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Research Article

List equitable coloring of planar graphs without 4- and 6-cycles when $\Delta(G) = 5$

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Abstract

A graph G is k list equitably colorable, if for any given k-uniform list assignment L, G is L-colorable and each color appears on at most $\lceil \frac{|V(G)|}{k} \rceil$ vertices. In 2009, Li and Bu obtained that for planar graph G, if $\Delta(G) \geq 6$ and without 4- and 6-cycles, then G is $\Delta(G)$ list equitably colorable. In order to further prove the conjecture of list equitable coloring, in this paper, we focus on planar graph with $\Delta(G) = 5$, and prove that if G is a planar graph without 4- and 6-cycles, then G is $\Delta(G)$ list equitably colorable.

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1. Introduction

The terminology and notation used but undefined in this paper can be found in [1]. Let $G = (V, E)$ be a graph. Let $d_G(x)$ or simply $d(x)$, denote the number of edges incident with the vertex (face) x in G. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, maximum degree, and minimum degree of G, respectively. Particularly, we use $F(G)$ to denote the face set of G when G is a plane graph. A vertex (resp. face) x is called a k-*vertex* (resp. k-*face*), k^+-*vertex* (resp. k^+-*face*), k^--vertex (resp. k^--*face*), if $d(x) = k$, $k \leq d(x) \leq \Delta(G)$, $k - 1 \leq d(x) \leq k$. A $(d_1, d_2, \cdots, d_n)$-face f is such that $d_1, d_2, \cdots, d_n$ are the degrees of vertices incident with the face f, respectively. In the following, let $f_i(v)$ denote the number of i-faces incident with v for each $v \in V(G)$. Let $n_i(f)$ denote the number of i-vertices which are incident with f. We use $n_i(v)$ to denote the number of i-vertices which are adjacent to v. A graph G is 3-degenerate if every subgraph has a vertex of degree at most 3.

A proper k-coloring of a graph G is a mapping π from the vertex set $V(G)$ to the set of colors $\{1, 2, \cdots, k\}$ such that $\pi(x) \neq \pi(y)$ for every edge $xy \in E(G)$. For a graph G and a list assignment L assigned to each vertex $v \in V(G)$ a set $L(v)$ of acceptable colors, an L-coloring of G is a proper vertex coloring such that for every $v \in V(G)$ the color on v belongs to $L(v)$. If G has an L-coloring, we call G is L-colorable. A list assignment L for G is k-*uniform* if $|L(v)| = k$ for all $v \in V(G)$. A graph G is k list equitably colorable if,

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for any k-uniform list assignment L, G is L-colorable and each color appears on at most $\lceil \frac{|V(G)|}{k} \rceil$ vertices.

In 2003, Kostochka, Pelsmajer and West investigated the list equitable coloring of graphs. They proposed the following conjectures [7].

Corollary 1.1. *Every graph G is k list equitably colorable whenever $k > \Delta(G)$.*

Corollary 1.2. *If G is a connected graph with maximum degree at least 3, then G is $\Delta(G)$ list equitably colorable, unless G is a complete graph or is $K_{k,k}$ for some odd k.*

It has been proved that Conjecture 1.1 holds for graphs with $\Delta(G) \leq 3$ in [10, 11] and graphs with $\Delta(G) \leq 7$ in [6]. Kostochka, Pelsmajer and West proved that a graph G is k list equitably colorable if either $G \neq K_{k+1}, K_{k,k}$ (with k odd in $K_{k,k}$) and $k \geq \max\{\Delta(G), \frac{|V(G)|}{2}\}$, or G is a connected interval graph and $k \geq \Delta(G)$ or G is a 2-degenerate graph and $k \geq \max\{\Delta(G), 5\}$ in [7]. Pelsmajer proved that every graph is k list equitably colorable for any $k \geq \frac{\Delta(G)(\Delta(G)-1)}{2}+2$ in [10]. Zhang and Wu [12] confirmed Conjecture 1.2 for series-parallel graphs. Dong et. [3] verified Conjecture 1.2 for graphs with bounded maximum average degree. In recent years, several groups of authors provided partial affirmative answers to Conjecture 1.2 for some planar graphs without some short cycles in [2, 4, 9, 13–15]. More results can be seen in [8].

In the present paper, we focus on the planar graphs without 4- and 6-cycles. In 2009, Li and Bu proved that if G is a planar graph without 4- and 6-cycles and $k \geq \max\{6, \Delta(G)\}$, then G is k list equitably colorable. In particular, every planar graph G with $\Delta(G) \geq 6$ and without 4- and 6-cycles is $\Delta(G)$ list equitably colorable. We improve the above results and obtain that the condition $k \geq \max\{6, \Delta(G)\}$ could be changed into a weaker condition $k \geq \max\{5, \Delta(G)\}$. In particular, we prove that every planar graph G with $\Delta(G) \geq 5$ and without 4- and 6-cycles are $\Delta(G)$ list equitably colorable.

2. Preliminary

First let us introduce some lemmas.

Lemma 2.1 ([5])**.** *Every planar graph without 6-cycles is 3-degenerate.*

Let G be a planar graph without 4- and 6-cycles. Then G has the following property.

Lemma 2.2. *Any 3-cycle is not adjacent to any other 3-cycle. Any 3-cycle is not adjacent to any other 5-cycle. Two 5-cycles which have common 2-vertices are not adjacent, i.e., the configurations F_1 (in which two 5-cycles have a common 2-vertex) and F_2 (in which two 5-cycles have two common 2-vertices) in Figure 1 don't exist in G.*

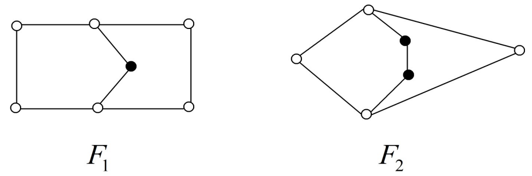

Figure 1

Lemma 2.3. *Let G be a connected planar graph without 4- and 6-cycles. If $\Delta(G) \leq 5$ and $|V(G)| \geq 5$, then G has at least one of the structures isomorphic to the configurations in Figure 2.*

Proof. Suppose to the contrary that G does not contain the structures isomorphic to the configurations H_1, H_2, $\ldots$, H_{13} in Figure 2.

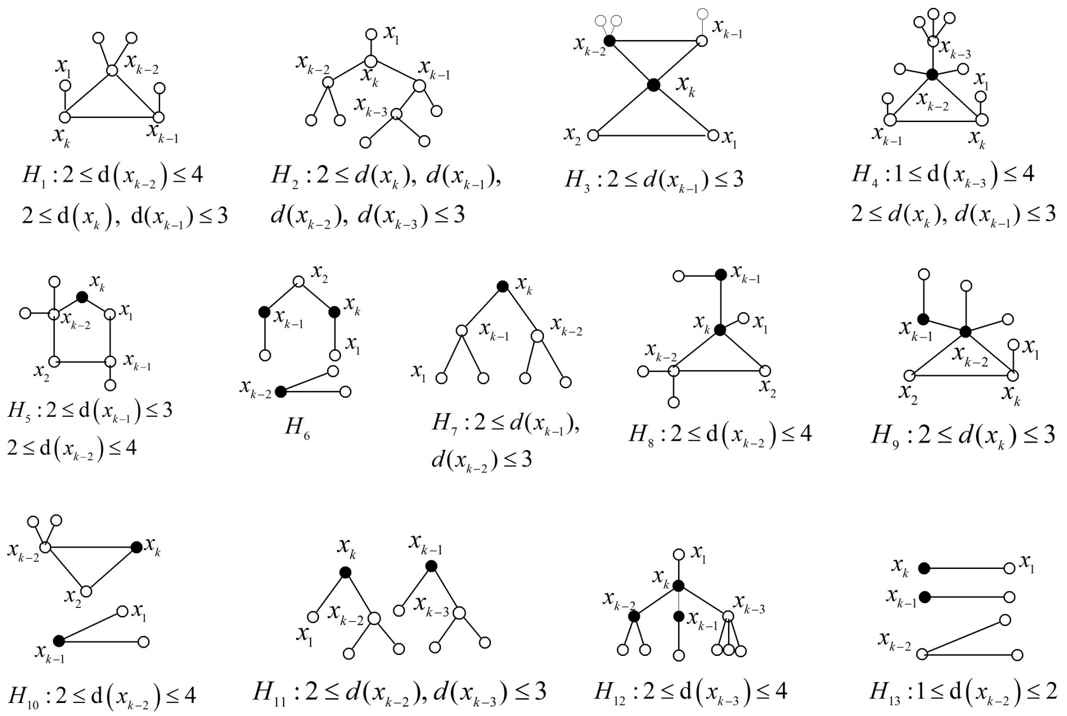

Figure 2

Each configuration depicted in Figure 2 is such that : (1)the vertices labelled x_k, x_{k-1}, x_{k-2} are separate and other vertices in one configuration may overlap if they have the same degree and multiple edges can not be resulted in, (2)the solid vertices have no incident edges other than the ones shown, and (3) except for being specially pointed, the degree of each hollow vertex may be any integer from $[d, 5]$, where d is the number of edges incident with the hollow vertex shown in the configuration, (4) the order of the vertices on the boundary of H_5 can be rearranged.

Since G contains no structure isomorphic to the configuration H_1, G has the following property.

Fact 2.4. *Each 3-face in G is a $(3^-, 3^-, 5)$-, $(3^-, 4^+, 4^+)$- or $(4^+, 4^+, 4^+)$-face.*

Since G contains no structure isomorphic to the configuration H_2, G has the following property.

Fact 2.5. *Each 5-face in G is a $(3^-, 3^-, 3^-, 4^+, 4^+)$-, $(3^-, 3^-, 4^+, 4^+, 4^+)$-, $(3^-, 4^+, 4^+, 4^+, 4^+)$- or $(4^+, 4^+, 4^+, 4^+, 4^+)$-face.*

Since G contains no configuration H_3, G has the following property.

Fact 2.6. *For each $v \in V(G)$, if $d(v) = 4$ and v is incident with two 3-faces, then each of the two 3-faces is not a $(3^-, 4, 4)$-face.*

Since G contains no configuration H_4, G has the following property.

Fact 2.7. *For each 5-vertex $v \in V(G)$, if v is incident with a $(3^-, 3^-, 5)$-face, then $n_5(v) = 3$.*

Since G contains no configuration H_5, G has the following property.

Fact 2.8. *Let f be a 5-face. If f is incident with a 2-vertex, then f is a $(2, 3^-, 5, 5, 5)$- or $(2, 4^+, 4^+, 4^+, 4^+)$-face.*

In the following, we divide the proof into the following four cases by Lemma 2.1. In each case, we will use discharging method to get a contradiction. By Euler's formula $|V|-|E|+|F|=2$ and $\sum_{v\in V(G)} d(v)=\sum_{f\in F(G)} d(f)=2|E|$, we have

$$\sum_{v\in V(G)}(2d(v)-6)+\sum_{f\in F(G)}(d(f)-6)=-6(|V|-|E|+|F|)=-12.$$

Define an initial charge function w on $V(G)\cup F(G)$ by setting $w(v)=2d(v)-6$ if $v\in V(G)$ and $w(f)=d(f)-6$ if $f\in F(G)$, so that $\sum_{x\in V(G)\cup F(G)} w(x)=-12$.

Now redistribute the charge according to the following discharging rules.

- $R1$. **Suppose f is a 3-face. Then for a face f containing a 4-vertex v, let v gives $\frac{3}{2}$ to f if f is a $(3^-,4,4)$-face; let v gives 0 to f if f is a $(4,5,5)$-face; otherwise, let v gives 1 to f.**
  For a face f containing a 5-vertex v, let v gives 3 to f if f is a $(3^-,3^-,5)$-face; let v gives 2 to f if f is a $(3^-,4,5)$-face; let v gives $\frac{3}{2}$ to f if f is a $(3^-,5,5)$-face or a $(4,4^+,5)$-face; let v gives 1 to f if f is a $(5,5,5)$-face.

- $R2$. **Suppose f is a 5-face. Then for a face f containing a 4-vertex v, let v gives $\frac{1}{2}$ to f if f is a $(3,3,3,4,4)$-face; otherwise, let v gives $\frac{1}{3}$ to f.**
  For a face f containing a 5-vertex v, let v gives $\frac{1}{3}$ to f if f is a $(5,5,5,2^+,2^+)$-face; otherwise, let v gives $\frac{3}{4}$ to f.

In the following, let us check the new charge of each element x for $x\in V(G)\cup F(G)$. Let the new charge of each element x be $w'(x)$ for each $x\in V(G)\cup F(G)$.

Case 1 $\delta(G)=3$.

Let v be a vertex in $V(G)$. **Suppose** $d(v)=3$. Then $w'(v)=w(v)=0$.

Suppose $d(v)=4$. Then $w(v)=2$ and $f_3(v)\le 2$ by Lemma 2.2. If $f_3(v)=2$, then $f_5(v)=0$ for the reason that G contains no 6-cycles. By Fact 2.6, any of the two 3-faces which are incident with v is not a $(3^-,4,4)$-face. We have $w'(v)\ge 2-1\times 2=0$ by $R1$. If $f_3(v)=1$, then $f_5(v)\le 1$ for the reason that G contains no 6-cycles. We have $w'(v)\ge 2-\frac{3}{2}-\frac{1}{2}=0$ by $R1$ and $R2$. Otherwise, i.e. $f_3(v)=0$, then $f_5(v)\le 4$. We have $w'(v)\ge 2-\frac{1}{2}\times 4=0$ by $R2$.

Suppose $d(v)=5$. Then $w(v)=4$ and $f_3(v)\le 2$ by Lemma 2.2.

Subcase 1-1 $f_3(v)=2$. Then $f_5(v)=0$. If one of the 3-faces which are incident with v is a $(3^-,3^-,5)$-face, then the other 3-face is a $(5,5,5)$-face by Fact 2.7. We have $w'(v)\ge 4-3-1=0$ by $R1$. Otherwise, we have $w'(v)\ge 4-2\times 2=0$ by $R1$.

Subcase 1-2 $f_3(v)=1$. Then $f_5(v)\le 2$ by Lemma 2.2. If the 3-face which is incident with v is a $(3^-,3^-,5)$-face, then each 5-face which is incident with v is a $(5,5,5,2^+,2^+)$-face by Fact 2.7. We have $w'(v)\ge 4-3-\frac{1}{3}\times 2=\frac{1}{3}>0$ by $R1$ and $R2$. Otherwise, We have $w'(v)\ge 4-2-\frac{3}{4}\times 2=\frac{1}{2}>0$ by $R1$ and $R2$.

Subcase 1-3 $f_3(v)=0$. Then $f_5(v)\le 5$. We have $w'(v)\ge 4-\frac{3}{4}\times 5=\frac{1}{4}>0$ by $R2$.

Let f be a face in $F(G)$. **Suppose** $d(f)=3$. Then $w(f)=-3$, $n_3(f)\le 2$ by Fact 2.4.

If f is a $(3^-,3^-,5)$-face, then $w'(f)\ge -3+3=0$ by $R1$. If f is a $(3^-,4,4)$-face, then $w'(f)=-3+\frac{3}{2}\times 2=0$ by $R1$. If f is a $(3^-,4,5)$-face, then $w'(f)=-3+1+2=0$ by $R1$. If f is a $(3^-,5,5)$-face, then $w'(f)=-3+\frac{3}{2}\times 2=0$ by $R1$. If f is a $(4,4,4)$-face, then $w'(f)=-3+1\times 3=0$ by $R1$. If f is a $(4,4,5)$-face, then $w'(f)=-3+1\times 2+\frac{3}{2}=\frac{1}{2}>0$ by $R1$. If f is a $(4,5,5)$-face, then $w'(f)=-3+\frac{3}{2}\times 2=0$ by $R1$. If f is a $(5,5,5)$-face, then $w'(f) = -3 + 1 \times 3 = 0$ by $R1$.

Suppose $d(f) = 5$. Then $w(f) = -1$ and $n_3(f) \leq 3$ by Fact 2.5.

If f is a $(3,3,3,4,4)$-face, then $w'(f) = -1 + \frac{1}{2} \times 2 = 0$ by $R2$. Otherwise, $w'(f) \geq -1 + \frac{1}{3} \times 3 = 0$ by $R2$.

Suppose $d(f) > 6$. Then $w'(f) = w(f) = d(f) - 6 > 0$.

From the above discussion, we have that $w'(x) \geq 0$ for each $x \in V(G) \cup F(G)$. Thus $\sum_{x \in V(G) \cup F(G)} w'(x) \geq 0$, a contradiction to $\sum_{x \in V(G) \cup F(G)} w(x) = -12$.

Case 2 $\delta(G) = 2$ and there are at most two 2-vertices in G.

By Fact 2.4, the 3-faces which are incident with 2-vertices may be $(2, 3^-, 5)$- or $(2, 4^+, 4^+)$-faces. By Fact 2.8, the 5-faces which are incident with 2-vertices may be $(2, 3^-, 5, 5, 5)$- or $(2, 4^+, 4^+, 4^+, 4^+)$-faces.

For each $v \in V(G)$ ($f \in F(G)$, resp.), if $d(v) \neq 2$ ($d(f) \neq 5$, resp.), then the discussion on the discharging is the same as that in Case 1. Let v be a vertex with $d(v) = 2$. Then $w'(v) = w(v) = -2$. Let f be a face with $d(f) = 5$. If f is a $(2, 3^-, 5, 5, 5)$-face, then $w'(f) = -1 + \frac{1}{3} \times 3 = 0$ by $R2$. If f is a $(2, 4^+, 4^+, 4^+, 4^+)$-face, then $w'(f) = -1 + \frac{1}{3} \times 4 = \frac{1}{3} > 0$ by $R2$. Otherwise, the discussion on the discharging is the same as that in Case 1.

Clearly, we can guarantee the new charge of each element $x \in V(G) \cup F(G)$ is larger than or equal to zero except for the 2-vertices. Thus $\sum_{x \in V(G) \cup F(G)} w'(x) \geq -2 \times 2 = -4$, a contradiction to $\sum_{x \in V(G) \cup F(G)} w(x) = -12$.

Case 3 $\delta(G) = 2$ and there are at least three 2-vertices in G.

Since G contains no structures isomorphic to the configurations H_6, H_7, ..., H_{12}, G has the following properties.

Fact 2.9. *Any vertex v is adjacent to at most one 2-vertex.*

Fact 2.10. *Any 2-vertex v is adjacent to at most one 3^--vertex.*

Fact 2.11. *For each $v \in V(G)$ with $d(v) = 4$, if v is adjacent to a 2-vertex, then the 3-face which is incident with v is a $(4,5,5)$-face.*

Fact 2.12. *For each $v \in V(G)$ with $d(v) = 5$, if v is adjacent to a 2-vertex u, then any 3-face which is incident with v and not incident with u does not contain a 3-vertex.*

Fact 2.13. *Every 3-face in G which is incident with a 2-vertex is a $(2,5,5)$-face.*

Fact 2.14. *There is at most one 2-vertex which is adjacent to a 3^--vertex in G except that two 2-vertices are adjacent to each other.*

Fact 2.15. *For each $v \in V(G)$ with $d(v) = 4$, if v is adjacent to a 2-vertex, then it is adjacent to at most one 3-vertex. Furthermore, if v is adjacent to a 2-vertex and a 3-vertex, then the other neighbors of v are 5-vertices.*

We call a 2-vertex a *special 2-vertex* if it is adjacent to a 3^--vertex, otherwise a simple 2-vertex. Clearly, there are at most two special 2-vertices in G by Fact 2.14.

Now redistribute the charge according to the following discharging rules.

- $R1'$, $R2'$ are the same as $R1$, $R2$, respectively.

- $R3'$. Suppose v is a 2-vertex incident with a vertex u. Then u gives v charge 1 if $d(u) = 4$ or 5.

Let $v \in V(G)$. **Suppose** $d(v) = 2$. Then $w(v) = -2$. If v is a special 2-vertex, then we have $w'(v) = w(v) + 1 = -1$ by Fact 2.10 and $R3'$. Otherwise, we have $w'(v) = -2 + 1 \times 2 = 0$ by $R3'$.

In the following discussion, if v is not adjacent to any 2-vertex, then the discussion is similar to the corresponding situation in Case 2. Thus we only focus on the situation in which v is adjacent to a 2-vertex by Fact 2.9.

Suppose $d(v) = 3$. Then $w'(v) = w(v) = 0$.

Suppose $d(v) = 4$. Then $w(v) = 2$ and $f_3(v) \leq 1$ by Fact 2.13 and Lemma 2.2.
Subcase 3-1 $f_3(v) = 1$. Then $f_5(v) \leq 1$ and the 3-face which is incident with v is a $(4, 5, 5)$-face by Fact 2.11. We have $w'(v) \geq 2 - \frac{1}{2} - 1 = \frac{1}{2} > 0$ by $R1'$, $R2'$ and $R3'$.
Subcase 3-2 $f_3(v) = 0$. Then $f_5(v) \leq 3$ for the reason that G contains no 6-cycle.
Subcase 3-2.1 $f_5(v) = 3$. By Fact 2.15, we have $n_3(v) \leq 1$. If $n_3(v) = 1$, then $n_5(v) = 2$ by Fact 2.15. We have that the 5-faces which are incident with v are not $(3, 3, 3, 4, 4)$-faces. Thus we have that $w'(v) = 2 - \frac{1}{3} \times 3 - 1 = 0$ by $R2'$ and $R3'$. Otherwise, i.e., $n_3(v) = 0$. Clearly, the 5-faces which are incident with v are not $(3, 3, 3, 4, 4)$-faces. We have that $w'(v) = 2 - \frac{1}{3} \times 3 - 1 = 0$ by $R2'$ and $R3'$.
Subcase 3-2.2 $f_5(v) \leq 2$. By $R2'$ and $R3'$, we have that $w'(v) \geq 2 - \frac{1}{2} \times 2 - 1 = 0$.

Suppose $d(v) = 5$. Then $w(v) = 4$ and $f_3(v) \leq 2$.
If $f_3(v) = 2$, then $f_5(v) = 0$. We have $w'(v) \geq 4 - \frac{3}{2} \times 2 - 1 = 0$ by $R1'$, $R3'$ and Fact 2.12.
If $f_3(v) = 1$, then $f_5(v) \leq 2$. We have $w'(v) \geq 4 - \frac{3}{2} - \frac{3}{4} \times 2 - 1 = 0$ by $R1'$, $R2'$, $R3'$ and Fact 2.12.
If $f_3(v) = 0$, then $f_5(v) \leq 4$ for the reason that G contains no 6-cycle. We have $w'(v) \geq 4 - \frac{3}{4} \times 4 - 1 = 0$ by $R2'$ and $R3'$.

Let $f \in F(G)$. **Suppose** $d(f) = 3$. Then $w(f) = -3$. If $n_2(f) \neq 0$, then f is a $(2, 5, 5)$-face by Fact 2.13. We have that $w'(f) \geq -3 + 2 \times \frac{3}{2} = 0$ by $R1'$. Otherwise, the discussion is the same as the corresponding situation when $d(f) = 3$ in Case 2.

Suppose $d(f) = 5$. Then $w(f) = -1$. By Fact 2.8, we have $n_2(f) \leq 2$. Furthermore, if $n_2(f) = 2$, then f is a $(2, 2, 5, 5, 5)$-face. We have that $w'(f) \geq -1 + \frac{1}{3} \times 3 = 0$ by $R2'$. If $n_2(f) = 1$, then f is a $(2, 3, 5, 5, 5)$- or $(2, 4^+, 4^+, 4^+, 4^+)$-face. Thus $w'(f) = -1 + \frac{1}{3} \times 3 = 0$ or $w'(f) \geq -1 + \frac{1}{3} \times 4 = \frac{1}{3} > 0$ by $R2'$. If $n_2(f) = 0$, then the discussion is the same as the situation when $d(f) = 5$ in Case 2.

Suppose $d(f) > 6$. Then $w'(f) = w(f) = d(f) - 6 > 0$.
From the above discussion, It is clear that for each $x \in V(G) \cup F(G)$ if x is not a special 2-vertex, then $w'(x) \geq 0$. Thus we have

$$\sum_{x \in V(G) \cup F(G)} w'(x) \geq -1 \times 2 = -2, \tag{2.1}$$

a contradiction to $\sum_{x \in V(G) \cup F(G)} w(x) = -12$.

Case 4 $\delta(G) = 1$.

Suppose that there is one 1-vertex in G.

First, we assume that there is at most two 2-vertices in G. The discussion on the discharging is the same as that in Case 2. Clearly, we can guarantee the new charge of each element $x \in V(G) \cup F(G)$ is larger than or equal to zero except for the 1-vertex and 2-vertices. For each $v \in V(G)$, if $d(v) = 1$, then $w'(v) = w(v) = -4$. If $d(v) = 2$, then $w'(v) = w(v) = -2$. Thus $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -4 + (-2) \times 2 = -8$, a contradiction to $\sum_{x\in V(G)\cup F(G)} w(x) = -12$.

Now we assume that there are at least three 2-vertices in G. The discussion is the same as that in Case 3. Clearly, we have $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -4-2 = -6$ by Equation (2.1), a contradiction to $\sum_{x\in V(G)\cup F(G)} w(x) = -12$.

Suppose that there are at least two 1-vertices in G.

If there are two 1-vertices in G, then there is neither 2-vertex nor other 1-vertex in G for the reason that G contains no structure isomorphic to the configuration H_{13}. The discussion on the discharging is the same as that in Case 1. Now, we have $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -4 \times 2 = -8$, a contradiction to $\sum_{x\in V(G)\cup F(G)} w(x) = -12$. □

Lemma 2.16 ([9]). *If G is a planar graph without 4- and 6-cycles and $k \geq \max\{6, \Delta(G)\}$, then G is k list equitably colorable.*

Lemma 2.17 ([10,11]). *Every graph G with maximum degree $\Delta(G) \leq 3$ is k list equitably colorable whenever $k \geq \Delta(G) + 1$.*

Lemma 2.18 ([7]). *Let G be a graph with a k-uniform list assignment L. Let $S = \{v_1, v_2, \ldots, v_k\}$, where $\{v_1, v_2, \ldots, v_k\}$ are distinct vertices in G. If $G - S$ has an equitable L-coloring and $|N_G(v_i) - S| \leq k - i$ for $1 \leq i \leq k$, then G has an equitable L-coloring.*

3. Proof of the main result

First, we prove an important lemma.

Lemma 3.1. *If G is a planar graph without 4- and 6-cycles and $k \geq \max\{5, \Delta(G)\}$, then G is k list equitably colorable.*

Proof. By Lemma 2.16, we only need to focus on the situation where $\Delta(G) \leq 5$. Let G be a counterexample with the fewest vertices. If each component of G has at most four vertices, then $\Delta(G) \leq 3$. So G is k list equitably colorable by Lemma 2.17. Otherwise, there is at least one component with at least five vertices. By Lemma 2.3, G has one of the structures H_1, H_2, …, H_{13}. In the following, we show how to find the set S in Lemma 2.18. For convenience, let S' be the set of the labelled vertices of the configuration contained in G in which the vertices are labelled as they are in Figure 2. If G has H_1, H_7 and H_{13}, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_1\}$. If G has H_2, H_4, H_{11} and H_{12}, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_1\}$. If G has H_3, H_5, H_6, H_8, H_9 and H_{10}, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_2, x_1\}$. We use x_i in S' to denote v_i in S in which $i \in \{1, 2, \ldots, k\}$. Next, we go to find the remaining unspecified vertices in the set S of Lemma 2.18, i.e., adding the vertices between x_1 and x_{k-2} if $S' = \{x_k, x_{k-1}, x_{k-2}, x_1\}$, adding the vertices between x_1 and x_{k-3} if $S' = \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_1\}$, adding the vertices between x_2 and x_{k-2} if $S' = \{x_k, x_{k-1}, x_{k-2}, x_2, x_1\}$. By Lemma 2.1, G is 3-degenerate. Starting from the set S', we choose a vertex with the minimum degree in the graph obtained from G by deleting the vertices in the collection S' as the vertex with the maximum subscript in $S \setminus S'$ and put it into set S'. Repeating the above steps, until all of the vertices of $S \setminus S'$

are found, i.e. $S' = S$. By the minimality of G, we have $G - S$ is k list equitably colorable. By Lemma 2.18, it follows that G is k list equitably colorable, a contradiction. □

By Lemma 3.1, we have our main theorem.

Theorem 3.2. *Let G be a planar graph without 4- and 6-cycles. If $\Delta(G) \geq 5$, then G is $\Delta(G)$ list equitably colorable.*

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