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Chen Inequalities for Isotropic Submanifolds in Pseudo-Riemannian Space Forms

Marius Mirea

(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

The class of isotropic submanifolds in pseudo-Riemannian manifolds is a distinguished family of submanifolds; they have been studied by several authors. In this article we establish Chen inequalities for isotropic immersions. An example of an isotropic immersion for which the equality case in the Chen first inequality holds is given.

Keywords: *Pseudo-Riemannian manifold, isotropic immersion, isotropic submanifold, spacelike submanifold, Chen inequalities.*
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1. Introduction

The class of isotropic submanifolds in pseudo-Riemannian manifolds represents a distinguished family of submanifolds. The concept of isotropic submanifold of a Riemannian manifold was introduced by O'Neill [6], who studied the general properties of this class of submanifolds. Roughly speaking, a pseudo-Riemannian submanifold is isotropic if the geometry of the submanifold is the same regardless of directions. The isotropic submanifolds generalize totally geodesic submanifolds and totally umbilical submanifolds.

On the other hand, one of the basic problems in the geometry of submanifolds is to find optimal relationships between the intrinsic invariants (for example, sectional curvature, scalar curvature, Ricci curvature) and the main extrinsic invariant (namely, the squared mean curvature) of a submanifold in a space form. In this respect, B.Y. Chen [2], [3] defined the δ-invariants, known as Chen invariants, and established geometric inequalities for these invariants, which are known as Chen inequalities, in particular Chen first inequality. Such inequalities for different submanifolds in various ambient spaces were obtained by many authors.

In the present paper, we establish a Chen first inequality and Chen inequalities (general case) for spacelike isotropic submanifolds in pseudo-Riemannian space forms. An explicit example of an isotropic submanifold which satisfies identically the equality case in the Chen first inequality is given.

2. Preliminaries

Let M_t^n be a submanifold of a pseudo-Riemannian manifold $(\tilde{M}_s^m, g)$. We denote by R and $\tilde{R}$ the Riemannian curvature tensors of M and $\tilde{M}$, respectively. The sectional curvature of a nondegenerate plane section π at a point $p \in M_t^n$ is given by

$$K(\pi) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g^2(X,Y)},$$

where $X, Y \in \pi$ are linearly independent vectors.

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** Corresponding author*

The Gauss and Weingarten formulae are:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

for vector fields X, Y tangent to M_t^n and ξ normal to M_t^n, where ∇ and $\tilde{\nabla}$ are the Levi-Civita connections on M and $\tilde{M}$, respectively, and h is the second fundamental form of M. A and D are its shape operator and normal connection.

The Gauss equation is

$$(\tilde{R}(X,Y)Z)^t = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X, \ \forall X,Y,Z \in \Gamma(TM),$$

and the mean curvature vector is defined by $H = \frac{1}{n}\sum_{i=1}^n g(e_i,e_i)h(e_i,e_i)$, for any orthonormal basis $\{e_1,\dots,e_n\}$ of $T_pM_t^n$.

We recall the definition of the Chen first invariant. Let $p \in M_t^n$ and

$$(\inf K)(p) = \inf\{K(\pi) \mid \pi \subset T_pM_t^n \text{ 2-plane}\} \in \mathbb{R}.$$

We denote by τ the scalar curvature of M_t^n. Then the Chen first invariant is

$$\delta_M = \tau - \inf K.$$

The concept of an isotropic submanifold of a Riemannian manifold was extended to submanifolds in pseudo-Riemannian manifolds by Cabrerizo et al. [1].

Let $\phi : M_t^n \to \tilde{M}_s^m$ be an isometric immersion. ϕ is called *isotropic* [1] at the point $p \in M_t^n$ if $g(h(u,u),h(u,u)) = \lambda(p) \in \mathbb{R}$, for any unit tangent vector $u \in T_pM_t^n$. This means that $\lambda(p)$ is well defined, i.e., it does not depend on the choice of the unit vector u. An isotropic immersion is one which is isotropic everywhere. λ is called the isotropy function of ϕ.

In the following Section we prove two geometric inequalities for spacelike isotropic submanifolds. We then give an explicit example for which the equality case is attained.

3. Chen first inequality for spacelike isotropic submanifolds

In [4] we obtained a Chen-Ricci inequality for spacelike isotropic submanifolds in pseudo-Riemannian space forms. Next, we establish Chen first inequality for such submanifolds.

Theorem 3.1. *Let M_0^n be a spacelike isotropic submanifold of dimension n in a pseudo-Riemannian space form $\tilde{M}_s^{n+s}(c)$ of dimension $n+s$, index s and sectional curvature c. Then, for any plane section π at a point $p \in M_0^n$, we have:*

$$\tau - K(\pi) \leq \frac{3}{4}n^2 g(H,H) - \frac{n^2+2n+4}{4}\lambda + \frac{(n-2)(n+1)}{2}c.$$

Furthermore, the equality holds for a plane section π at a point $p \in M_0^n$ if and only if there exists an orthonormal basis $\{e_1, e_2\} \subset \pi$ such that $h(e_1,e_2) = 0$, where h is the second fundamental form of M_0^n.

Proof. We use the following result from [1]:

$$3A_{h(X,Y)}Z = \lambda[g(X,Y)Z + g(Y,Z)X + g(X,Z)Y] + R(Z,X)Y - (\tilde{R}(Z,X)Y)^t + R(Z,Y)X - (\tilde{R}(Z,Y)X)^t,$$

for any vectors $X,Y,Z \in T_pM_0^n, p \in T_pM_0^n$.

By applying the metric tensor g we obtain:

$$\begin{aligned} 3g(A_{h(X,Y)}Z,W) =& \lambda[g(X,Y)g(Z,W) + g(Y,Z)g(X,W) + g(X,Z)g(Y,W)] + \\ &+ g(R(Z,X)Y,W) - g(\tilde{R}(Z,X)Y,W) + g(R(Z,Y)X,W) - g(\tilde{R}(Z,Y)X,W). \end{aligned} \tag{3.1}$$

Let $p \in M_0^n,\ \pi \subset T_pM_0^n,\ \{X,Y\} \subset \pi$ an orthonormal basis. In equation (3.1) we take $X = W,\ Y = Z$ and we get:

$$\begin{aligned}3g(h(X,Y),h(X,Y)) =&\lambda[g(X,Y)g(X,Y)+g(Y,Y)g(X,X)+g(X,Y)g(Y,X)]+\\&+g(R(Y,X)Y,X)-g(\tilde{R}(Y,X)Y,X)+g(R(Y,Y)X,X)-g(\tilde{R}(Y,Y)X,X),\end{aligned}$$

which implies

$$3g(h(X,Y),h(X,Y)) = \lambda + c - K(\pi).$$

Therefore

$$K(\pi) + 3g(h(X,Y),h(X,Y)) = \lambda + c.$$

In the case under consideration, M_0^n a spacelike submanifold of $\tilde{M}_s^{n+s}(c)$, $h(X,Y)$ is necessarily timelike or 0, i.e. $(h(X,Y),h(X,Y)) \le 0$. Hence

$$K(\pi) \ge \lambda + c, \tag{3.2}$$

Let $\{e_1,\dots,e_n\}$ be an orthonormal basis in $T_pM_0^n$, with $e_1, e_2 \in \pi$. In the equation (3.1) put $X = Y = e_i,\ Z = W = e_j$. It follows that

$$\begin{aligned}3g(h(e_i,e_i),h(e_j,e_j)) =&\lambda[g(e_i,e_i)g(e_j,e_j)+g(e_i,e_j)g(e_i,e_j)+g(e_i,e_j)g(e_i,e_j)]+\\&+g(R(e_j,e_i)e_i,e_j)-g(\tilde{R}(e_j,e_i)e_i,e_j)+g(R(e_j,e_i)e_i,e_j)-g(\tilde{R}(e_j,e_i)e_i,e_j),\end{aligned}$$

or equivalently,

$$3g(h(e_i,e_i),h(e_j,e_j)) = \lambda(1+2\delta_{ij}) + \underbrace{2g(R(e_j,e_i)e_i,e_j)}_{2K(e_i\wedge e_j)} - \underbrace{2g(\tilde{R}(e_j,e_i)e_i,e_j)}_{2\tilde{K}(e_i\wedge e_j)\,=2c}.$$

By summation after i and j in the above equation we get

$$3\sum_{i,j=1}^{n} g(h(e_i,e_i),h(e_j,e_j)) = 2\sum_{i,j=1}^{n} K(e_i\wedge e_j) - 2\sum_{i,j=1}^{n}\tilde{K}(e_i\wedge e_j) + \sum_{i,j=1}^{n}\lambda(1+2\delta_{ij}),$$

which implies

$$3g(nH,nH) = (n^2+2n)\lambda + 4\tau - 2n(n-1)c.$$

Equivalently, we get

$$\tau = \frac{3}{4}n^2 g(H,H) - \frac{n(n+2)}{4}\lambda + \frac{n(n-1)}{2}c. \tag{3.3}$$

We subtract the inequality (3.2) from the equality (3.3) and we obtain

$$\tau - K(\pi) \le \frac{3}{4}n^2 g(H,H) - \frac{n(n+2)}{4}\lambda + \frac{n(n-1)}{2}c - \lambda - c,$$

which is equivalent to

$$\tau - K(\pi) \le \frac{3}{4}n^2 g(H,H) - \frac{n^2+2n+4}{4}\lambda + \frac{(n-2)(n+1)}{2}c. \tag{3.4}$$

Equality holds if and only if $K(\pi) = \lambda + c$, which happens if and only if $h(e_1,e_2) = 0$. □

In a similar way, we can prove the following result.

Theorem 3.2. *Let M^n be a Riemannian isotropic submanifold of dimension n in a Riemannian space form $\tilde{M}^{n+s}(c)$ of dimension $n+s$ and sectional curvature c. Then*

$$\tau - K(\pi) \ge \frac{3}{4}n^2 g(H,H) - \frac{n^2+2n+4}{4}\lambda + \frac{(n-2)(n+1)}{2}c.$$

Furthermore, the equality holds for a plane section π at a point $p \in M^n$ if and only if there exists an orthonormal basis $\{e_1,e_2\} \subset \pi$ such that $h(e_1,e_2) = 0$, where h is the second fundamental form of M^n.

4. An example for the equality case

Let $\gamma : \mathbb{R} \to \mathbb{C}$ be the smooth plane curve

$$\gamma(t) = \sqrt{\cosh(2t)}\exp\{i\arctan(\tanh t)\}.$$

It may be shown [1] that the mapping $\phi : \mathbb{R} \times S^{n-1} \to \mathbb{C}^n, \phi(t, x_1, \ldots, x_n) = \gamma(t)(x_1, \ldots, x_n)$ is an isometric immersion when $\mathbb{R} \times S^{n-1}$ is equipped with the induced metric

$$g = \cosh 2t(dt^2 + g_0),$$

where g_0 denotes the standard Riemannian metric on S^{n-1} (induced by the standard Euclidean inner product on $\mathbb{R}^n$).

We compute the sectional curvature of $M^n = \text{Im}\ \phi$.

As in [1], let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM^n, with

$$\begin{aligned} e_1 &= \frac{1}{|\gamma|}(\partial_t, 0), \\ e_i &= \frac{1}{|\gamma|}(0, \overline{e}_n), \end{aligned}$$

where $\{\overline{e}_2, \ldots, \overline{e}_n\}$ is an orthonormal basis in the corresponding tangent space to S^{n-1}.

We also obtain the local orthonormal basis $\{e_i, Je_i\}$ of $\mathbb{C}^n$ by this construction.

Remark 4.1. We note that in the above definitions of the vectors e_i, the corresponding vectors $\overline{e}_i$ on the sphere are completely arbitrary, except for the condition that they form an orthonormal basis. Therefore any set of vectors $\{\overline{e}_2, \ldots, \overline{e}_n\}$ that form an orthonormal basis on S^{n-1} determine an orthonormal basis $\{e_1, \ldots, e_n\}$ on T_pM^n and it is obvious that we can also go in the opposite direction, $\{e_1, \ldots, e_n\} \to \{\overline{e}_2, \ldots, \overline{e}_n\}$, by taking $\overline{e}_i$ to be the last $n-1$ components of $|\gamma|e_i$; therefore there is a $1:1$ correspondence between the bases $\{e_1, \ldots, e_n\}$ in T_pM^n with $e_1 = \frac{1}{\gamma}(\partial_t, 0, ..., 0)$ and the bases $\{\overline{e}_2, \ldots, \overline{e}_n\}$ in $T_{\overline{p}}S^{n-1}$.

Remark 4.2. With respect to the basis $\{e_i, Je_i\}$ the second fundamental form h of ϕ has the expression [1], [5]:

$$\begin{cases} h(e_1, e_1) = aJe_1, h(e_2, e_2) = \ldots = h(e_n, e_n) = -aJe_1, \\ h(e_1, e_j) = -aJe_j, h(e_j, e_k) = 0, 2 \leq j \neq k \leq n, \end{cases}$$

where $a = -(\cosh 2t)^{-\frac{3}{2}}$.

This implies that ϕ is isotropic with isotropy function $\lambda = a^2$.

Lemma 4.1. *The sectional curvature of M^n is*

$$K(\pi) = (\cos^2\alpha - 2\sin^2\alpha)a^2,\ \alpha \in [0, \frac{\pi}{2}],$$

where π is a plane section tangent to M^n at a point p and α is the angle between π and $span\{e_2, \ldots, e_n\}$.

Proof. We split the proof into three cases.

Case 1. Let $\pi \subset T_pM^n$, $\pi = span\{e_1, u\}$, with $u \in span\{e_2, \ldots, e_n\}$. We may assume $\pi = span\{e_1, e_2\}$. Then:

$$\begin{aligned} K(e_1 \wedge e_2) &= < h(e_1, e_1), h(e_2, e_2) > - < h(e_1, e_2), h(e_1, e_2) > \\ &= < aJe_1, -aJe_1 > - < -aJe_2, -aJe_2 > \\ &= -a^2 - a^2 = -2a^2. \end{aligned}$$

Case 2. Let $U = span\{e_2, \ldots, e_n\}$ and $\pi \subset U$ be a plane section. We may assume $\pi = span\{e_2, e_3\}$. Then:

$$\begin{aligned} K(e_2 \wedge e_3) &= < h(e_2, e_2), h(e_3, e_3) > - < h(e_2, e_3), h(e_2, e_3) > \\ &= < -aJe_1, -aJe_1 > - < 0, 0 >= a^2. \end{aligned}$$

Case 3. Let $U = span\{e_2, \dots, e_n\}$ and π be a 2-dimensional plane determined by 2 vectors x and y with $x \notin U$, $y \notin U$.

Since $\underbrace{\dim \pi}_{=2} + \underbrace{\dim U}_{=n-1} = n + 1$, it follows that π and $U = span\{e_2, \dots, e_n\}$ intersect in a 1-dimensional subspace. Therefore there exists a vector $u \in \pi \cap span\{e_2, \dots, e_n\}$. We may take u unit without loss of generality.

If $e_1 \in \pi$, as $e_1 \perp u$ and u is unit, we may take $u = e_2$ and this corresponds to Case 1. Therefore we may assume $e_1 \notin \pi$.

Let v be another vector in π such that $\{u, v\}$ is an orthonormal basis of π. The plane $span\{e_1, v\}$ intersects $U = span\{e_2, \dots, e_n\}$ (for the same reason as above) in another 1-dimensional vector space, which we take to be spanned by the unit vector w.

By construction $u \perp v$, by hypothesis $u \perp e_1$, therefore $u \perp span\{e_1, v\}$. As $w \in span\{e_1, v\}$ we obtain that $u \perp w$. Therefore $\pi = span\{u, v\}$ with $\{u, v\}$ orthonormal basis and $v = \sin\alpha e_1 + \cos\alpha w$, where $u \perp w$, $u \perp e_1$. α is obviously the angle between the 2-dimensional plane π and the $(n-1)$-dimensional subspace U.

We have shown that any plane π which is not included in $span\{e_2, \dots, e_n\}$ and does not pass through e_1 is spanned by two orthogonal unit vectors u and v, with $v = \sin\alpha e_1 + \cos\alpha w$, $u \perp w$, and both u and w are in $span\{e_2, \dots, e_n\}$. Obviously $e_1 \perp w$ and $\alpha \in (0, \frac{\pi}{2})$. Using the Remark 4.1 we may take $u = e_3$ and $w = e_2$, therefore $\pi = span\{\sin\alpha e_1 + \cos\alpha e_2, e_3\}$.

We can also compute $K(\pi)$ in this case:

$$\begin{aligned}
K(\pi) &= < h(\sin\alpha e_1 + \cos\alpha e_2, \sin\alpha e_1 + \cos\alpha e_2), h(e_3, e_3) >= \\
&\quad - < h(\sin\alpha e_1 + \cos\alpha e_2, e_3), h(\sin\alpha e_1 + \cos\alpha e_2, e_3) >= \\
&= < \sin^2\alpha \underbrace{h(e_1, e_1)}_{aJe_1} + 2\sin\alpha\cos\alpha \underbrace{h(e_1, e_2)}_{-aJe_2} + \cos^2\alpha \underbrace{h(e_2, e_2)}_{-aJe_1}, \underbrace{h(e_3, e_3)}_{-aJe_1} > - \\
&\quad - < \sin\alpha \underbrace{h(e_1, e_3)}_{-aJe_3} + \cos\alpha \underbrace{h(e_2, e_3)}_{=0}, \sin\alpha \underbrace{h(e_1, e_3)}_{-aJe_3} + \cos\alpha \underbrace{h(e_2, e_3)}_{=0} >= \\
&= (\cos^2\alpha - \sin^2\alpha) a^2 - \sin^2\alpha a^2 = \\
&= (\cos^2\alpha - \ 2\sin^2\alpha) a^2.
\end{aligned}$$

We get $K(\pi) = (\cos^2\alpha - \ 2\sin^2\alpha)a^2$, $\alpha \in (0, \frac{\pi}{2})$.

The above three cases above exhaust the possibilities. We notice that in fact Case 1 corresponds to the angle $\alpha = \frac{\pi}{2}$ and Case 2 to the angle $\alpha = 0$.

Therefore we have in general $K(\pi) = (\cos^2\alpha - \ 2\sin^2\alpha)a^2$, $\alpha \in [0, \frac{\pi}{2}]$. □

We apply the results of Section 3 to M^n.

In the case of ϕ defined above $h(e_2, e_3) = 0$, so the equality case is obtained. We also show by direct computation that is indeed true.

We have $M^n = \operatorname{Im}\phi$ and $\tilde{M} = \mathbb{C}^n$. We compute the scalar curvature τ of M^n.

$$\begin{aligned}
\tau &= \sum_{1 \le i < j \le n} K(e_i \wedge e_j) = \sum_{j=2}^{n} \underbrace{K(e_1 \wedge e_j)}_{\text{Case 1 of Remark 4.2}} + \sum_{2 \le i < j \le n} \underbrace{K(e_i \wedge e_j)}_{\text{Case 2 of Remark 4.2}} = \\
&= (n-1)K(e_1 \wedge e_{2)} + \frac{(n-2)(n-1)}{2} K(e_2 \wedge e_3) = -2(n-1)a^2 + \frac{(n-2)(n-1)}{2} a^2 = \\
&= (n-1)a^2 \left(\frac{n-2}{2} - 2 \right) = \frac{(n-6)(n-1)}{2} a^2.
\end{aligned}$$

We compute H:

$$\begin{aligned}
H &= \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i) = \frac{1}{n}(h(e_1, e_1) + h(e_2, e_2) + \ldots + h(e_n, e_n)) = \\
&= \frac{1}{n}(aJe_1 - aJe_1 - \ldots - aJe_1) = \frac{2-n}{n} aJe_1.
\end{aligned}$$

The inequality to check is

$$\tau - K(\pi) \geq \frac{3}{4}n^2 g(H,H) - \frac{n^2+2n+4}{4}\lambda + \frac{(n-2)(n+1)}{2}c.$$

Because $\lambda = g(h(e_1,e_1),h(e_1,e_1)) = a^2$ and $K(\mathbb{C}^n) = 0$, we have

$$\frac{(n-6)(n-1)}{2}a^2 - K(\pi) \geq \frac{3}{4}n^2\frac{(2-n)^2}{n^2}g(aJe_1,aJe_1) - \frac{n^2+2n+4}{4}a^2,$$

i.e.,

$$-K(\pi) \geq \frac{3}{4}(2-n)^2a^2 - \frac{n^2+2n+4}{4}a^2 - \frac{(n-6)(n-1)}{2}a^2,$$

equivalent with

$$K(\pi) \leq a^2.$$

But $K(\pi) = (\cos^2\alpha - 2\sin^2\alpha)a^2 \leq a^2$ for any π, so the inequality is checked, and it is also clear that the equality case is attained for $\alpha = 0$.

5. Chen inequalities: general case

We generalize the Theorems 3.1 and 3.2.

Let $k \in \mathbb{N}$ and $n_1, \ldots, n_k \geq 2$ be integers such that $n_1 < n$, $n_1 + \ldots + n_k \leq n$. We consider mutually orthogonal subspaces $L_1, \ldots, L_k \subset T_pM$, with $\dim(L_i) = n_i$, $i = 1, ..., k$. Denote by $\tau(L_j)$ the scalar curvature of L_j.

Theorem 5.1. *Let M_0^n be a spacelike isotropic submanifold of dimension n in a pseudo-Riemannian space form $\tilde{M}_s^{n+s}(c)$ of dimension $n+s$, index s and sectional curvature c. Then, for any $p \in M_0^n$ and any mutually orthogonal subspaces $L_1, ..., L_k$, we have:*

$$\tau - \sum_{i=1}^{k}\tau(L_i) \leq \frac{3}{4}n^2 g(H,H) - \frac{n(n+2)}{4}\lambda + \frac{n(n-1)}{2}c - (\lambda+c)\sum_{i=1}^{k}\frac{n_i(n_i-1)}{2}.$$

The equality case is attained at a point $p \in M_0^n$ for the subspaces $L_1, ..., L_k$ if and only if for any $j = 1, ..., k$, $h(X,Y) = 0$, for all orthonormal vectors $X, Y \in L_j$.

Theorem 5.2. *Let M^n be a Riemannian isotropic submanifold of dimension n in a Riemannian space form $\tilde{M}^{n+s}(c)$ of dimension $n+s$ and sectional curvature c. Then, for any $p \in M^n$ and any mutually orthogonal subspaces $L_1, ..., L_k$, we have:*

$$\tau - \sum_{i=1}^{k}\tau(L_i) \geq \frac{3}{4}n^2 g(H,H) - \frac{n(n+2)}{4}\lambda + \frac{n(n-1)}{2}c - (\lambda+c)\sum_{i=1}^{k}\frac{n_i(n_i-1)}{2}.$$

The equality case is attained at a point $p \in M^n$ for the subapsaces $L_1, ..., L_k$ if and only if for any $j = 1, ..., k$, $h(X,Y) = 0$, for all orthonormal vectors $X, Y \in L_j$.

We prove Theorem 5.2. Theorem 5.1 has an analogous proof.

Proof. Let $i = 1, ..., k$ and $\{e_1, \ldots, e_{n_i}\}$ be an orthonormal basis of L_i. Then

$$\tau(L_i) = \sum_{1\leq a<b\leq n_i} K(e_a \wedge e_b) \leq \sum_{1\leq a<b\leq n_i}(\lambda+c) = \frac{n_i(n_i-1)}{2}(\lambda+c). \tag{5.1}$$

Therefore

$$\sum_{i=1}^{k}\tau(L_i) \leq \sum_{i=1}^{k}\frac{n_i(n_i-1)}{2}(\lambda+c). \tag{5.2}$$

Subtracting the above equation from the equation (3.3), we obtain

$$\tau - \sum_{i=1}^{k} \tau(L_i) \geq \frac{3}{4} n^2 g(H,H) - \frac{n(n+2)}{4}\lambda + \frac{n(n-1)}{2} c - (\lambda + c) \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}. \tag{5.3}$$

The equality holds in the equation (5.3) if and only if we have the equality in the equation (5.2), which means that $K(e_a \wedge e_b) = \lambda + c$, for any $1 \leq a < b \leq n_i$. We know from the (3.2) that this is equivalent to $h(e_a, e_b) = 0$, for all $1 \leq a < b \leq n_i$. □

Remark 5.1. The equality case is attained in the example if n is large enough, as is obvious from the expression of the second fundamental form h in Remark 4.2. Computations completely similar to those preceding Theorem 5.1 also show this directly.

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Affiliation

Marius Mirea
Address: University of Bucharest, Doctoral School of Mathematics, 010014 Bucharest, Romania
E-mail: marius-ion.mirea@my.fmi.unibuc.ro
ORCID ID: 0009-0006-3973-2892