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Research Article

On the Thales Theorem in the Iso-Taxicab Plane

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Abstract
There are non-Euclidean geometries that are both accessible in a very concrete form and close to Euclidean geometry in their basic structure. In this paper, we study the Iso-taxicab analogues of the intercept theorem, also known as Thales's theorem is an important theorem in elementary geometry.

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1. Introduction

The taxicab plane geometry is non-Euclidean since it fails to satisfy the side-angle-side axiom but satisfies all the remaining twelve axioms of the Euclidean plane geometry. Since the taxicab plane geometry has a different distance function from the Euclidean distance, it seems interesting to study the taxicab analogues of the topics that include the concept of distance in the Euclidean geometry. Pythagorean theorem, Stewart theorem and a median property, trigonometric functions, norm and lengths under rotations in the non-Euclidean geometries were given in [1, 2, 3, 4, 5, 6, 7].

Iso-taxicab geometry also is a non-Euclidean geometry defined by K. O. Sowell in 1989 in [8]. In this geometry presented by Sowell three distance functions arise depending on the relative positions of the points A and B. There are three axes at the origin; the $x-$axis, the $y-$axis and the y' $-$axis, having 60^0 angle which each other. These three axes separate the plane into six regions. The Iso-taxicab distance between any two points calculated in three different orientations according to the regions, as the following Figure 1:

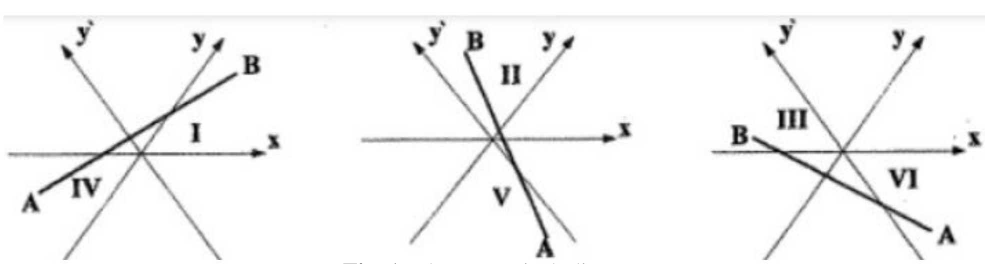

Fig. 1. The Iso-taxicab distance

Let $A=(x_1,y_1)$ and $B=(x_2,y_2)$ be any two points in $\mathbb{R}^2$.

i. If the points have I-IV orientation, then

$$d_I(A,B)=|x_1-x_2|+|y_1-y_2|.$$

ii. If the points have a II-V orientation or lie on a line parallel to the $y-$ or $y'-$ axis, then

$$d_I(A,B) = |y_1 - y_2|.$$

iii. If the points have a III-VI orientation or lie on a line parallel to the $x-$axis, then

$$d_I(A,B) = |x_1 - x_2|.$$

Trigonometry on the Iso-taxicab Geometry was given in [9].

A family of distances, $d_{\pi n}$, that includes taxicab, Chinese- Checker and Iso-taxi distances, was introduced and it was shown that the group of isometries of the plane with $d_{\pi n}$ metric is the semi-direct product of D_{2n} and $T(2)$ in [10]. The trigonometric functions and the versions of some Euclidean theorems in the plane $\mathbb{R}^2_{\pi 3}$ were given in [11, 12].

The definition of $d_{\pi n}-$distances family is given as follows:

Definition 1. *(See [10]) Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be any two points in $\mathbb{R}^2$, a family of $d_{\pi n}$ distances is defined by;*

$$d_{\pi n}(A,B) = \frac{1}{\sin\frac{\pi}{n}}\left(\left|\sin\frac{k\pi}{n} - \sin\frac{(k-1)\pi}{n}\right||x_1 - x_2| + \left|\cos\frac{(k-1)\pi}{n} - \cos\frac{k\pi}{n}\right||y_1 - y_2|\right),$$

where

$$\begin{cases} 1 \le k \le \left[\frac{n-1}{2}\right], k \in \mathbb{Z} & , \quad \tan\frac{(k-1)\pi}{n} \le \left|\frac{y_2-y_1}{x_2-x_1}\right| \le \tan\frac{k\pi}{n}, \\ k = \left[\frac{n+1}{2}\right] & , \quad \tan\frac{\left[\frac{n-1}{2}\right]\pi}{n} \le \left|\frac{y_2-y_1}{x_2-x_1}\right| < \infty \text{ or } x_1 = x_2. \end{cases}$$

For $n = 3$ and accordingly $k = 1$, $k = 2$, we obtain the formula of $d_{\pi 3}$-distance between the points A and B according to the inclination in the plane $\mathbb{R}^2_{\pi 3}$ as

$$d_{\pi 3}(A,B) = \frac{1}{\sin\frac{\pi}{3}}\left(\left|\sin\frac{k\pi}{3} - \sin\frac{(k-1)\pi}{3}\right||x_1 - x_2| + \left|\cos\frac{(k-1)\pi}{3} - \cos\frac{k\pi}{3}\right||y_1 - y_2|\right),$$

where

$$\begin{cases} k = 1 & , \quad 0 \le \left|\frac{y_2-y_1}{x_2-x_1}\right| \le \tan\frac{\pi}{3}, \\ k = 2 & , \quad \tan\frac{\pi}{3} \le \left|\frac{y_2-y_1}{x_2-x_1}\right| < \infty \text{ or } x_1 = x_2, \end{cases}$$

or

$$d_{\pi 3}(A,B) = \begin{cases} |x_1 - x_2| + \frac{1}{\sqrt{3}}|y_1 - y_2| & , \quad 0 \le \left|\frac{y_2-y_1}{x_2-x_1}\right| \le \sqrt{3}, \\ \frac{2}{\sqrt{3}}|y_1 - y_2| & , \quad \sqrt{3} \le \left|\frac{y_2-y_1}{x_2-x_1}\right| < \infty \text{ or } x_1 = x_2. \end{cases}$$

Theorem 2. *(See [13]) The shortest distance of a point $P_0 = (x_0, y_0)$ to the line l given by*

$$ax + by + c = 0,$$

in the plane $\mathbb{R}^2_{\pi 3}$ is

$$d_{\pi 3}(P_0, l) = \rho\left(\frac{-1}{m}\right) d_E(P_0, l).$$

Corollary 3. *If the points A, B, X are collinear in the Iso-taxicab plane then*

$$\frac{d_{\pi 3}(A,X)}{d_{\pi 3}(B,X)} = \frac{d_E(A,X)}{d_E(B,X)}.$$

There are two theorems of Thales in elementary geometry, one known as Thales' theorem having to do with a triangle inscribed in a circle and having the circle's diameter as one leg, the other theorem being also called the intercept theorem. The intercept theorem, basic proportionality theorem or side splitter theorem also known as Thales's theorem, are an important theorem in elementary geometry about the ratios of various line segments that are created if two intersecting lines are intercepted by a pair of parallels. It is equivalent to the theorem about ratios in similar triangles. Although it is traditionally attributed to Greek mathematician Thales and its first known proof appear Euclid's elements, it was known to the ancient Babylonians and Egyptians, although its first known proof appears in Euclid's elements.

2. The version Iso-taxicab of Thales theorem

In this section, we give the version Iso-taxicab of Thales' Theorem.

Theorem 4. *Let d_1, d_2 and d_3 be three parallel straight lines and two lines l_1, l_2 intersect at the point A in the Iso-taxicab plane. Then the Iso-taxicab lengths of opposite sides of the triangles AA_iB_i, $i = 1,2,3$ are proportional*

$$\frac{d_{\pi 3}(A,B_2)}{d_{\pi 3}(A,B_3)} = \frac{d_{\pi 3}(A,A_2)}{d_{\pi 3}(A,A_3)} = \frac{d_{\pi_3}(B_2,A_2)}{d_{\pi 3}(B_3,A_3)},$$

and

$$\frac{d_{\pi 3}(A,B_2)}{d_{\pi 3}(A,B_1)} = \frac{d_{\pi 3}(A,A_2)}{d_{\pi 3}(A,A_1)} = \frac{d_{\pi 3}(B_2,A_2)}{d_{\pi 3}(A_1,B_1)},$$

where the line d_i intersect the lines l_1 and l_2 at the points A_i and B_i, $i = 1,2,3$, respectively.

Proof. Since the points A, B_2, B_3 and A, A_2, A_3 are collinear, from Corollary 3

$$\frac{d_{\pi 3}(A,B_2)}{d_{\pi 3}(A,B_3)} = \frac{d_E(A,B_2)}{d_E(A,B_3)},$$

and

$$\frac{d_{\pi 3}(A,A_2)}{d_{\pi 3}(A,A_3)} = \frac{d_E(A,A_2)}{d_E(A,A_3)}$$

are found. In this case, from the Thales relation in the plane $\mathbb{R}^2$

$$\frac{d_E(A,B_2)}{d_E(A,B_3)} = \frac{d_E(A,A_2)}{d_E(A,A_3)},$$

it can be written

$$\frac{d_{\pi 3}(A,B_2)}{d_{\pi 3}(A,B_3)} = \frac{d_{\pi 3}(A,A_2)}{d_{\pi 3}(A,A_3)}.$$

Now, let the slope of the line d_1 be m. Since $d_1 \parallel d_2 \parallel d_3$, the slopes of the lines d_2 and d_3 are also m. In accordance with the equality

$$d_{\pi 3}(A,B) = \rho(m) d_E(A,B).$$

i. For $0 \leq |m| \leq \sqrt{3}$, since the equalities

$$d_{\pi 3}(B_2,A_2) = \left(\frac{1}{\sqrt{1+m^2}} + \frac{|m|}{\sqrt{3}\sqrt{1+m^2}}\right) d_E(B_2,A_2), \tag{1}$$

$$d_{\pi 3}(B_3,A_3) = \left(\frac{1}{\sqrt{1+m^2}} + \frac{|m|}{\sqrt{3}\sqrt{1+m^2}}\right) d_E(B_3,A_3),$$

the equality

$$\frac{d_{\pi 3}(B_2,A_2)}{d_{\pi 3}(B_3,A_3)} = \frac{\left(\frac{1}{\sqrt{1+m^2}} + \frac{|m|}{\sqrt{3}\sqrt{1+m^2}}\right) d_E(B_2,A_2)}{\left(\frac{1}{\sqrt{1+m^2}} + \frac{|m|}{\sqrt{3}\sqrt{1+m^2}}\right) d_E(B_3,A_3)} = \frac{d_E(B_2,A_2)}{d_E(B_3,A_3)}$$

is obtained.

ii. For $\sqrt{3} \leq |m|$, from the equalities

$$d_{\pi 3}(B_2, A_2) = \left(\frac{2|m|}{\sqrt{3}\sqrt{1+m^2}}\right) d_E(B_2, A_2), \tag{2}$$

$$d_{\pi 3}(B_3, A_3) = \left(\frac{2|m|}{\sqrt{3}\sqrt{1+m^2}}\right) d_E(B_3, A_3),$$

the equality

$$\frac{d_{\pi 3}(B_2, A_2)}{d_{\pi 3}(B_3, A_3)} = \frac{\left(\frac{2|m|}{\sqrt{3}\sqrt{1+m^2}}\right) d_E(B_2, A_2)}{\left(\frac{2|m|}{\sqrt{3}\sqrt{1+m^2}}\right) d_E(B_3, A_3)} = \frac{d_E(B_2, A_2)}{d_E(B_3, A_3)}$$

is found.

iii. For $|m| \to \infty$, the equalities

$$d_{\pi 3}(B_2, A_2) = \left(\frac{2}{\sqrt{3}}\right) d_E(B_2, A_2), \tag{3}$$

$$d_{\pi_3}(B_3, A_3) = \left(\frac{2}{\sqrt{3}}\right) d_E(B_3, A_3)$$

give

$$\frac{d_{\pi 3}(B_2, A_2)}{d_{\pi 3}(B_3, A_3)} = \frac{\left(\frac{2}{\sqrt{3}}\right) d_E(B_2, A_2)}{\left(\frac{2}{\sqrt{3}}\right) d_E(B_3, A_3)} = \frac{d_E(B_2, A_2)}{d_E(B_3, A_3)}.$$

In conclusion we obtain

$$\frac{d_{\pi 3}(A, B_2)}{d_{\pi 3}(A, B_3)} = \frac{d_{\pi 3}(A, A_2)}{d_{\pi 3}(A, A_3)} = \frac{d_{\pi_3}(B_2, A_2)}{d_{\pi 3}(B_3, A_3)}.$$

Similarly, since the points A, B_2, B_1 and A, A_2, A_1 are collinear then we use Corollary 3

$$\frac{d_{\pi 3}(A, B_2)}{d_{\pi 3}(A, B_1)} = \frac{d_E(A, B_2)}{d_E(A, B_1)}$$

and

$$\frac{d_{\pi 3}(A, A_2)}{d_{\pi 3}(A, A_1)} = \frac{d_E(A, A_2)}{d_E(A, A_1)}$$

are found. In this case, from the Thales relation in the plane $\mathbb{R}^2$

$$\frac{d_E(A, B_2)}{d_E(A, B_1)} = \frac{d_E(A, A_2)}{d_E(A, A_1)}$$

it can be written

$$\frac{d_{\pi 3}(A, B_2)}{d_{\pi 3}(A, B_1)} = \frac{d_{\pi 3}(A, A_2)}{d_{\pi 3}(A, A_1)}.$$

Now, let the slope of the line d_1 be m. Since $d_1 \parallel d_2 \parallel d_3$, the slope of the line d_2 is also m. In accordance with the equality

$$d_{\pi 3}(A, B) = \rho(m) d_E(A, B).$$

i. For $0 \leq |m| \leq \sqrt{3}$, since the equalities in the equation (1) and

$$d_{\pi 3}(A_1,B_1) = \left(\frac{1}{\sqrt{1+m^2}} + \frac{|m|}{\sqrt{3}\sqrt{1+m^2}}\right) d_E(A_1,B_1)$$

the equality

$$\frac{d_{\pi 3}(B_2,A_2)}{d_{\pi 3}(A_1,B_1)} = \frac{\left(\frac{1}{\sqrt{1+m^2}} + \frac{|m|}{\sqrt{3}\sqrt{1+m^2}}\right) d_E(B_2,A_2)}{\left(\frac{1}{\sqrt{1+m^2}} + \frac{|m|}{\sqrt{3}\sqrt{1+m^2}}\right) d_E(A_1,B_1)} = \frac{d_E(B_2,A_2)}{d_E(A_1,B_1)}$$

is obtained.

ii. For $\sqrt{3} \leq |m|$, from the equalities in the equation (2) and

$$d_{\pi 3}(A_1,B_1) = \left(\frac{2|m|}{\sqrt{3}\sqrt{1+m^2}}\right) d_E(A_1,B_1)$$

the equality

$$\frac{d_{\pi 3}(B_2,A_2)}{d_{\pi 3}(A_1,B_1)} = \frac{\left(\frac{2|m|}{\sqrt{3}\sqrt{1+m^2}}\right) d_E(B_2,A_2)}{\left(\frac{2|m|}{\sqrt{3}\sqrt{1+m^2}}\right) d_E(A_1,B_1)} = \frac{d_E(B_2,A_2)}{d_E(A_1,B_1)}$$

is found.

iii. For $|m| \to \infty$, the equalities in the equation (3) and

$$d_{\pi_3}(A_1,B_1) = \left(\frac{2}{\sqrt{3}}\right) d_E(B_3,A_3)$$

give

$$\frac{d_{\pi 3}(B_2,A_2)}{d_{\pi 3}(A_1,B_1)} = \frac{\left(\frac{2}{\sqrt{3}}\right) d_E(B_2,A_2)}{\left(\frac{2}{\sqrt{3}}\right) d_E(A_1,B_1)} = \frac{d_E(B_2,A_2)}{d_E(A_1,B_1)}.$$

In conclusion we obtain

$$\frac{d_{\pi 3}(A,B_2)}{d_{\pi 3}(A,B_1)} = \frac{d_{\pi_3}(A,A_2)}{d_{\pi 3}(A,A_1)} = \frac{d_{\pi 3}(B_2,A_2)}{d_{\pi 3}(A_1,B_1)}.$$

■

3. Conclusions

This paper examines the Iso-taxicab analogues of the intercept theorem, also known as Thales's Theorem, in the context of non-Euclidean geometries. This research contributes to the field of the elementary geometry by shedding light on the applicability and generalizability of the fundamentals theorem in various geometric contexts.

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