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# Existence for a Nonlocal Porous Medium Equations of Kirchhoff Type with Logarithmic Nonlinearity

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Abstract. We study the Dirichlet problem for the nonlocal parabolic equation of the Kirchhoff type

$$u_t - a\left(\|u\|^p_{L^p(\Omega)}\right)\sum_{i=1}^{n} D_i\left(|u|^{p-2}\, D_i u\right) + b(x,t)\,|u|^{\alpha(x,t)-2}\, u\log|u| = f(x,t) \quad \text{in } Q_T = \Omega\times(0,T),$$

where $p \geq 2$, $T > 0$, $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a smooth bounded domain. The coefficient $a(\cdot)$ is real-valued function defined on $\mathbb{R}_+$. It is shown that the problem has a weak solution under appropriate and general conditions on $a(\cdot)$, $\alpha(\cdot,\cdot)$ and $b(\cdot)$.

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**Keywords:** Kirchhoff-type equation, nonlocal, existence, logarithmic nonlinearity.

## 1. Introduction

This paper deals with the existence of the solution for the following nonlocal nonlinear parabolic Dirichlet-type boundary value problem with logarithmic nonlinearity.

$$\begin{cases} u_t - a\left(\|u\|^p_{L^p(\Omega)}\right)\sum\limits_{i=1}^{n} D_i\left(|u|^{p-2}\, D_i u\right) + b(x,t)\,|u|^{\alpha(x,t)-2}\, u\log|u| = f(x,t), \\ u(x,0) = 0 = u_0(x), \quad u\,|_{\Gamma_T} = 0 \end{cases} \tag{1.1}$$

where $p \geq 2$ and $(x,t) \in Q_T := \Omega\times(0,T)$, $T > 0$, $\Gamma_T := \partial\Omega\times[0,T]$, $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with Lipschitz boundary, $D_i \equiv \partial/\partial x_i$, $a(\cdot)$ is real-valued function defined on $\mathbb{R}_+$ and $b(x,t)$, $\alpha(x,t)$ are measurable functions defined on $Q_T$.

One of the main feature of problem (1.1) is the presence of the term $a(\|u\|^p_{L^p(\Omega)})$, which is said to be nonlocal since it depends not only on the point in $Q_T$ where the equation is evaluated, but on the norm of the whole solution. Here we note that for the functions $u(t) : (0,T) \mapsto L^p(\Omega)$

$$\|u(t)\|^p_{L^p(\Omega)} = \|u(t)\|^p_p = \int_\Omega |u(x,t)|^p\, dx.$$

Such problems are usually called of Kirchhoff-type, as they are generalizations of the Kirchhoff equation, originally proposed in [20]. More specifically, Kirchhoff proposed the following model

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$$\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where $\rho, \rho_0, h, L, E$ are constants. This nonlocal model extends the classical D'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations.

There are numerous nonlocal mathematical models of Kirchhoff type studied by many authors to express the processes in physics and engineering see, e.g., [1,6,9–11,25,35] and references therein. For example, nonlocal PDEs arise in mathematical modelling of migration of a population to describe the density of some biological species are worked in [12, 16], nonlocal models obtained from combustion theory is considered in [3].

The questions of existence, uniqueness and asymptotic behavior of solutions of the initial and boundary-value problems for the equations

$$u_t - a(l(u))\Delta u = f, \qquad u_t - a(\|\nabla u\|^2_{L^2(\Omega)})\Delta u = f,$$

were studied in the series of works [9–11] with a continuous function $a$ whose argument $l(u)$ was a linear continuous functional on $L^2(\Omega)$, or a continuously differentiable function $a$ of the argument $\|\nabla u\|^2_{L^2(\Omega)}$. In these works, the equation is nondegenerate: $a$ is assumed to be bounded away from zero so there exist positive constants $0 < m \le M < \infty$ such that

$$m \le a(s) \le M, \quad \forall s \in \mathbb{R}.$$

In recent years, logarithmic nonlinearity appears frequently in partial differential equations which describes important physical phenomena (see [4, 7, 8, 18, 21, 23]) and the references therein). This type of nonlinearity was introduced in the nonrelativistic wave equations [17]. Moreover, the logarithmic nonlinearity appears in several branches of physics such as nuclear physics [36], optics and Q-ball dynamics in theoretical physics [15].

It was Chen et al. [7, 8], who first carried out the research on logarithmic source. They studied the following semilinear heat equation with logarithmic nonlinearity in [7]:

$$u_t - \Delta u = u \log |u|$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with zero Dirichlet boundary condition. By using the logarithmic Sobolev inequality, they proved the existence of global weak solution and showed that the power nonlinearity is a critical condition of blow-up in finite time for the solutions of the considered problem.

There are a few papers devoted to study on Kirchhoff-type equations with logarithmic nonlinearity [5,14,27,33,37]. The first result due to Ding and Zhou [14]. They considered the following fractional Kirchhoff-type parabolic problem with logarithmic nonlinearity:

$$\begin{cases} u_t + M([u]_s^2)\mathcal{L}_K u = |u|^{p-2} u \log |u|, & \text{in } \Omega \times \mathbb{R}^+ \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $0 < s < 1$, $\mathcal{L}_K$ is a nonlocal integro-differential operator which generalizes the fractional Laplace operator $(-\Delta)^s$. They combined the Galerkin approximation method and the potential well to prove the existence of a global weak solution with subcritical and critical states.

The boundary-value problems including the equations of type (1.1) is known as Newtonian filtration equation which can be given in the following general form:

$$u_t = \Delta\varphi(u) + h.$$

Equation (1.1) is a parabolic equation with implicit degeneracy which is so called the porous medium equation [19, 22, 34], i.e.,

$$u_t = \Delta \left( |u|^{m-1} u \right) + h,$$

where $m > 1$. This equation is parabolic for $u$ different from 0 and degenerates when $u = 0$. Under condition $m > 1$, above equation describes the non-stationary flow of a compressible Newtonian fluid in a porous medium under polytropic conditions.

Equations with variable nonlinearity and nonlocal equations of Kirchhoff type with logarithmic source appear in numerous applications and are actively studied as mentioned above. Inspired by the above works, in the present article we concern a class of the evolution equations which combine both features.

We generalize the results mentioned above to the Kirchhoff type porous medium equation by considering the equation with nonlocal diffusion:

$$a(\|u\|^p_{L^p(\Omega)})\Delta\left(|u|^{p-2}\,u\right) \text{ for all } p \geq 2.$$

It is important to emphasize that by rearranging the diffusion part of the equation (1.1), we have

$$u_t = a\left(\|u\|^p_{L^p(\Omega)}\right)\Delta\left(|u|^{p-2}\,u\right) + F\left(x,t,u,f\right). \tag{1.2}$$

To the best of our knowledge, there are no papers dealing with porous medium equations of type (1.2) providing a nonlocal coefficient $a(\|u\|^p_{L^p(\Omega)})$ and forcing term $F$ whose argument depends on $|u|$ with variable nonlinearity and also logarithmic source.

We apply the general solvability theorem [30], see Theorem 2.8, to prove the existence of weak solution of (1.1). We study problem (1.1) on the domain of the operator generated by addressed problem and verify the existence of sufficiently smooth solution of the problem under more general (weak) conditions. Essentially we show that problem (1.1) has a solution in the space

$$S_0 := L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}\left(\Omega\right)\right) \cap W^{1,q}\left(0,T;W^{-1,q}\left(\Omega\right)\right) \cap \{u : u\left(x,0\right) = 0\},$$

where

$$\mathring{S}_{1,(p-2)q,q}\left(\Omega\right) := \left\{u \in L^1\left(\Omega\right) : \sum_{i=1}^{n}\left(\int_{\Omega} |u|^{(p-2)q}\,|D_i u|^q\,dx\right) < \infty\right\} \cap \{u\,|_{\partial\Omega} \equiv 0\}.$$

Apart from linear boundary value problems, the sets generated by nonlinear problems are subsets of linear spaces which do not have the linear structure (see [26, 28–32] and references therein).

## 2. Function Spaces and Notations

We first present some basic facts from the theory of the Generalized Lebesgue spaces which are so called Orlicz-Lebesgue space. The more details about these spaces can be found in [2, 13].

Let $\Omega$ be a Lebesgue measurable subset of $\mathbb{R}^n$ such that $|\Omega| > 0$. (Throughout the paper, we denote by $|\Omega|$ the Lebesgue measure of $\Omega$). Let $\alpha\left(x,t\right) \geq 1$ be a measurable bounded function defined on the cylinder $Q_T = \Omega \times (0,T)$, i.e.,

$$1 \leq \alpha^- \equiv \underset{Q_T}{ess}\inf|\alpha\left(x,t\right)| \leq \underset{Q_T}{ess}\sup|\alpha\left(x,t\right)| \equiv \alpha^+ < \infty. \tag{2.1}$$

Then, on the set of all functions on $Q_T$ define the functional $\zeta_\alpha$ and $\|\cdot\|_{L^{\alpha(x,t)}(Q_T)}$ by

$$\zeta_\alpha\left(u\right) \equiv \int_{Q_T} |u|^{\alpha(x,t)}\;dxdt$$

and

$$\|u\|_{L^{\alpha(x,t)}(Q_T)} \equiv \inf\left\{\lambda > 0 |\; \zeta_\alpha\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

The Generalized Lebesgue space is defined as follows:

$$L^{\alpha(x,t)}\left(Q_T\right) := \{u : u \text{ is a measurable real-valued function in } Q_T,\ \zeta_\alpha\left(u\right) < \infty\}.$$

The space $L^{\alpha(x,t)}\left(Q_T\right)$ becomes a Banach space under the norm $\|.\|_{L^{\alpha(x,t)}(Q_T)}$ which is so-called Luxemburg norm.

**Lemma 2.1.** *Let* $0 < |\Omega| < \infty$, $\alpha_1$, $\alpha_2$ *fulfill* (2.1). *Then,*

$$L^{\alpha_1(x,t)}\left(Q_T\right) \subset L^{\alpha_2(x,t)}\left(Q_T\right) \iff \alpha_2\left(x,t\right) \leq \alpha_1\left(x,t\right) \text{ for a.e } \left(x,t\right) \in Q_T.$$

**Lemma 2.2.** *The dual space of* $L^{\alpha(x,t)}\left(Q_T\right)$ *is* $L^{\alpha^*(x,t)}\left(Q_T\right)$ *if and only if* $\alpha \in L^\infty\left(Q_T\right)$. *The space* $L^{\alpha(x,t)}\left(Q_T\right)$ *is reflexive if and only if*

$$1 < \alpha^- \leq \alpha^+ < \infty,$$

*here* $\alpha^*\left(x,t\right) \equiv \frac{\alpha(x,t)}{\alpha(x,t)-1}$.

For $u \in L^{\alpha(x,t)}(Q_T)$ and $v \in L^{\alpha^*(x,t)}(Q_T)$ where $\alpha$, $\alpha^*$ satisfy (2.1) and $\frac{1}{\alpha(x,t)} + \frac{1}{\alpha^*(x,t)} = 1$, the following inequalities hold:

$$\int_{Q_T} |uv| \, dxdt \le 2 \|u\|_{L^{\alpha(x,t)}(Q_T)} \|v\|_{L^{\alpha^*(x,t)}(Q_T)}$$

and

$$\min\{\|u\|^{\alpha^-}_{L^{\alpha(x,t)}(Q_T)}, \|u\|^{\alpha^+}_{L^{\alpha(x,t)}(Q_T)}\} \le \zeta_\alpha(u) \le \max\{\|u\|^{\alpha^-}_{L^{\alpha(x,t)}(Q_T)}, \|u\|^{\alpha^+}_{L^{\alpha(x,t)}(Q_T)}\}. \tag{2.2}$$

We introduce certain nonlinear function spaces (pn-spaces) which are complete metric spaces and directly connected to the problem under consideration. We also give some embedding results for these spaces [28–32] (see also references cited therein).

**Definition 2.3.** Let $\gamma \ge 0$, $\beta \ge 1$, $\varrho = (\varrho_{1,..,}\varrho_n)$ is multi-index, $m \in \mathbb{Z}^+$, $\Omega \subset \mathbb{R}^n$ $(n \ge 1)$ is bounded domain with Lipschitz boundary.

$$S_{m,\gamma,\beta}(\Omega) \equiv \left\{ u \in L^1(\Omega) \mid [u]^{\gamma+\beta}_{S_{m,\gamma,\beta}(\Omega)} \equiv \sum_{0 \le |\varrho| \le m} \left( \int_\Omega |u|^\gamma |D^\varrho u|^\beta \, dx \right) < \infty \right\}$$

in particularly,

$$\mathring{S}_{1,\gamma,\beta}(\Omega) \equiv \left\{ u \in L^1(\Omega) \mid [u]^{\gamma+\beta}_{\mathring{S}_{1,\gamma,\beta}(\Omega)} \equiv \sum_{i=1}^{n} \left( \int_\Omega |u|^\gamma |D_i u|^\beta \, dx \right) < \infty \right\} \cap \{u \mid_{\partial\Omega} \equiv 0\}$$

and for $p \ge 1$,

$$L^p\left(0,T; \mathring{S}_{1,\gamma,\beta}(\Omega)\right) \equiv \left\{ u \in L^1(Q_T) \mid [u]^p_{L^p(0,T;\mathring{S}_{1,\gamma,\beta}(\Omega))} \equiv \int_0^T [u]^p_{\mathring{S}_{1,\gamma,\beta}(\Omega)} \, dt < \infty \right\}.$$

These spaces are called pn-spaces.[1]

**Theorem 2.4.** *Let $\gamma \ge 0, \beta \ge 1$ then $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) \equiv |t|^{\frac{\gamma}{\beta}} t$ is a homeomorphism between $S_{1,\gamma,\beta}(\Omega)$ and $W^{1,\beta}(\Omega)$.*

**Theorem 2.5.** *The following embeddings hold:*

(i) *Let $\gamma, \gamma_1 \ge 0$ and $\beta_1 \ge 1$, $\beta \ge \beta_1$, $\frac{\gamma_1}{\beta_1} \ge \frac{\gamma}{\beta}$, $\gamma_1 + \beta_1 \le \gamma + \beta$. Then, we have*

$$\mathring{S}_{1,\gamma,\beta}(\Omega) \subseteq \mathring{S}_{1,\gamma_1,\beta_1}(\Omega).$$

(ii) *Let $\gamma \ge 0, \beta \ge 1$, $n > \beta$ and $\frac{n(\gamma+\beta)}{n-\beta} \ge r$. Then, there is a continuous embedding*

$$\mathring{S}_{1,\gamma,\beta}(\Omega) \subset L^r(\Omega).$$

*Furthermore for $\frac{n(\gamma+\beta)}{n-\beta} > r$ the embedding is compact.*

(iii) *If $\gamma \ge 0, \beta \ge 1$ and $p \ge \gamma + \beta$, then*

$$W_0^{1,p}(\Omega) \subset \mathring{S}_{1,\gamma,\beta}(\Omega)$$

*holds.*

Similar problem to (1.1) was studied in [24]. Differently from this article we consider logarithmic nonlinearity in the reaction part of the equation (1.1). The presence of the logarithmic nonlinearity caused some difficulties to obtain energy inequalities to apply Theorem 2.8. In order to handle this situation the following two lemmas (Lemma 2.6 and Lemma 2.7) will be used to get the required estimates. For the proof of these lemmas, we refer [28] and references cited there.

Let us denote the function set $M(\Omega)$ to the family of all measurable functions $\alpha : \Omega \longrightarrow [1,\infty]$ and the set $M_0(\Omega)$ is defined as,

$$M_0(\Omega) := \{\alpha \in M(\Omega) : \ 1 \le \alpha^- \le \alpha(x) \le \alpha^+ < \infty, \text{ a.e. } x \in \Omega\},$$

[1] $S_{1,\gamma,\beta}(\Omega)$ is a complete metric space with the following metric: $\forall u, v \in S_{1,\gamma,\beta}(\Omega)$

$$d_{S_{1,\gamma,\beta}}(u,v) = \left\| |u|^{\frac{\gamma}{\beta}} u - |v|^{\frac{\gamma}{\beta}} v \right\|_{W^{1,\beta}(\Omega)}$$

where $\alpha^- := \underset{\Omega}{ess}\inf |\alpha(x)|$, $\alpha^+ := \underset{\Omega}{ess}\sup |\alpha(x)|$.

**Lemma 2.6.** *Assume that* $\alpha \in M_0(\Omega)$ *and* $\beta \geq 1$, $\sigma > 0$. *Then, for every* $u \in L^{\alpha(\cdot)+\sigma}(\Omega)$

$$\int_\Omega |u|^{\alpha(x)} \left|\log|u|\right|^\beta \, dx \leq M_1 \int_\Omega |u|^{\alpha(x)+\sigma} \, dx + M_2$$

*is fulfilled. Here* $M_1 \equiv M_1(\sigma, \beta) > 0$ *and* $M_2 \equiv M_2(\sigma, \beta, |\Omega|) > 0$ *are constants.*

**Lemma 2.7.** *Let* $\tilde{\varepsilon} > 0$ *and* $\beta_1 : \Omega \to [\tilde{\varepsilon}, \infty)$ *be a measurable function which satisfy* $\tilde{\varepsilon} \leq \beta_1^- \leq \beta_1(x) \leq \beta_1^+ < \infty$ *and* $\alpha$, $\beta \in M_0(\Omega)$. *Then, the inequality*

$$\int_\Omega |u|^{\alpha(x)} |\ln|u||^{\beta(x)} \, dx \leq C_1 \int_\Omega |u|^{\alpha(x)+\beta_1(x)} \, dx + C_2, \ \forall u \in L^{\alpha(\cdot)+\beta_1(\cdot)}(\Omega)$$

*holds. Here* $C_1 \equiv C_1(\tilde{\varepsilon}, \beta^+) > 0$ *and* $C_2 \equiv C_2(\tilde{\varepsilon}, \beta^+, |\Omega|) > 0$ *are constants.*

In the following, we present the general solvability theorem [30](see also for similar theorems [29, 32]). We will employ this theorem to demonstrate the existence of a weak solution of problem (1.1).

**Theorem 2.8.** *Let X and Y be Banach spaces with dual spaces* $X^*$ *and* $Y^*$, *respectively, Y be a reflexive Banach space,* $M_0 \subseteq X$ *be a weakly complete "reflexive" pn-space,* $X_0 \subseteq M_0 \cap Y$ *be a separable vector topological space. Let the following conditions be fulfilled:*

(i) $\xi : S_0 \longrightarrow L^q(0, T; Y)$ *is a weakly compact (weakly continuous) mapping, where*

$$S_0 := L^p(0, T; M_0) \cap W^{1,q}(0, T; Y) \cap \{x(t) : x(0) = 0\}$$

$1 < \max\{q, q'\} \leq p < \infty$, $q' = \frac{q}{q-1}$;

(ii) *there is a linear continuous operator* $A : W^{s,m}(0, T; X_0) \longrightarrow W^{s,m}(0, T; Y^*)$, $s \geq 0$, $m \geq 1$ *such that A commutes with* $\frac{\partial}{\partial t}$ *and the conjugate operator* $A^*$ *has ker*$(A^*) = 0$;

(iii) *operators* $\xi$ *and A generate, in generalized sense, a coercive pair on space* $L^p(0, T; X_0)$, *i.e. there exist a number* $r > 0$ *and a function* $\Psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *such that* $\Psi(\tau)/\tau \nearrow \infty$ *as* $\tau \nearrow \infty$ *and for any* $x \in L^p(0, T; X_0)$ *such that* $[x]_{L^p(M_0)} \geq r$ *following inequality holds:*

$$\int_0^T \langle \xi(t, x(t)), Ax(t) \rangle \, dt \geq \Psi\left([x]_{L^p(M_0)}\right);$$

(iv) *there exists some constants* $C_0 > 0$, $C_1, C_2 \geq 0$ *and* $\nu > 1$ *such that the inequalities*

$$\int_0^T \langle \eta(t), A\eta(t) \rangle \, dt \geq C_0 \|\eta\|^\nu_{L^q(0,T;Y)} - C_2,$$

$$\int_0^t \left\langle \frac{\partial x}{\partial \tau}, Ax(\tau) \right\rangle d\tau \geq C_1 \|x\|^\nu_Y(t) - C_2, \quad a.e. \ t \in [0, T]$$

*hold for any* $x \in W^{1,p}(0, T; X_0)$ *and* $\eta \in L^p(0, T; X_0)$.

*Assume that that conditions (i)-(iv) are fulfilled. Then, the Cauchy problem*

$$\frac{dx}{d\tau} + \xi(t, x(t)) = y(t), \quad y \in L^q(0, T; Y); \ x(0) = 0$$

*is solvable in* $S_0$ *in the following sense*

$$\int_0^T \left\langle \frac{dx}{d\tau} + \xi(t, x(t)), y^*(t) \right\rangle dt = \int_0^T \langle y(t), y^*(t) \rangle, \quad \forall y^* \in L^{q'}(0, T; Y^*),$$

*for any* $y \in L^q(0,T;Y)$ *satisfying the inequality*

$$\sup\left\{\frac{1}{[x]_{L^p(0,T;M_0)}}\int_0^T \langle y(t), Ax(t)\rangle\, dt : x \in L^p(0,T;X_0)\right\} < \infty.$$

## 3. Assumptions and the Main Result

Suppose that following conditions are fulfilled for problem (1.1):

**(U.1)** Let $p \geq 2$, $a(\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists positive constants $0 < m \leq M < \infty$ such that

$$m \leq a(s) \leq M, \quad \forall s \in \mathbb{R}. \tag{3.1}$$

**(U.2)** Assume $\alpha : Q_T \to \mathbb{R}$ is a measurable function and satisfies $1 < \alpha^- \leq \alpha(x,t) \leq \alpha^+ < p$. Also, the coefficient $b(x,t)$ is a measurable function defined on $Q_T$ that holds $|b(x,t)| \leq B_0$.

We study problem (1.1) for the functions $f \in L^q\left(0,T;W^{-1,q}(\Omega)\right)$, where the dual space is defined $W^{-1,q}(\Omega) := \left(W_0^{1,p}(\Omega)\right)^*$ for $q := \frac{p}{p-1}$.

Let us denote $S_0$ by

$$S_0 := L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right) \cap W^{1,q}\left(0,T;W^{-1,q}(\Omega)\right) \cap \{u : u(x,0) = 0\}.$$

The solution of the problem (1.1) is understood in the following sense:

**Definition 3.1.** A function $u : Q_T \to \mathbb{R}$ is called a weak solution of problem (1.1) if

(i) $u \in S_0$;

(ii) for every test-function $\eta \in L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right) \cap W^{1,q}\left(0,T;W^{-1,q}(\Omega)\right)$

$$\int_{Q_T} u_t\eta\, dz + \sum_{i=1}^n \int_{Q_T}\left(a(\|u\|_p^p)\,|u|^{p-2} D_i u\right) D_i\eta\, dz + \int_{Q_T} b(x,t)|u|^{\alpha(x,t)-2}u\log|u|\eta\, dz = \int_{Q_T} f\eta\, dz.$$

We now present the main result of this paper:

**Theorem 3.2.** *Let the conditions* **(U.1)-(U.2)** are fulfilled. Then, for all $f \in L^q\left(0,T;W^{-1,q}(\Omega)\right)$ problem (1.1) has a weak solution in the space $S_0$ and $\partial u/\partial t$ belongs to $L^q\left(0,T;W^{-1,q}(\Omega)\right)$.

We introduce the following mappings in order to apply Theorem 2.8 to prove Theorem 3.2.

$$\begin{aligned}
&\xi : S_0 \longrightarrow L^q\left(0,T;W^{-1,q}(\Omega)\right),\\
&\xi(u) := -a(\|u\|_p^p)\sum_{i=1}^n D_i\left(|u|^{p-2} D_i u\right) + b(x,t)|u|^{\alpha(x,t)-2}u\log|u|,\\
&A : L^p\left(0,T;W_0^{1,p}(\Omega)\right) \subset S_0 \longrightarrow L^p\left(0,T;W_0^{1,p}(\Omega)\right),\\
&A(u) := u.
\end{aligned}$$

We prove several lemmas to show that all conditions of Theorem 2.8 are fulfilled under the conditions of Theorem 3.2.

**Lemma 3.3.** *Under the conditions of Theorem 3.2, $\xi$ and $A$ generate a "coercive pair" on* $L^p\left(0,T;W_0^{1,p}(\Omega)\right)$.

*Proof.* Since $A \equiv Id$, being "coercive pair" equals to order coercivity of $\xi$ on $L^p\left(0,T;W_0^{1,p}(\Omega)\right)$.

For $u \in L^p\left(0,T;W_0^{1,p}(\Omega)\right)$, we have the following:

$$\langle\xi(u), u\rangle_{Q_T} = \sum_{i=1}^n\left(\int_0^T a(\|u\|_p^p)\int_\Omega |u|^{p-2}|D_i u|^2\, dxdt\right) + \int_{Q_T} b(x,t)|u|^{\alpha(x,t)}\log|u|\, dz.$$

By using (3.1) we have

$$\langle \xi(u), u\rangle_{Q_T} \geq m \sum_{i=1}^{n} \left( \int_0^T \int_\Omega |u|^{p-2} |D_i u|^2 \, dz \right) - \int_{Q_T} |b(x,t)||u|^{\alpha(x,t)}| \log|u|| \, dz.$$

From Definition 2.3 and condition **(U.2)** we obtain

$$\langle \xi(u), u\rangle_{Q_T} \geq m \, [u]^p_{L^p\left(0,T;\mathring{S}_{1,(p-2),2}(\Omega)\right)} - B_0 \int_{Q_T} |u|^{\alpha(x,t)} | \log|u|| \, dz. \tag{3.2}$$

If we consider the embedding

$$\mathring{S}_{1,(p-2),2}(\Omega) \subset \mathring{S}_{1,(p-2)q,q}(\Omega)$$

to estimate pseudo-norm in (3.2) and as $\alpha^+ < p$, utilizing Lemma 2.6 for some $\sigma > 0$ such that $\alpha^+ < \alpha^+ + \sigma \leq p$ to estimate the second integral in (3.2), then we get

$$\langle \xi(u), u\rangle_{Q_T} \geq mC \, [u]^p_{L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)} - C_0 \int_{Q_T} |u|^{\alpha(x,t)+\sigma} \, dz - C_1. \tag{3.3}$$

Using (2.2) to estimate the integral right-hand side of (3.3), we obtain

$$\langle \xi(u), u\rangle_{Q_T} \geq mC \, [u]^p_{L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)} - C_0(\|u\|^{\alpha^+ + \sigma}_{L^{\alpha(x,t)+\sigma}(Q_T)} + 1) - C_2. \tag{3.4}$$

By applying Young's inequality in (3.4) for $\epsilon \in (0, 1)$, we have

$$\langle \xi(u), u\rangle_{Q_T} \geq mC \, [u]^p_{L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)} - \epsilon \|u\|^p_{L^{\alpha(x,t)+\sigma}(Q_T)} - C_3. \tag{3.5}$$

By considering the embedding

$$L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right) \subset L^p(Q_T) \subset L^{\alpha(x,t)+\sigma}(Q_T) \tag{3.6}$$

into (3.5) to estimate the second norm and choosing $\epsilon$ sufficiently small, we obtain

$$\langle \xi(u), u\rangle_{Q_T} \geq C_4 \, [u]^p_{L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)} - C_3. \tag{3.7}$$

Here, $C_3 = C_3(p, \alpha^+, \sigma, B_0, |\Omega|)$, $C_4 = C_4(p, m, \alpha^+, B_0, |\Omega|)$ are positive constants. So from (3.7) the proof is completed. □

**Lemma 3.4.** *Under the conditions of Theorem 3.2, $\xi$ is bounded from $S_0$ into $L^q\left(0,T;W^{-1,q}(\Omega)\right)$.*

*Proof.* We define the mappings

$$\xi_1(u) := -a(\|u\|_p^p) \sum_{i=1}^{n} D_i\left(|u|^{p-2} D_i u\right),$$
$$\xi_2(u) := b(x,t)|u|^{\alpha(x,t)-2} u \log|u|.$$

We need to show that, these mappings are both bounded from $L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)$ into $L^q\left(0,T;W^{-1,q}(\Omega)\right)$.

Let us show that $\xi_1$ is bounded: For $u \in L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)$ and $v \in L^p\left(0,T;W_0^{1,p}(\Omega)\right)$

$$\left|\langle \xi_1(u), v\rangle_{Q_T}\right| \leq \sum_{i=1}^{n} \left( \int_0^T a(\|u\|_p^p) \int_\Omega |u|^{p-2} |D_i u| |D_i v| \, dxdt \right).$$

Applying Hölder's inequality and by (3.1) we find,

$$\left|\langle \xi_1(u), v\rangle_{Q_T}\right| \leq M\left[\sum_{i=1}^{n}\left(\int_0^T\int_\Omega |u|^{(p-2)q}|D_i u|^q\,dxdt\right)\right]^{\frac{1}{q}}\left[\sum_{i=1}^{n}\left(\int_0^T\int_\Omega |D_i v|^p\,dxdt\right)\right]^{\frac{1}{p}}$$
$$= M[u]^{p-1}_{L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)}\|v\|_{L^p\left(0,T;W_0^{1,p}(\Omega)\right)}.$$

By the last inequality, boundedness of $\xi_1$ is obtained.

On the other hand, for the boundeness of $\xi_2$ from (3.6) and Theorem 2.5, it is sufficient to show that $\xi_2(u)$ is bounded in $L^{\alpha^*(x,t)}(Q_T)$ where $\alpha^*$ is conjugate of $\alpha$ i.e. $\alpha^*(x,t) := \frac{\alpha(x,t)}{\alpha(x,t)-1}$.

$$\zeta_{\alpha^*}(\xi_2(u)) = \int_{Q_T}\left(|b(x,t)||u|^{\alpha(x,t)-1}|\log|u||\right)^{\alpha^*(x,t)}dz$$
$$\leq C_5\int_{Q_T}|u|^{\alpha(x,t)}|\log|u||^{\alpha^*(x,t)}dz.$$

By applying Lemma 2.7 to the last integral above, we get

$$\begin{aligned}\zeta_{\alpha^*}(\xi_2(u)) &\leq C_6\int_{Q_T}|u|^{\alpha(x,t)+\sigma}dz + C_7\\ &\leq C_6(\|u\|^{\alpha^++\sigma}_{L^{\alpha(x,t)+\sigma}(Q_T)}+1)+C_7.\end{aligned}\tag{3.8}$$

Estimating the right side of (3.8) by the help of embedding (3.6), we obtain

$$\zeta_{\alpha^*}(\xi_2(u)) \leq C_8[u]^p_{L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)} + C_9,$$

here $C_8 = C_8(\alpha^+, B_0, \sigma, p)$, $C_9 = C_9(\alpha^+, \sigma, B_0, p, |\Omega|) > 0$ are constants.

That yields $\xi_2 : L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right) \to L^{\alpha^*(x,t)}(Q_T) \subset L^q\left(0,T;W^{-1,q}(\Omega)\right)$ is bounded. □

**Lemma 3.5.** *Under the conditions of Theorem 3.2, $\xi$ is weakly compact from $S_0$ into $L^q\left(0,T;W^{-1,q}(\Omega)\right)$.*

*Proof.* We first prove the weak compactness of $\xi_1$, where $\xi_1(u) := -a(\|u\|_p^p)\sum_{i=1}^n D_i\left(|u|^{p-2}D_i u\right)$. Let $\{u_m(x,t)\}_{m=1}^\infty \subset S_0$ be bounded and $u_m \overset{S_0}{\rightharpoonup} \tilde{u}_0$. It is sufficient to find a subsequence of $\left\{u_{m_j}\right\}_{m=1}^\infty \subset \{u_m\}_{m=1}^\infty$ which satisfies $\xi_1\left(u_{m_j}\right) \overset{L^q\left(0,T;W^{-1,q}(\Omega)\right)}{\rightharpoonup} \xi_1(\tilde{u}_0)$.

For a.e. $t \in (0,T)$, $u_m(\cdot,t) \in \mathring{S}_{1,(p-2)q,q}(\Omega)$ and by using the one-to-one correspondence between the classes (Theorem 2.4)

$$\mathring{S}_{1,(p-2)q,q}(\Omega) \underset{\varphi^{-1}}{\overset{\varphi}{\longleftrightarrow}} W_0^{1,q}(\Omega)$$

with the homeomorphism

$$\varphi(\tau) \equiv |\tau|^{p-2}\tau, \quad \varphi^{-1}(\tau) \equiv |\tau|^{-\frac{p-2}{p-1}}\tau,$$

for all $m \geq 1$ we have

$$|u_m|^{p-2}u_m \in L^q\left(0,T;W_0^{1,q}(\Omega)\right) \text{ is bounded.}$$

Due to the fact $L^q\left(0,T;W_0^{1,q}(\Omega)\right)$ is a reflexive space, there exists a subsequence $\left\{u_{m_j}\right\}_{m=1}^\infty \subset \{u_m\}_{m=1}^\infty$ such that

$$\left|u_{m_j}\right|^{p-2}u_{m_j} \overset{L^q\left(0,T;W_0^{1,q}(\Omega)\right)}{\rightharpoonup} \zeta. \tag{3.9}$$

Now, we show that $\zeta = |\tilde{u}_0|^{p-2}\tilde{u}_0$. According to compact embedding [32],

$$L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right) \cap W^{1,q}\left(0,T;W^{-1,q}(\Omega)\right) \hookrightarrow L^p(Q_T) \tag{3.10}$$

we have

$$\exists \left\{u_{m_{j_k}}\right\}_{m=1}^{\infty} \subset \left\{u_{m_j}\right\}_{m=1}^{\infty}, u_{m_{j_k}} \overset{L^p(Q_T)}{\to} \tilde{u_0}$$

which implies

$$u_{m_{j_k}} \underset{a.e}{\overset{Q_T}{\to}} \tilde{u_0}$$

by the continuity of $\varphi(\cdot)$, we get

$$\left|u_{m_{j_k}}\right|^{p-2} u_{m_{j_k}} \underset{a.e}{\overset{Q_T}{\to}} |\tilde{u_0}|^{p-2} \tilde{u_0}$$

that yields $\zeta = |\tilde{u_0}|^{p-2} \tilde{u_0}$.

By the the compact embedding (3.10) and continuity of $a(\cdot)$ we have

$$a(\|u_{m_{j_k}}(t)\|_p^p) \to a(\|\tilde{u_0}(t)\|_p^p) \quad \text{a.e. in } (0,T). \tag{3.11}$$

Thus, by using (3.9) and (3.11) together with the boundedness of $\xi_1$ from Lemma 3.4, we deduce that for each $v \in L^p\left(0,T;W_0^{1,p}(\Omega)\right)$

$$\begin{aligned}\langle \xi_1\left(u_{m_{j_k}}\right), v\rangle_{Q_T} &= \sum_{i=1}^{n}\int_0^T a(\|u_{m_{j_k}}\|_p^p)\langle -D_i\left(\left|u_{m_{j_k}}\right|^{p-2} D_i u_{m_{j_k}}\right), v\rangle_\Omega\, dt \\ &\underset{m_j\nearrow\infty}{\longrightarrow} \sum_{i=1}^{n}\int_0^T a(\|\tilde{u_0}(t)\|_p^p)\langle -D_i\left(|\tilde{u_0}|^{p-2} D_i\tilde{u_0}\right), v\rangle_\Omega\, dt = \langle \xi_1(\tilde{u_0}), v\rangle_{Q_T},\end{aligned}$$

whence, the result is obtained.

We now show the weak compactness of $\xi_2$: By Lemma 3.4

$$\xi_2 : L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right) \to L^{\alpha^*(x,t)}(Q_T)$$

is bounded. Thus, for $m \geq 1$, $\xi_2(u_m) = \left\{b(x,t)|u_m|^{\alpha(x,t)-2}u_m \log|u_m|\right\}_{m=1}^{\infty} \subset L^{\alpha^*(x,t)}(Q_T)$.

From Lemma 2.2, $L^{\alpha^*(x,t)}(Q_T)$ $(1 < (\alpha^*)^- < \infty)$ is a reflexive space, so $\{u_m\}_{m=1}^{\infty}$ has a subsequence $\left\{u_{m_j}\right\}_{m=1}^{\infty}$ such that

$$|u_m|^{\alpha(x,t)-2}u_m \log|u_m| \overset{L^{\alpha^*(x,t)}(Q_T)}{\rightharpoonup} \psi.$$

We deduce from the compact embedding (3.10) that

$$\exists \left\{u_{m_{j_k}}\right\}_{m=1}^{\infty} \subset \left\{u_{m_j}\right\}_{m=1}^{\infty}, u_{m_{j_k}} \overset{L^p(Q_T)}{\to} \tilde{u_0}.$$

Thus,

$$u_{m_{j_k}} \underset{a.e}{\overset{Q_T}{\to}} \tilde{u_0}.$$

Accordingly, for almost $(x,t) \in Q_T$ the continuity of $|\tau|^{\alpha(x,t)-2}\tau\log|\tau|$ with respect to $\tau$ implies that

$$|u_{m_{j_k}}|^{\alpha(x,t)-2}u_{m_{j_k}} \log|u_{m_{j_k}}| \underset{a.e}{\overset{Q_T}{\to}} |\tilde{u_0}|^{\alpha(x,t)-2}\tilde{u_0}\log|\tilde{u_0}|,$$

so, we arrive at $\psi = |\tilde{u_0}|^{\alpha(x,t)-2}\tilde{u_0}\log|\tilde{u_0}|$ i.e. $\xi_2(u_{m_{j_k}}) \overset{L^q\left(0,T;W^{-1,q}(\Omega)\right)}{\rightharpoonup} \xi_2(\tilde{u_0})$. Thus, we conclude that $\xi$ is weakly compact from $S_0$ into $L^q\left(0,T;W^{-1,q}(\Omega)\right)$. □

Now, we give the proof of main theorem of this section.

*Proof of Theorem 3.2*. Since $A = Id$, obviously it is a linear bounded map and satisfies the conditions (ii) of Theorem 2.8. Furthermore for any $u \in W_0^{1,p}(Q_T)$ the following inequalities are valid:

$$\int_0^T \langle u,u\rangle_\Omega\, dt = \int_0^T \|u\|_{L^2(\Omega)}^2\, dt \geq K\|u\|^2_{L^q\left(0,T;W^{-1,q}(\Omega)\right)}$$

and

$$\int_0^t \left\langle \frac{\partial u}{\partial \tau}, u \right\rangle_\Omega d\tau = \frac{1}{2} \|u(t)\|^2_{L^2(\Omega)} \geq K\frac{1}{2} \|u(t)\|^2_{W^{-1,q}(\Omega)},$$

a.e. $t \in [0, T]$ (constant $K > 0$ comes from embedding inequality). Thus, condition (iv) of Theorem 2.8 is satisfied as well. Consequently from Lemma 3.3-Lemma 3.5, it follows that the mappings $\xi$ and $A$ fulfill all the conditions of Theorem 2.8. Employing this theorem to problem (1.1), we find that (1.1) is solvable in $S_0$ for any $f \in L^q\left(0, T; W^{-1,q}(\Omega)\right)$ satisfying the following inequality

$$\sup\left\{ \frac{1}{[u]_{L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)}} \int_0^T \langle f, u\rangle_\Omega \, dt : u \in L^p\left(0, T; W_0^{1,p}(\Omega)\right) \right\} < \infty.$$

Considering the norm definition of $f$ in $L^q\left(0, T; W^{-1,q}(\Omega)\right)$, we conclude that (1.1) is solvable in $S_0$ for any $f \in L^q\left(0, T; W^{-1,q}(\Omega)\right)$. In order to complete the proof, it remains to remark that (1.1) can be written in the form

$$\frac{\partial u}{\partial t} = f(x, t) - F(x, t, u, D_i u),$$

and under the conditions of Theorem 3.2, right hand belongs to $L^q\left(0, T; W^{-1,q}(\Omega)\right)$ which implies
$\partial u/\partial t \in L^q\left(0, T; W^{-1,q}(\Omega)\right)$.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

## Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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