *Let $k \in \mathbb{N}$ and s be an even positive integer. Then, there exists a hypergraph $\mathcal{F}$ with $\chi\left(\mathrm{KG}^2(\mathcal{F}, s)\right) = k$ in such a way that*

$$\mathrm{cd}^2(\mathcal{F}, l) = \mathrm{ecd}^2(\mathcal{F}, l) = k(2l - s + 1)$$

for each l in $\{\frac{s}{2}+1, \frac{s}{2}+2, \ldots, s\}$.

Proof. Let us regard some pairwise disjoint sets $A_1, A_2, \ldots, A_k$ with

$$|A_1| = |A_2| = \cdots = |A_k| = s + 1.$$

Now, define a hypergraph $\mathcal{F}$ with

$$V(\mathcal{F}) := A_1 \cup A_2 \cup \cdots \cup A_k \quad \text{and} \quad E(\mathcal{F}) := \{A_1, A_2, \ldots, A_k\}.$$

Let $\{X_0, X_1, X_2\}$ be a partition of $V(\mathcal{F})$ that satisfies the following two properties :

- $|X_0| = \mathrm{cd}^2(\mathcal{F}, l)$,
- If $A_i \in \{A_1, A_2, \ldots, A_k\}$ and $X_j \in \{X_1, X_2\}$, then $A_i \not\subseteq_l X_j$.

We aim to show that $|X_0| \geq k(2l - s + 1)$.
Let $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2\}$. Since $A_i \nsubseteq_l X_j$, we must have

$$|A_i \setminus X_j| \geq l + 1.$$

Hence,

$$|A_i \cap X_j| = |A_i| - |A_i \setminus X_j| = (s+1) - |A_i \setminus X_j| \leq (s+1) - (l+1) = s - l.$$

Thus, $|A_i \cap X_j| \leq s - l$ for each i in $\{1, 2, \ldots, k\}$ and each j in $\{1, 2\}$. Hence,

$$|X_1| = |X_1 \cap V(\mathcal{F})| = \left| X_1 \bigcap \left(\bigcup_{i=1}^{k} A_i \right) \right| \leq \sum_{i=1}^{k} |X_1 \cap A_i| \leq \sum_{i=1}^{k} (s - l) = k(s - l).$$

Similarly, $|X_2| \leq k(s - l)$. We conclude that

$$|X_1 \cup X_2| = |X_1| + |X_2| \leq 2k(s - l).$$

Therefore,

$$|X_0| = |V(\mathcal{F})| - |X_1 \cup X_2| = k(s+1) - |X_1 \cup X_2| \geq k(s+1) - 2k(s-l) = k(2l - s + 1).$$

We conclude that

$$\mathrm{cd}^2(\mathcal{F}, l) \geq k(2l - s + 1).$$

Now, we claim that $\mathrm{ecd}^2(\mathcal{F}, l) \leq k(2l - s + 1)$. In this regard, for each i in $\{1, \ldots, k\}$, let A_{i_1} and A_{i_2} be two disjoint subsets of A_i, each of size $s - l$. More precisely,

$$A_{i_1} \cup A_{i_2} \subseteq A_i \quad \text{and} \quad A_{i_1} \cap A_{i_2} = \varnothing, \quad \text{and also,} \quad |A_{i_1}| = |A_{i_2}| = s - l.$$

Now, define a partition $\{Y_0, Y_1, Y_2\}$ of $V(\mathcal{F})$ as follows :

$$Y_1 := \bigcup_{i=1}^{k} A_{i_1} \quad \text{and} \quad Y_2 := \bigcup_{i=1}^{k} A_{i_2} \quad \text{and} \quad Y_0 := V(\mathcal{F}) \setminus (Y_1 \cup Y_2).$$

We have $Y_1 \cap Y_2 = \varnothing$ and $|Y_1| = |Y_2| = k(s - l)$. Also,

$$|Y_0| = |V(\mathcal{F})| - |Y_1 \cup Y_2| = k(s+1) - 2k(s-l) = k(2l - s + 1).$$

Also, if $A_i \in \{A_1, A_2, \ldots, A_k\}$, then because of $A_i \cap Y_1 = A_{i_1}$, we have

$$|A_i \setminus Y_1| = |A_i \setminus (A_i \cap Y_1)| = |A_i| - |A_{i_1}| = (s+1) - (s-l) = l + 1.$$

Thus, $|A_i \setminus Y_1| = l + 1$. Hence, $|A_i \setminus Y_1| \nleq l$; and therefore, $A_i \nsubseteq_l Y_1$. Similarly, we have $A_i \nsubseteq_l Y_2$.
It follows that $\{Y_0, Y_1, Y_2\}$ is a partition of $V(\mathcal{F})$ that satisfies the following three properties

- $|Y_0| = k(2l - s + 1)$;
- $|Y_1| = |Y_2|$;
- $e \nsubseteq_l Y_1$ and $e \nsubseteq_l Y_2$ for each hyperedge e of $V(\mathcal{F})$.

So, $\mathrm{ecd}^2(\mathcal{F}, l) \leq |Y_0| = k(2l - s + 1)$; and therefore, $\mathrm{ecd}^2(\mathcal{F}, l) \leq k(2l - s + 1)$; as claimed.
We conclude that $k(2l - s + 1) \leq \mathrm{cd}^2(\mathcal{F}, l) \leq \mathrm{ecd}^2(\mathcal{F}, l) \leq k(2l - s + 1)$; which implies

$$\mathrm{cd}^2(\mathcal{F}, l) = \mathrm{ecd}^2(\mathcal{F}, l) = k(2l - s + 1);$$

which is desired. □

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