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Research Article

King operators which preserve x^j

ZOLTÁN FINTA*

ABSTRACT. We prove the unique existence of the functions r_n $(n = 1, 2, \dots)$ on $[0, 1]$ such that the corresponding sequence of King operators approximates each continuous function on $[0, 1]$ and preserves the functions $e_0(x) = 1$ and $e_j(x) = x^j$, where $j \in \{2, 3, \dots\}$ is fixed. We establish the essential properties of r_n, and the rate of convergence of the new sequence of King operators will be estimated by the usual modulus of continuity. Finally, we show that the introduced operators are not polynomial and we obtain quantitative Voronovskaja type theorems for these operators.

Keywords: Bernstein operator, King operator, Korovkin theorem, modulus of continuity, polynomial operator.

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1. INTRODUCTION

Let Π_n be the space of all algebraic polynomials of degree not greater than n. The Bernstein operators $B_n : C[0, 1] \to \Pi_n$ are given by

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where $n = 1, 2, \dots$, $x \in [0, 1]$, $f \in C[0, 1]$ and $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. For $j = 0, 1, 2, \dots$, we denote by e_j the power function $e_j(x) = x^j$, $x \in [0, 1]$. It is well-known [6, p. 3] that

$$(B_n e_0)(x) = 1,\ (B_n e_1)(x) = x,\ (B_n e_2)(x) = x^2 + \frac{1}{n} x(1-x),\ x \in [0, 1]. \tag{1.2}$$

Studying the connection between regular summability matrices and convergent positive linear operators, King [14, pp. 204-205] introduced the operators $V_n : C[0, 1] \to C[0, 1]$ defined by

$$(V_n f)(x) = \sum_{k=0}^{n} p_{n,k}(r_n^*(x)) f\left(\frac{k}{n}\right), \tag{1.3}$$

where

$$r_n^*(x) = \begin{cases} x^2, & \text{if } n = 1 \\ -\frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1} x^2 + \frac{1}{4(n-1)^2}}, & \text{if } n = 2, 3, \dots \end{cases}. \tag{1.4}$$

Taking into account (1.1)-(1.3), we have $(V_n f)(x) = (B_n f)(r_n^*(x))$, $x \in [0, 1]$ and $V_n e_0 = e_0$, $V_n e_2 = e_2$. The uniform convergence $\lim_{n\to\infty} V_n f = f$ and a quantitative estimation are also discussed in [14, p. 204 and p. 206]. We mention that in [8] we obtained direct and converse

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*Corresponding author: Zoltán Finta; fzoltan@math.ubbcluj.ro
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approximation theorems for (1.3). The existence of a sequence of linear positive bounded *polynomial* operators on $C[0,1]$, possessing e_0 and e_2 as fixed points, was proved in [9]. Main results concerning certain King type modifications of the Bernstein operators and the Szász-Mirakyan operators were presented in the survey paper [1].

Replacing $f\left(\frac{k}{n}\right)$ in (1.1) with $f\left(\sqrt[j]{\frac{k(k-1)\ldots(k-j+1)}{n(n-1)\ldots(n-j+1)}}\right)$, $n \geq j \geq 2$, Aldaz, Kounchev and Render [3, p. 12, Proposition 11] defined a new King type operator, which preserves the functions e_0 and e_j, where $j \in \{2,3,\ldots\}$ is fixed. In [10], we proved that there exist infinitely many sequences of Bernstein type operators $(L_n)_{n\geq 1}$, which approximate each continuous function on $[0,1]$ and have the functions e_0 and e_j as fixed points, where $j \in \{1,2,\ldots\}$ is given and

$$(L_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x)\lambda_{n,k}(f), \quad f \in C[0,1]$$

and $\lambda_{n,k} \in C^*[0,1]$ are bounded positive linear functionals. Further properties of the Bernstein type operators of Aldaz, Kounchev and Render were obtained in the papers [2], [4], [5] and [13]. In [11], among others, we studied the approximation properties of the operators $U_n : C[0,1] \to C[0,1]$ defined by

$$(U_n f)(x) = \sum_{k=0}^{n} p_{n,k}(r_n(x)) f\left(\frac{k}{n}\right), \tag{1.5}$$

where the functions $r_n \in C[0,1]$ were constructed such that U_n preserves the functions e_0 and e_{2i}, with $i \in \{1,2,\ldots\}$ given. The main goal of the present paper is to prove the unique existence of the functions $r_n : [0,1] \to [0,1]$ $(n = 1,2,\ldots)$ such that the corresponding King operators given by (1.5) approximate each continuous function on $[0,1]$ and satisfy the conditions $U_n e_0 = e_0$ and $U_n e_j = e_j$, where $j \in \{2,3,\ldots\}$ is fixed. The essential properties of r_n $(n = 1,2,\ldots)$ will be established. A necessary and sufficient condition is given for the uniform convergence of $(U_n f)_{n\geq 1}$ to $f \in C[0,1]$. The quantitative estimates for the operators (1.5) are obtained with the aid of the usual modulus of continuity. Finally, we show that U_n cannot be polynomial operator of degree n, and we obtain a quantitative Voronovskaja type theorem for the operators (1.5).

2. The construction of r_n

At first we prove the following lemma.

Lemma 2.1. *Let $f, g : [a,b] \to [\alpha,\beta]$ be strictly increasing and continuous functions such that $f(a) = \alpha = g(a)$, $f(b) = \beta = g(b)$ and $f(u) \leq g(u)$ for all $u \in [a,b]$. Then, the inverse mappings $f^{-1}, g^{-1} : [\alpha,\beta] \to [a,b]$ exist and are strictly increasing and continuous on $[\alpha,\beta]$ such that $g^{-1}(v) \leq f^{-1}(v)$ for all $v \in [\alpha,\beta]$.*

Proof. The existence of f^{-1} and g^{-1} is the consequence of the following *continuous inverse theorem*: if $\varphi : [a,b] \to \mathbb{R}$ is a strictly increasing and continuous function then the inverse mapping $\varphi^{-1} : [\varphi(a),\varphi(b)] \to [a,b]$ exists and is strictly increasing and continuous on $[\varphi(a),\varphi(b)]$. Consequently $f^{-1}, g^{-1} : [\alpha,\beta] \to [a,b]$ are strictly increasing and continuous on $[\alpha,\beta]$. Moreover, for every $v \in [\alpha,\beta]$ there exists a unique $u \in [a,b]$ such that $v = f(u)$. Then $g^{-1}(v) = g^{-1}(f(u)) \leq g^{-1}(g(u)) = u = f^{-1}(v)$, because $f(u) \leq g(u)$ and g^{-1} is strictly increasing. □

The next result contains the construction of the functions r_n $(n = 1,2,\ldots)$.

Theorem 2.1. *For every $n = 1, 2, \ldots$, there exists the unique function $r_n : [0,1] \to [0,1]$ such that*

$$\sum_{k=0}^{n} p_{n,k}(r_n(x)) \left(\frac{k}{n}\right)^j = x^j \tag{2.6}$$

for all $x \in [0,1]$, being $j \in \{2, 3, \ldots\}$ fixed.

Proof. If $n = 1$ then the function $r_1(x) = x^j$, $x \in [0,1]$ satisfies the equality

$$p_{1,0}(r_1(x)) \cdot 0 + p_{1,1}(r_1(x)) \cdot 1 = x^j, \quad x \in [0,1].$$

Let $n \geq 2$ and consider the function $\phi_n : [0,1] \to \mathbb{R}$,

$$\phi_n(y) = (B_n e_j)(y) = \sum_{k=0}^{n} p_{n,k}(y) \left(\frac{k}{n}\right)^j.$$

By (1.1)-(1.2), we have $\phi_n(0) = 0$, $\phi_n(1) = 1$ and $0 \leq \phi_n(y) \leq (B_n e_0)(y) = 1$ for every $y \in [0,1]$. Because

$$(B_n f)'(y) = n \sum_{k=0}^{n-1} p_{n-1,k}(y) \left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right]$$

(see [6, p. 305, (2.2)]), we get

$$\begin{aligned}
\phi_n'(y) = (B_n e_j)'(y) &= n \sum_{k=0}^{n-1} p_{n-1,k}(y) \left[\left(\frac{k+1}{n}\right)^j - \left(\frac{k}{n}\right)^j \right] \\
&= n \left\{ (1-y)^{n-1} \left(\frac{1}{n}\right)^j + \binom{n-1}{1} y (1-y)^{n-2} \left[\left(\frac{2}{n}\right)^j - \left(\frac{1}{n}\right)^j \right] + \ldots \right. \\
&\left. + \binom{n-1}{n-2} y^{n-2} (1-y) \left[\left(\frac{n-1}{n}\right)^j - \left(\frac{n-2}{n}\right)^j \right] + y^{n-1} \left[1 - \left(\frac{n-1}{n}\right)^j \right] \right\} \\
&> 0
\end{aligned} \tag{2.7}$$

for all $y \in [0,1]$. Thus $\phi_n : [0,1] \to [0,1]$ is a strictly increasing and continuous function. But the function e_j is also strictly increasing and continuous on $[0,1]$ such that $e_j(0) = 0$ and $e_j(1) = 1$, therefore if $x \in [0,1]$ is arbitrary then the equation $\phi_n(y) = x^j$ has a unique solution $y = r_n(x)$. In view of the continuous inverse theorem, there exists the strictly increasing and continuous inverse mapping ϕ_n^{-1}. Then

$$r_n(x) = (\phi_n^{-1} \circ e_j)(x), \quad x \in [0,1] \tag{2.8}$$

and satisfies (2.6). Moreover $0 = r_n(0) \leq r_n(x) \leq r_n(1) = 1$ for all $x \in [0,1]$. □

The essential properties of r_n $(n = 1, 2, \ldots)$ are gathered in the following theorem.

Theorem 2.2. *Let $r_n : [0,1] \to [0,1]$ $(n = 1, 2, \ldots)$ be the function defined by (2.6). Then*

a) *r_n is strictly increasing and continuous function on $[0,1]$;*
b) *$x^j \leq r_n(x) \leq r_{n+1}(x) \leq x$ for all $x \in [0,1]$;*
c) *$\lim_{n\to\infty} r_n(x) = x$ for all $x \in [0,1]$;*
d) *r_n is differentiable on $[0,1]$.*

Proof. *a)* By (2.8), we have that $r_n(x) = (\phi_n^{-1} \circ e_j)(x)$, $x \in [0,1]$, where ϕ_n^{-1} is a strictly increasing and continuous function on $[0,1]$. Hence, we obtain that r_n is also a strictly increasing and continuous function on $[0,1]$.

b) In view of (1.2), we have $\sum_{k=0}^{n} p_{n,k}(r_n(x))\frac{k}{n} = r_n(x)$. Using (2.6) and Jensen's inequality on $[0,1]$ for the convex function e_j, we get

$$x^j = \sum_{k=0}^{n} p_{n,k}(r_n(x))\left(\frac{k}{n}\right)^j \geq \left(\sum_{k=0}^{n} p_{n,k}(r_n(x))\frac{k}{n}\right)^j = (r_n(x))^j, \quad x \in [0,1].$$

Hence $r_n(x) \leq x$, $x \in [0,1]$.

Because $(B_n f)(y) > (B_{n+1} f)(y)$, $0 < y < 1$ for any strictly convex function f on $[0,1]$ (see [6, p. 310, Corollary 4.2]), we obtain $\phi_n(y) = (B_n e_j)(y) > (B_{n+1} e_j)(y) = \phi_{n+1}(y)$ for $y \in (0,1)$. But $\phi_n(0) = 0 = \phi_{n+1}(0)$ and $\phi_n(1) = 1 = \phi_{n+1}(1)$, therefore $\phi_n(y) \geq \phi_{n+1}(y)$, $y \in [0,1]$. Due to Lemma 2.1, we have $\phi_n^{-1}(x) \leq \phi_{n+1}^{-1}(x)$, $x \in [0,1]$. In particular $\phi_n^{-1}(x^j) \leq \phi_{n+1}^{-1}(x^j)$, $x \in [0,1]$, i.e. $r_n(x) \leq r_{n+1}(x)$, $x \in [0,1]$, because of (2.8). But $r_1(x) = x^j$, $x \in [0,1]$, thus $x^j \leq r_n(x)$, $x \in [0,1]$.

c) Because $p_{n,k}$ $(k = 0, 1, \ldots, n)$ are polynomials of degree n, we have, by Taylor's formula for $x, y \in [0,1]$ that

$$p_{n,k}(y) = p_{n,k}(x) + \frac{1}{1!}p'_{n,k}(x)(y-x) + \frac{1}{2!}p''_{n,k}(x)(y-x)^2 + \ldots + \frac{1}{n!}p^{(n)}_{n,k}(x)(y-x)^n.$$

Hence, in view of (2.6) and (1.1),

$$\begin{aligned} x^j - (B_n e_j)(x) &= \sum_{k=0}^{n} p_{n,k}(r_n(x))\left(\frac{k}{n}\right)^j - \sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k}{n}\right)^j \\ &= \sum_{k=0}^{n} \left[p_{n,k}(r_n(x)) - p_{n,k}(x)\right]\left(\frac{k}{n}\right)^j \\ &= \sum_{k=0}^{n} \left\{\sum_{i=1}^{n} \frac{1}{i!}p^{(i)}_{n,k}(x)(r_n(x)-x)^i\right\}\left(\frac{k}{n}\right)^j \\ &= \sum_{i=1}^{n} \frac{1}{i!}(r_n(x)-x)^i \sum_{k=0}^{n} p^{(i)}_{n,k}(x)\left(\frac{k}{n}\right)^j = \sum_{i=1}^{n} \frac{1}{i!}(r_n(x)-x)^i (B_n e_j)^{(i)}(x). \end{aligned} \tag{2.9}$$

On the other hand the Bernstein polynomial $B_n P$ of a polynomial P of degree m is itself a polynomial of degree m, if $n \geq m$ (see [6, p. 306]). Then $(B_n e_j)^{(i)}(x) = 0$, $x \in [0,1]$ for $n \geq i > j$. By (2.9), we get for $n > j$ that

$$x^j - (B_n e_j)(x) = \sum_{i=1}^{j} \frac{1}{i!}(r_n(x)-x)^i (B_n e_j)^{(i)}(x). \tag{2.10}$$

It is known that $\lim_{n\to\infty} (B_n f)^{(i)}(x) = f^{(i)}(x)$, if $x \in [0,1]$ and $f \in C^i[0,1]$ (see [6, p. 306, Theorem 2.1]). Thus

$$\lim_{n\to\infty} (B_n e_j)^{(i)}(x) = e_j^{(i)}(x) = j(j-1)\ldots(j-i+1)x^{j-i}, \tag{2.11}$$

where $x \in [0,1]$ and $i \in \{1, 2, \ldots, j\}$. Furthermore, in view of *b*), the sequence $(r_n(x))_{n\geq 1}$ is convergent for all $x \in [0,1]$: there exists

$$\lim_{n\to\infty} r_n(x) = r(x), \quad x \in [0,1]. \tag{2.12}$$

Combining (2.10)-(2.12), we find that

$$\begin{aligned}
0 &= \lim_{n\to\infty}(x^j - (B_n e_j)(x)) \\
&= \sum_{i=1}^{j} \frac{1}{i!} \lim_{n\to\infty}(r_n(x)-x)^i \lim_{n\to\infty}(B_n e_j)^{(i)}(x) \\
&= \sum_{i=1}^{j} \frac{1}{i!}(r(x)-x)^i j(j-1)\dots(j-i+1)x^{j-i} = \sum_{i=1}^{j}\binom{j}{i}(r(x)-x)^i x^{j-i} \\
&= -x^j + \sum_{i=0}^{j}\binom{j}{i}(r(x)-x)^i x^{j-i} = -x^j + (r(x)-x+x)^j = -x^j + (r(x))^j.
\end{aligned}$$

Hence $r(x) = x$, $x \in [0,1]$, thus $\lim_{n\to\infty} r_n(x) = x$.

d) Because $\phi_n'(y) > 0$, $y \in [0,1]$ (see (2.7)) and $r_n(x) = \phi_n^{-1}(x^j)$, $x \in [0,1]$ (see (2.8)), it follows that r_n is a differentiable function on $[0,1]$. Moreover

$$r_n'(x) = (\phi_n^{-1})'(x^j)\cdot(x^j)' = \frac{jx^{j-1}}{(\phi_n')'(r_n(x))} = \frac{jx^{j-1}}{(B_n e_j)'(r_n(x))}, \quad x \in [0,1], \tag{2.13}$$

because $\phi_n(r_n(x)) = x^j$. □

Remark 2.1. *Due to (1.4), we have for all $x \in [0,1]$ that*

$$(r_n^*)'(x) = \begin{cases} 2x, & \text{if } n = 1 \\ \frac{n}{n-1}x\left(\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}\right)^{-\frac{1}{2}}, & \text{if } n = 2,3,\dots \end{cases}.$$

The same result can be obtained from (2.13) for $j = 2$.

Indeed, by (2.7), (1.2) and (1.4), we have for $x \in [0,1]$ and $n \geq 2$ that

$$\begin{aligned}
(B_n e_2)'(r_n^*(x)) &= n\sum_{k=0}^{n-1} p_{n-1,k}(r_n^*(x))\left[\left(\frac{k+1}{n}\right)^2 - \left(\frac{k}{n}\right)^2\right] \\
&= \frac{2(n-1)}{n}\sum_{k=0}^{n-1} p_{n-1,k}(r_n^*(x))\frac{k}{n-1} + \frac{1}{n}\sum_{k=0}^{n-1} p_{n-1,k}(r_n^*(x)) \\
&= \frac{2(n-1)}{n} r_n^*(x) + \frac{1}{n} \\
&= \frac{2(n-1)}{n}\sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}.
\end{aligned}$$

Hence, by (2.13),

$$(r_n^*)'(x) = \frac{2x}{(B_n e_2)'(r_n^*(x))} = \frac{n}{n-1}x\left(\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}\right)^{-\frac{1}{2}}.$$

3. The approximation properties of U_n

The operators $U_n : C[0,1] \to C[0,1]$ given by (1.5) are positive linear and $(U_n f)(0) = f(0)$ and $(U_n f)(1) = 1$, because $r_n(0) = 0$ and $r_n(1) = 1$. Moreover, by (1.2) and (2.6), we have $U_n e_0 = e_0$ and $U_n e_j = e_j$. In the following theorem, we study the convergence $U_n f \to f$ in the uniform norm defined by $\|f\| = \sup\{|f(x)| : x \in [0,1]\}$, $f \in C[0,1]$.

Theorem 3.3. *$\lim_{n\to\infty}\|U_nf-f\|=0$ for each $f\in C[0,1]$ if and only if $\lim_{n\to\infty}\|r_n-e_1\|=0$, where r_n $(n=1,2,\ldots)$ are defined by (2.6).*

Proof. Using (1.5), (1.1) and (1.2), we obtain

$$(U_ne_0)(x)=1,\ (U_ne_1)(x)=r_n(x),\ (U_ne_2)(x)=(r_n(x))^2+\frac{1}{n}r_n(x)(1-r_n(x)). \tag{3.14}$$

Hence

$$\|U_ne_0-e_0\|=0, \tag{3.15}$$

$$\|U_ne_1-e_1\|=\|r_n-e_1\| \tag{3.16}$$

and

$$\|U_ne_2-e_2\|\le\|r_n^2-e_1^2\|+\frac{1}{4n}\le 2\|r_n-e_1\|+\frac{1}{4n}, \tag{3.17}$$

because $r_n(x)\in[0,1]$ for $x\in[0,1]$ (see Theorem 2.1).

On the other hand, the statements $a)$, $b)$ and $c)$ of Theorem 2.2, and Dini's theorem (see e.g. [15, p. 150, 7.13. Theorem]) imply that

$$\lim_{n\to\infty}\|r_n-e_1\|=0. \tag{3.18}$$

Combining (3.15)-(3.18), in view of Korovkin theorem [6, pp. 8-10], we obtain the assertion of our theorem. □

The next result contains pointwise and uniform quantitative estimates for U_n $(n=1,2,\ldots)$, using the usual modulus of continuity of $f\in C[0,1]$ given by

$$\omega(f;\delta)=\sup\{|f(u)-f(v)|: u,v\in[0,1], |u-v|<\delta\},\ \delta>0.$$

Theorem 3.4. *Let $(U_n)_{n\ge1}$ be the sequence of operators defined by (1.5). Then for every $f\in C[0,1]$, we have*

a) $|(U_nf)(x)-f(x)|\le 2\omega\left(f;\sqrt{(r_n(x)-x)^2+\frac{1}{n}r_n(x)(1-r_n(x))}\right),\ n\ge1, x\in[0,1];$

b)

$$|(U_nf)(x)-f(x)|\le\begin{cases}6\,\omega\left(f;\sqrt{\frac{x(1-x)}{n}}\right), & \text{if } j=2\\ 2(1+\sqrt{C(j)})\,\omega\left(f;\frac{\sqrt{x(1-x)}}{\sqrt[2j]{n}}\right), & \text{if } j=3,4,\ldots\end{cases},$$

where $n\ge j$, $x\in[0,1]$ and

$$C(j)=(j-1)\sqrt[j]{\frac{j(j-1)^2}{8}}+j; \tag{3.19}$$

c)

$$\|U_nf-f\|\le\begin{cases}6\,\omega\left(f;\frac{1}{\sqrt{n}}\right), & \text{if } j=2\\ 2(1+\sqrt{C(j)})\,\omega\left(f;\frac{1}{\sqrt[2j]{n}}\right), & \text{if } j=3,4,\ldots\end{cases},$$

where $n\ge j$ and $C(j)$ is defined by (3.19).

Proof. $a)$ For any sequence $(L_n)_{n\geq 1}$ of positive linear operators on $C[a,b]$, it is known [7, p. 30] that for $f\in C[a,b]$ and $x\in[a,b]$, we have

$$|(L_nf)(x)-f(x)|\leq |f(x)|\cdot|(L_ne_0)(x)-e_0(x)| \\ +\omega(f;\delta)\left[(L_ne_0)(x)+\frac{1}{\delta}\left((L_ne_0)(x)\right)^{1/2}\cdot\left((L_n(e_1-xe_0)^2)(x)\right)^{1/2}\right].$$

In our case $[a,b]=[0,1]$ and $U_ne_0=e_0$ (see (3.14)), thus

$$|(U_nf)(x)-f(x)|\leq\left[1+\delta^{-1}\left((U_n(e_1-xe_0)^2)(x)\right)^{1/2}\right]\omega(f;\delta). \tag{3.20}$$

But, in view of (3.14), we have

$$\begin{aligned}(U_n(e_1-xe_0)^2)(x)&=(U_ne_2)(x)-2x(U_ne_1)(x)+x^2(U_ne_0)(x)\\ &=(r_n(x)-x)^2+\frac{1}{n}r_n(x)(1-r_n(x)).\end{aligned} \tag{3.21}$$

Choosing $\delta=\left((r_n(x)-x)^2+\frac{1}{n}r_n(x)(1-r_n(x))\right)^{1/2}$ in (3.20), we get the required estimate.
$b)$ We will prove the following estimates below:

$$\left(U_n(e_1-xe_0)^2\right)(x)\leq\begin{cases}\frac{4}{n}x(1-x), & \text{if } j=2\\ \frac{C(j)}{\sqrt[j]{n}}x(1-x), & \text{if } j\geq 3\end{cases}, \tag{3.22}$$

where $x\in[0,1]$ is arbitrary. Hence, by (3.20) and the property $\omega(f;\lambda\delta)\leq(1+\lambda)\omega(f;\delta)$, $\lambda>0$, we get for $\delta=\left(\left(U_n(e_1-xe_0)^2\right)(x)\right)^{1/2}$ that

$$\begin{aligned}|(U_nf)(x)-f(x)|&\leq 2\,\omega\left(f;\left(\left(U_n(e_1-xe_0)^2\right)(x)\right)^{1/2}\right)\\ &\leq\begin{cases}2\,\omega\left(f;2\sqrt{\frac{x(1-x)}{n}}\right), & \text{if } j=2\\ 2\,\omega\left(f;\sqrt{C(j)}\frac{\sqrt{x(1-x)}}{\sqrt[2j]{n}}\right), & \text{if } j\geq 3\end{cases}\\ &\leq\begin{cases}6\,\omega\left(f;\frac{\sqrt{x(1-x)}}{\sqrt{n}}\right), & \text{if } j=2\\ 2(1+\sqrt{C(j)})\,\omega\left(f;\frac{\sqrt{x(1-x)}}{\sqrt[2j]{n}}\right), & \text{if } j\geq 3\end{cases}\end{aligned}$$

which was to be proved.

Now let us prove (3.22). Using Theorem 2.2 b), we have for $x\in[0,1]$ that

$$r_n(x)(1-r_n(x))\leq x(1-x^j)=x(1-x)(1+x+\ldots+x^{j-1})\leq jx(1-x). \tag{3.23}$$

For $j=2$, we have in view of [8, p. 87, Lemma 1, d)] that $0\leq x-r_n(x)\leq\frac{2}{n}(1-x)$. Hence

$$(x-r_n(x))^2=(x-r_n(x))(x-r_n(x))\leq\frac{2}{n}x(1-x). \tag{3.24}$$

Then (3.21), (3.24) and (3.23) imply $\left(U_n(e_1-xe_0)^2\right)(x)\leq\frac{2}{n}x(1-x)+\frac{2}{n}x(1-x)=\frac{4}{n}x(1-x)$.

Let $j\geq 3$ and $n\geq j$. By Theorem 2.1 and [11, pp. 102-103, Lemma 1 and Lemma 2], the polynomial $\phi_n(y)\equiv P_{n,j}(y)=\sum_{k=0}^{n}p_{n,k}(y)\left(\frac{k}{n}\right)^j=a_0y^j+a_1y^{j-1}+\ldots+a_{j-1}y$ satisfies the following conditions:

$$P_{n,j}(r_n(x)) = x^j;$$
$$a_0 = \frac{1}{n^{j-1}}(n-1)(n-2)\dots(n-j+1);\ a_1,\dots,a_{j-1} > 0;\ a_0+a_1+\dots+a_{j-1} = 1;$$
$$0 \le 1 - a_0 \le \frac{j(j-1)}{2n}.$$

Hence

$$\begin{aligned}
0 \le x^j - (r_n(x))^j &= P_{n,j}(r_n(x)) - (r_n(x))^j \\
&= \sum_{k=0}^{j-1} a_k (r_n(x))^{j-k} - (r_n(x))^j \\
&= (a_0 - 1)(r_n(x))^j + \sum_{k=1}^{j-1} a_k (r_n(x))^{j-k} \\
&= -\sum_{k=1}^{j-1} a_k (r_n(x))^j + \sum_{k=1}^{j-1} a_k (r_n(x))^{j-k} \\
&= \sum_{k=1}^{j-1} a_k (r_n(x))^{j-k} \left[1 - (r_n(x))^k\right] \\
&= \sum_{k=1}^{j-1} a_k (r_n(x))^{j-k} (1 - r_n(x)) \left[1 + r_n(x) + \dots + (r_n(x))^{k-1}\right] \\
&\le r_n(x)(1-r_n(x)) \sum_{k=1}^{j-1} k a_k \le (j-1) r_n(x)(1-r_n(x)) \sum_{k=1}^{j-1} a_k \\
&= (j-1) r_n(x)(1-r_n(x))(1-a_0) \\
&\le \frac{j(j-1)^2}{2n} r_n(x)(1-r_n(x)) \le \frac{j(j-1)^2}{8n}.
\end{aligned}$$

Using $(u-v)^{2j} \le (u^j - v^j)^2$, $u, v \in [0,1]$ (see [11, p. 103, Lemma 2, (b)]), we find that $(x - r_n(x))^{2j} \le (x^j - (r_n(x))^j)^2 \le \left(\frac{1}{8n} j(j-1)^2\right)^2$, i.e.

$$0 \le x - r_n(x) \le \sqrt[j]{\frac{1}{8n} j(j-1)^2}. \tag{3.25}$$

At the same time, due to Theorem 2.2 b), we obtain

$$0 \le x - r_n(x) \le x - x^j = x(1-x^{j-1}) = x(1-x)(1+x+\dots+x^{j-2}) \le (j-1)x(1-x). \tag{3.26}$$

Hence, in view of (3.21), (3.25), (3.26) and (3.23), we get

$$\begin{aligned}
\left(U_n(e_1 - xe_0)^2\right)(x) &= (x - r_n(x))(x - r_n(x)) + \frac{1}{n} r_n(x)(1-r_n(x)) \\
&\le \sqrt[j]{\frac{1}{8n} j(j-1)^2}(j-1)x(1-x) + \frac{j}{n} x(1-x) \le \frac{C(j)}{\sqrt[j]{n}} x(1-x).
\end{aligned}$$

c) Because $x(1-x) \le 1$ for $x \in [0,1]$, the estimates formulated in c) follow from the statement of b). □

Remark 3.2. *By Theorem 2.1, we have* $U_n f \equiv V_n f$ *for* $j = 2$*. Then* $V_n e_0 = e_0$ *and* $V_n e_2 = e_2$*, thus, by (3.21), we get* $(V_n(e_1 - xe_0)^2)(x) = 2x(x - r_n^*(x))$*. Applying Theorem 3.4, we obtain*

$$|(V_n f)(x) - f(x)| \leq 2\,\omega\left(f; \sqrt{2x(x - r_n^*(x))}\right), \; n \geq 1, \; x \in [0,1];$$

$$|(V_n f)(x) - f(x)| \leq 6\,\omega\left(f; \sqrt{\frac{x(1-x)}{n}}\right), \; n \geq 2, \; x \in [0,1];$$

$$\|V_n f - f\| \leq 6\,\omega\left(f; \frac{1}{\sqrt{n}}\right), \; n \geq 2.$$

For the first estimate see [14, p. 206, Theorem 3.1].

Furthermore, we have the following theorem.

Theorem 3.5. *Let* $U_n : C[0,1] \to C[0,1]$ $(n = 1, 2, \ldots)$ *be the operators given by (1.5) with* r_n *defined by (2.6). Then* U_n *cannot be polynomial operator of degree* n*: there exists* $f \in C[0,1]$ *such that* $U_n f \notin \Pi_n$*.*

Proof. Let $n \geq j$ and suppose that $U_n f \in \Pi_n$ for all $f \in C[0,1]$. Then $U_n e_1 = r_n \in \Pi_n$ due to (3.14). Furthermore $B_n e_j$ is a polynomial of degree j, because $n \geq j$, and thus $(B_n e_j)(y) = a_0 y^j + a_1 y^{j-1} + \ldots + a_{j-1} y$, where $a_0 > 0$ (see [11, p. 102, Lemma 1]). Taking into account (2.6), we have

$$x^j = (U_n e_j)(x) = (B_n e_j)(r_n(x)) = a_0 (r_n(x))^j + a_1 (r_n(x))^{j-1} + \ldots + a_{j-1} r_n(x).$$

In view of $r_n \in \Pi_n$ and $a_0 > 0$, we find that r_n is a first degree polynomial. By Theorem 2.1, we have $r_n(0) = 0$ and $r_n(1) = 1$, thus $r_n(x) = x$, $x \in [0,1]$. Hence $(U_n f)(x) = (B_n f)(r_n(x)) = (B_n f)(x)$, $x \in [0,1]$. But $U_n e_j = e_j$ (see (2.6)), therefore $B_n e_j = e_j$ on $[0,1]$, contradiction, because $(B_n f)(x) > f(x)$, $0 < x < 1$ for any strictly convex function f on $[0,1]$ (see [6, p. 310, Corollary 4.2]), in particular $B_n e_j > e_j$ on $(0,1)$.

If $1 \leq n < j$ and $U_n f \in \Pi_n$ for all $f \in C[0,1]$, then $U_n e_j = e_j \in \Pi_n$ due to (2.6). Hence $j \leq n$, contradiction. □

Finally, we have the following quantitative Voronovskaja type theorem for the operators (1.5). We mention that similar result was established for the Bernstein type operators of Aldaz, Kounchev and Render in [12].

Theorem 3.6. *Let* U_n $(n = 1, 2, \ldots)$ *be given by (1.5). Then*

a) $\left| n((U_n f)(x) - f(x)) + (f'(x) - x f''(x)) n(x - r_n(x)) \right| \leq 2(2 + \sqrt{39}) x(1-x) \omega\left(f''; \frac{1}{\sqrt{n}}\right)$
for all $x \in [0,1]$*,* $f \in C^2[0,1]$ *and* $j = 2$*, where*

$$0 \leq \liminf_{n\to\infty} n(x - r_n(x)) \leq \limsup_{n\to\infty} n(x - r_n(x)) \leq 2;$$

b) $\left| \sqrt[j]{n}((U_n f)(x) - f(x)) + f'(x) \sqrt[j]{n}(U_n(xe_0 - e_1))(x) - \frac{1}{2} f''(x) \sqrt[j]{n}(U_n(xe_0 - e_1)^2)(x) \right|$
$\leq \sqrt{C(j)}(\sqrt{C(j)} + \sqrt{C_1(j)}) x(1-x) \omega\left(f''; \frac{1}{2\sqrt[j]{n}}\right)$
for all $x \in [0,1]$*,* $f \in C^2[0,1]$ *and* $j \geq 3$*, where* $C(j)$ *is defined by (3.19),*

$$C_1(j) = \frac{3}{4} j^2 + \frac{119}{8} j + \frac{1}{4}(j-1)^2 \sqrt[j]{\frac{1}{64} j^2 (j-1)^4}$$

and

$$0 \leq \liminf_{n\to\infty} \sqrt[j]{n}(U_n(xe_0 - e_1))(x) \leq \limsup_{n\to\infty} \sqrt[j]{n}(U_n(xe_0 - e_1))(x) \leq \sqrt[j]{\frac{1}{8} j(j-1)^2},$$

$$0 \le \liminf_{n\to\infty} \sqrt[j]{n}(U_n(xe_0 - e_1)^2)(x) \le \limsup_{n\to\infty} \sqrt[j]{n}(U_n(xe_0 - e_1)^2)(x) \le \frac{1}{4}C(j).$$

Proof. For $f \in C^2[0,1]$ and $x, t \in [0,1]$, by Taylor's formula, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \int_x^t (f''(u) - f''(x))(t-u)\,du.$$

Hence

$$\begin{aligned}(U_n f)(x) = {} & f(x) + f'(x)(U_n(e_1 - xe_0))(x) + \frac{1}{2}f''(x)(U_n(xe_0 - e_1)^2)(x) \\ & + U_n\left(\int_x^t (f''(u) - f''(x))(t-u)\,du; x\right). \end{aligned} \tag{3.27}$$

Because

$$\begin{aligned} & \left| \int_x^t (f''(u) - f''(x))(t-u)\,du \right| \le \left| \int_x^t |f''(u) - f''(x)||t-u|\,du \right| \\ \le & \left| \int_x^t \omega(f''; |u-x|)\,|t-u|\,du \right| \le \left| \int_x^t (1 + \delta^{-1}|u-x|)\,\omega(f'';\delta)\,|t-u|\,du \right| \\ = & \omega(f'';\delta) \left| \int_x^t (|t-u| + \delta^{-1}|u-x||t-u|)\,du \right| \le \omega(f'';\delta)\left(|t-x|^2 + \delta^{-1}|t-x|^3\right), \end{aligned}$$

where $\delta > 0$, we get, by (3.27) and Hölder's inequality that

$$\begin{aligned} & \left| ((U_n f)(x) - f(x)) + f'(x)(U_n(xe_0 - e_1))(x) - \frac{1}{2}f''(x)(U_n(e_1 - xe_0)^2)(x) \right| \\ & \le \omega(f'';\delta)\left\{(U_n(e_1 - xe_0)^2)(x) + \delta^{-1}(U_n|e_1 - xe_0|^3)(x)\right\} \\ & \le \omega(f'';\delta) \\ & \times \left\{(U_n(e_1 - xe_0)^2)(x) + \delta^{-1}\left[(U_n(e_1 - xe_0)^2)(x)\right]^{1/2}\left[(U_n(e_1 - xe_0)^4)(x)\right]^{1/2}\right\}. \end{aligned} \tag{3.28}$$

Using the first four moments of the Bernstein polynomials [6, p. 304], we have

$$\begin{aligned} & (U_n(e_1 - xe_0)^4)(x) = \sum_{k=0}^n p_{n,k}(r_n(x))\left(\frac{k}{n} - x\right)^4 \\ = & \sum_{k=0}^n p_{n,k}(r_n(x))\left[\left(\frac{k}{n} - r_n(x)\right) + (r_n(x) - x)\right]^4 \\ = & \sum_{k=0}^n p_{n,k}(r_n(x))\left(\frac{k}{n} - r_n(x)\right)^4 + 4(r_n(x) - x)\sum_{k=0}^n p_{n,k}(r_n(x))\left(\frac{k}{n} - r_n(x)\right)^3 \\ & + 6(r_n(x) - x)^2\sum_{k=0}^n p_{n,k}(r_n(x))\left(\frac{k}{n} - r_n(x)\right)^2 \\ & + 4(r_n(x) - x)^3\sum_{k=0}^n p_{n,k}(r_n(x))\left(\frac{k}{n} - r_n(x)\right) + (r_n(x) - x)^4 \\ = & \frac{3}{n^2}(r_n(x))^2(1 - r_n(x))^2 + \frac{1}{n^3}\left[r_n(x)(1 - r_n(x)) - 6(r_n(x))^2(1 - r_n(x))^2\right] \\ & + 4(r_n(x) - x)\frac{1}{n^2}(1 - 2r_n(x))r_n(x)(1 - r_n(x)) + 6(r_n(x) - x)^2\frac{1}{n}r_n(x)(1 - r_n(x)) \\ & + (r_n(x) - x)^4. \end{aligned} \tag{3.29}$$

a) If $j=2$, then $r_n(x)(1-r_n(x))\le 2x(1-x)$, $x\in[0,1]$, due to (3.23). Hence, by (3.29) and (3.24),

$$\begin{aligned}
&(U_n(e_1-xe_0)^4)(x)\\
&\le \frac{12}{n^2}x^2(1-x)^2+\frac{2}{n^2}x(1-x)(1+6r_n(x)(1-r_n(x))\\
&+\frac{8}{n^2}x(1-x)(x-r_n(x))(1+2r_n(x))+\frac{12}{n}x(1-x)(x-r_n(x))^2+(x-r_n(x))^4\\
&\le \frac{3}{n^2}x(1-x)+\frac{2}{n^2}\left(1+\frac{3}{2}\right)x(1-x)\\
&+\frac{24}{n^2}x(1-x)+\frac{12}{n}x(1-x)\frac{2}{n}\frac{1}{4}+\frac{4}{n^2}x(1-x)\frac{1}{4}\\
&=\frac{39}{n^2}x(1-x). \qquad (3.30)
\end{aligned}$$

Then (3.28), (3.22) and (3.30) imply that

$$\begin{aligned}
&\left| n((U_nf)(x)-f(x))+f'(x)n(U_n(xe_0-e_1))(x)-\frac{1}{2}f''(x)n(U_n(e_1-xe_0)^2)(x)\right|\\
\le&\omega(f'';\delta)\left\{4x(1-x)+\delta^{-1}\frac{2\sqrt{39}}{\sqrt{n}}x(1-x)\right\}.
\end{aligned}$$

Choosing $\delta=\frac{1}{\sqrt{n}}$, and taking into account that $(U_n(xe_0-e_1))(x)=x-r_n(x)$ and $(U_n(e_1-xe_0)^2)(x)=2x(x-r_n(x))$, we obtain the desired estimate.

Furthermore, in view of [8, p. 87, Lemma 1, d)], we have $0\le x-r_n(x)\le\frac{2}{n}(1-x)\le\frac{2}{n}$, $x\in[0,1]$, thus $0\le\liminf\limits_{n\to\infty}n(x-r_n(x))\le\limsup\limits_{n\to\infty}n(x-r_n(x))\le 2$.

b) If $j\ge 3$, then (3.29), (3.23), (3.25) and (3.26) imply that

$$\begin{aligned}
&(U_n(e_1-xe_0)^4)(x)\\
\le&\frac{3}{n^2}j^2x^2(1-x)^2+\frac{1}{n^3}jx(1-x)(1+6r_n(x)(1-r_n(x))\\
+&\frac{4}{n^2}jx(1-x)(x-r_n(x))(1+2r_n(x))+\frac{6}{n}jx(1-x)(x-r_n(x))^2+(x-r_n(x))^4\\
\le&\frac{3j^2}{4n^2}x(1-x)+\frac{5j}{2n^3}x(1-x)\\
+&\frac{12j}{n^2}x(1-x)+\frac{3j}{8n}(j-1)^2x(1-x)+\sqrt[j]{\frac{1}{64n^2}j^2(j-1)^4}(j-1)^2\frac{1}{4}x(1-x)\\
\le&\frac{1}{\sqrt[j]{n^2}}x(1-x)\left\{\frac{3}{4}j^2+\frac{119}{8}j+\frac{1}{4}(j-1)^2\sqrt[j]{\frac{1}{64}j^2(j-1)^4}\right\}=\frac{C_1(j)}{\sqrt[j]{n^2}}x(1-x). \qquad (3.31)
\end{aligned}$$

Using (3.28), (3.22) and (3.31), we get

$$\begin{aligned}
&\left|\sqrt[j]{n}((U_nf)(x)-f(x))+f'(x)\sqrt[j]{n}(U_n(xe_0-e_1))(x)-\frac{1}{2}f''(x))\sqrt[j]{n}(U_n(e_1-xe_0)^2)(x)\right|\\
\le&\omega(f'';\delta)\left\{C(j)x(1-x)+\delta^{-1}\frac{\sqrt{C(j)}}{\sqrt[2j]{n}}\sqrt{C_1(j)}x(1-x)\right\}.
\end{aligned}$$

Choosing $\delta=\frac{1}{\sqrt[2j]{n}}$, we obtain the desired estimate.

Finally, by (3.25) and (3.22), we get

$$0 \leq \liminf_{n\to\infty} \sqrt[j]{n}(U_n(xe_0 - e_1))(x) \leq \limsup_{n\to\infty} \sqrt[j]{n}(U_n(xe_0 - e_1))(x) \leq \sqrt[j]{\frac{1}{8}j(j-1)^2}$$

and

$$0 \leq \liminf_{n\to\infty} \sqrt[j]{n}(U_n(xe_0 - e_1)^2)(x) \leq \limsup_{n\to\infty} \sqrt[j]{n}(U_n(xe_0 - e_1)^2)(x) \leq \frac{1}{4}C(j)$$

which completes the proof of the theorem. □

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ZOLTÁN FINTA
BABEŞ-BOLYAI UNIVERSITY
DEPARTMENT OF MATHEMATICS
1, M. KOGĂLNICEANU ST., 400084 CLUJ-NAPOCA, ROMANIA
ORCID: 0000-0003-2104-3483
E-mail address: fzoltan@math.ubbcluj.ro