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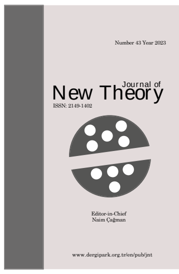

Chebyshev Collocation Method for the Fractional Fredholm Integro-Differential Equations

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Abstract — In this study, Chebyshev polynomials have been applied to construct an approximation method to attain the solutions of the linear fractional Fredholm integro-differential equations (IDEs). By this approximation method, the fractional IDE has been transformed into a linear algebraic equations system with the aid of the collocation points. In the method, the conformable fractional derivatives of the Chebyshev polynomials have been calculated in terms of the Chebyshev polynomials. Using the results of these calculations, the matrix relation for the conformable fractional derivatives of Chebyshev polynomials was attained for the first time in the literature. After that, the matrix forms have been replaced with the corresponding terms in the given fractional integro-differential equation, and the collocation points have been used to have a linear algebraic system. Furthermore, some numerical examples have been presented to demonstrate the preciseness of the method. It is inferable from these examples that the solutions have been obtained as the exact solutions or approximate solutions with minimum errors.

Keywords *Conformable fractional derivative, Chebyshev polynomials, numerical solutions*

Mathematics Subject Classification (2020) 26A33, 33C45

1. Introduction

The theory of fractional derivatives plays an impressive role in the field of the study of applied mathematics to analyze innumerable problems through the diverse areas of engineering and science, such as bioengineering, mathematical physics, astrophysics, hydrology, control theory, biophysics, statistical mechanics, thermodynamics, cosmology, and finance [1]. As much as the theory of fractional derivatives has drawn considerable attention among scientists, especially mathematicians, investigating the solution methods for the fractional linear and nonlinear IDEs has been the focus point continually in the last decades [2, 3]. The methods utilized to obtain the solutions of the Fredholm IDEs, fractional in the Caputo sense with the aid of the Chebyshev polynomials are given as the Chebyshev wavelet method of the second kind [4, 5] and least squares method [6, 7]. Besides, Chebyshev wavelet methods of the second kind [8-10] and the fourth kind [11] have been applied to attain the solutions of the fractional integro-differential equations of the Fredholm-Volterra type in the sense of the Caputo differentiation operator.

Moreover, investigating the exact and numerical solutions of the fractional integro-differential equations in the conformable sense is a fresh and strange field of investigation among applied mathematicians. Preliminarily, Bayram et al. [12] have applied the Sinc-collocation method, and Daşcıoğlu et al. [13] have used a collocation method based upon the Laguerre polynomials to attain the solutions of the linear fractional IDEs in the conformable sense. This method mentioned in [13] is an improvement of the method that had been used for

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the solutions of the linear Caputo fractional IDEs of the Volterra type [14] and Caputo fractional linear IDEs of the Fredholm type [15]. However, for the conformable fractional Fredholm IDEs, there has not been a method in the literature in the sense of Chebyshev polynomials. To this respect, in that study, a method predicated on the Chebyshev polynomials of the first kind is announced to obtain the numerical (in some cases exact) solutions of the linear conformable fractional integro-differential equation of the Fredholm type having the fractionality in the differential part as

$$\sum_{i=0}^{m} p_i(x)D^{\alpha_i}y(x) = g(x) + \lambda \int_{-1}^{1} K(x,t)y(t)dt, \quad -1 \le x \le 1 \tag{1}$$

with the initial conditions

$$y(0) = c_0 \tag{2}$$

where $l \in N, \lambda \in R$, $0 < \alpha_i \le 1$, $K(x,t)$, p_i, and g are given (known) functions, $y(x)$ stands for the unknown function to be found, and $D^{\alpha_i}y(x)$ represents the fractional derivative in the conformable sense of the unknown function $y(x)$.

In the present paper, Section 2 provides the basic definitions and their properties. Section 3 constitutes the fundamental matrix relations for each term in the fractional integro-differential equation provided in Equation 1. Section 4 presents a well-functional collocation method based on the Chebyshev polynomials. Section 5 resolves some numerical examples and exhibits their results to affirm the preciseness and effectiveness of the introduced method. Finally, the last section discusses the need for further research.

2. Preliminaries

This section provides some basic notions to be needed in the following sections.

Definition 2.1. [16] The conformable fractional derivative of a function f of the α-th order is described as

$$D^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \qquad t > 0, \quad \alpha \in (0,1)$$

where $f: [0, \infty) \to \mathbb{R}$. Here, if the function f is differentiable of the order α in the conformable sense in some open interval $(0, \alpha)$ and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then $\lim_{t \to 0^+} f^{(\alpha)}(t) = f^{(\alpha)}(0)$.

Since we have become familiar with the definition of the conformable fractional derivative, it is obvious that the notion of the conformable fractional derivative is the most analogous to the classical definition of the usual derivative. By the theorem below, we recognize the similarity between the conformable fractional derivative and the ordinary derivative:

Theorem 2.2. [16] Suppose that $\alpha \in (0,1]$ and the functions f and g are differentiable of the order α in the conformable sense at the point $t > 0$. Therefore, the following statements are satisfied.

i. $D^{\alpha}(af + bg) = aD^{\alpha}(f) + bD^{\alpha}(g)$, for all $a, b \in \mathbb{R}$

ii. $D^{\alpha}(t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$

iii. $D^{\alpha}(\lambda) = 0$ for all constant functions $f(t) = \lambda$

iv. $D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f)$

v. $D^{\alpha}\left(\frac{f}{g}\right) = \frac{gD^{\alpha}(f) - fD^{\alpha}(g)}{g^2}$

vi. If f is differentiable, then $D^{\alpha}(f)(t) = t^{1-\alpha}\frac{df}{dt}$

Proposition 2.3. [16] The obtained expression for the fractional derivative of the power function x^k in the conformable sense for $k \in \{0,1,2,\cdots\}$

$$D^\alpha x^k = \begin{cases} 0, & k < 1 \\ kx^{k-\alpha}, & k \geq 1 \end{cases}$$

The following theorem introduces the chain rule for the conformable fractional derivative:

Theorem 2.4. [17] Assume $f, g : (0,\infty) \to \mathbb{R}$ be the differentiable functions of the order α in the conformable sense where $0 < \alpha \leq 1$. Suppose $h(t) = f\big(g(t)\big)$. Then, the composite function $h(t)$ is differentiable of the order α in the conformable sense and, for all t with $t \neq 0$ and $g(t) \neq 0$,

$$D^\alpha(h)(t) = D^\alpha(f)\big(g(t)\big).D^\alpha(g)(t).g(t)^{\alpha-1}$$

For $t = 0$, we can use the following limit

$$D^\alpha(h)(0) = \lim_{t\to 0} D^\alpha(f)\big(g(t)\big).D^\alpha(g)(t).g(t)^{\alpha-1}$$

The fundamental goal of this research is to introduce a useful approximation method that will provide an approximate solution (in some cases an exact solution) of the fractional Fredholm integro-differential equation in Problem 1 under the Condition 2 in the type

$$y(x) \cong y_N(x) = \sum_{i=0}^{N} a_i T_i(x) \tag{3}$$

where the upper limit of the sum $N \geq 1$ is any selected positive integer, the term T_i stand for the Chebyshev polynomials of the first kind of the order i, and the coefficients a_i are unknown and to be determined. Afterward, we provide the definition of the Chebyshev polynomials:

Definition 2.5. [18] The Chebyshev polynomial of degree n of the first kind is a polynomial in variable x is denoted by $T_n(x)$ and defined as

$$T_n(x) = \cos n\theta, \quad \cos\theta = x, \quad -1 \leq x \leq 1$$

Moreover, these well-known Chebyshev polynomials satisfy the following recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n \in \{2,3,\cdots\}$$

together with the initial conditions $T_0(x) = 1$ and $T_1(x) = x$ recursively generates all the polynomials $\{T_n(x)\}$ efficiently.

Furthermore, the following properties present the relation between the Chebyshev polynomials and the power function:

Proposition 2.6. [18] The Chebyshev polynomials are provided in terms of the powers of x as

$$T_n(x) = \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \left[(-1)^k \sum_{j=k}^{\left\lfloor\frac{n}{2}\right\rfloor} \binom{n}{2j}\binom{j}{k}\right] x^{n-2k}$$

or

$$T_n(x) = \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} (-1)^k 2^{n-2k-1} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}$$

where $\left\lfloor\frac{n}{2}\right\rfloor$ denotes the integer part of $\frac{n}{2}$.

Proposition 2.7. [18] The famous Chebyshev series in the Chebyshev polynomials of the first kind of the power function x^n has been stated as

$$x^n = 2^{1-n} \sum_{i=0'}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{i} T_{n-2i}(x), \quad n \in \{0,1,2,\cdots\}$$

where the dashed sigma stands for that the ith term in the sum is to be halved if n is even and $i = \frac{n}{2}$; in other words, the term in $T_0(x)$, if there is one, is to be halved.

3. Elementary Matrix Formulas

In this part of the paper, we transform Equation 1 by formulating the matrix forms of the unknown function and the fractional derivative of that function in a conformable sense. First, we can formulate the approximate solution in Equation 3 as the product of the Chebyshev matrix $\mathrm{T}(x)$ and the coefficient matrix A by

$$y_N(x) = \mathrm{T}(x)\mathrm{A} \tag{4}$$

where the matrices are as follows:

$$\mathrm{T}(x) = [T_0(x) \quad T_1(x) \quad \cdots \quad T_N(x)] \text{ and } \mathrm{A} = [a_0 \quad a_1 \quad \cdots \quad a_N]^T$$

For that purpose, we prove a theorem that states the relation between the conformable fractional derivative of the Chebyshev polynomials and the Chebyshev polynomials of the first kind:

Theorem 3.1. Suppose that $T_i(x)$ denotes the ith order Chebyshev polynomial of the first kind. Then, the fractional derivative of the Chebyshev polynomial $T_i(x)$ in the conformable sense in terms of the Chebyshev polynomials of the first kind are constructed as:

$$D^\alpha T_0(x) = 0 \tag{5}$$

and otherwise

$$D^\alpha T_n(x) = x^{1-\alpha} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{j=k}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{j}{k} \binom{n}{2j} (n-2k) 2^{2k-n+2} \sum_{i=0'}^{\left\lfloor \frac{n-2k-1}{2} \right\rfloor} \binom{n-2k-1}{i} T_{n-2k-2i-1}(x) \tag{6}$$

where $\lfloor n \rfloor$ denotes the integer part of n and the dashed sigma ($\Sigma_{\prime}$) stands for that the ith term in the sum is to be halved if $n-2k-1$ is even and $i = \frac{n-2k-1}{2}$.

PROOF. We will originate with the expression of the Chebyshev polynomials in terms of the powers of x, and α-differentiate these polynomials as

$$D^\alpha T_n(x) = D^\alpha \left\{ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left[(-1)^k \sum_{j=k}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} \binom{j}{k} \right] x^{n-2k} \right\}$$

Since the conformable fractional derivative is linear, we have the equality

$$D^\alpha T_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left[(-1)^k \sum_{j=k}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} \binom{j}{k} \right] D^\alpha \left(x^{n-2k} \right)$$

Utilizing the conformable fractional derivative of the power function x^k, for $k \in \{0,1,2,\cdots\}$,

$$D^\alpha x^k = \begin{cases} 0, & k < 1 \\ kx^{k-\alpha}, & k \geq 1 \end{cases}$$

we obtain $D^\alpha T_0(x) = 0$ and

$$D^\alpha T_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \sum_{j=k}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j}\binom{j}{k}(n-2k)x^{n-2k-\alpha}, \quad n \in \{1,2,\cdots\}$$

At that point, we will take the term $x^{1-\alpha}$ out of the series since it is independent of the indices of the sums

$$D^\alpha T_n(x) = x^{1-\alpha} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \sum_{j=k}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j}\binom{j}{k}(n-2k)x^{n-2k-1}, \quad n \in \{1,2,\cdots\}$$

and utilizing the Chebyshev series of x^n mentioned with Property 2

$$x^n = 2^{1-n} \sum_{i=0'}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{i} T_{n-2i}(x), \quad n \in \{0,1,2,\cdots\}$$

where the dashed sigma stands for that the ith term in the sum is to be halved if n is even and $i = \frac{n}{2}$; in other words, the term in $T_0(x)$, if there is one, is to be halved; we get the statement of the formulas given by Equations 5 and 6, and the proof of Theorem 3.1 is accomplished. □

Theorem 3.2. Suppose that $\mathrm{T}(x)$ is a row matrix with $(N+1)$ columns and is called as Chebyshev matrix, and $D^\alpha \mathrm{T}(x)$ stands for the conformable fractional derivative of α-th order of the Chebyshev matrix $\mathrm{T}(x)$. Then, the matrix relation for the conformable fractional derivative of $\mathrm{T}(x)$ is attained as

$$D^\alpha \mathrm{T}(x) = 2x^{1-\alpha}\mathrm{T}(x)\mathrm{M} \tag{7}$$

where the $(N+1)$ dimensional square matrix M is characterized by odd N as

$$\mathrm{M} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \cdots & \frac{N}{2} \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and for even N

$$\mathrm{M} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \cdots & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & N \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Proof.

The explicit forms of the Chebyshev matrix $\mathrm{T}(x)$ and $D^{\alpha}\mathrm{T}(x)$ are

$$\mathrm{T}(x) = [T_0(x) \quad T_1(x) \quad \cdots \quad T_N(x)]$$

and

$$D^{\alpha}\mathrm{T}(x) = [D^{\alpha}T_0(x) \quad D^{\alpha}T_1(x) \quad \cdots \quad D^{\alpha}T_N(x)]$$

The statement of Theorem 3.1. is utilized to obtain the relation between the matrices above. Using Equations 5 and 6, the terms in $D^{\alpha}\mathrm{T}(x)$ can be expressed explicitly, for $n \in \{0,1,\dots,N\}$, as formulated below:

For $n = 0$,

$$D^{\alpha}T_0(x) = 0$$

For $n = 1$,

$$\begin{aligned} D^{\alpha}T_1(x) &= x^{1-\alpha}\sum_{k=0}^{\left\lfloor\frac{1}{2}\right\rfloor}\sum_{j=k}^{\left\lfloor\frac{1}{2}\right\rfloor}(-1)^k\binom{1}{2j}\binom{j}{k}(1-2k)2^{2k-1+2}\sum_{i=0'}^{\left\lfloor\frac{1-2k-1}{2}\right\rfloor}\binom{1-2k-1}{i}T_{1-2k-2i-1}(x) \\ &= x^{1-\alpha}T_0(x) \\ &= 2x^{1-\alpha}\left[\frac{1}{2}T_0(x)\right] \end{aligned}$$

For $n = 2$,

$$\begin{aligned} D^{\alpha}T_2(x) &= x^{1-\alpha}\sum_{k=0}^{\lfloor 1\rfloor}\sum_{j=k}^{\lfloor 1\rfloor}(-1)^k\binom{2}{2j}\binom{j}{k}(2-2k)2^{2k-2+2}\sum_{i=0'}^{\left\lfloor\frac{2-2k-1}{2}\right\rfloor}\binom{2-2k-1}{i}T_{2-2k-2i-1}(x) \\ &= 4x^{1-\alpha}T_1(x) \\ &= 2x^{1-\alpha}[2T_1(x)] \end{aligned}$$

For $j = N$ and odd N,

$$\begin{aligned} D^{\alpha}T_N(x) &= x^{1-\alpha}\sum_{k=0}^{\left\lfloor\frac{N}{2}\right\rfloor}\sum_{j=k}^{\left\lfloor\frac{N}{2}\right\rfloor}(-1)^k\binom{N}{2j}\binom{j}{k}(N-2k)2^{2k-N+2}\sum_{i=0'}^{\left\lfloor\frac{N-2k-1}{2}\right\rfloor}\binom{N-2k-1}{i}T_{N-2k-2i-1}(x) \\ &= 2x^{1-\alpha}\left[\frac{N}{2}T_0(x) + NT_2(x) + \ldots + NT_{N-1}(x)\right] \end{aligned}$$

and for even N

$$\begin{aligned} D^{\alpha}T_N(x) &= x^{1-\alpha}\sum_{k=0}^{\left\lfloor\frac{N}{2}\right\rfloor}\sum_{j=k}^{\left\lfloor\frac{N}{2}\right\rfloor}(-1)^k\binom{N}{2j}\binom{j}{k}(N-2k)2^{2k-N+2}\sum_{i=0'}^{\left\lfloor\frac{N-2k-1}{2}\right\rfloor}\binom{N-2k-1}{i}T_{N-2k-2i-1}(x) \\ &= 2x^{1-\alpha}[NT_1(x) + NT_3(x) + \ldots + NT_N(x)] \end{aligned}$$

It can be observed that the relation between the fractional derivative of the Chebyshev matrix in the conformable sense $D^{\alpha}\mathrm{T}(x)$ and the Chebyshev matrix $\mathrm{T}(x)$ is in the form as stated in Equation 7. This proves the theorem. ☐

After that, by applying Equations 4 and 7, the left-hand side of Equation 1 could be expressed as

$$D^{\alpha}y(x) \cong D^{\alpha}\mathrm{T}(x)\mathrm{A} = x^{1-\alpha}\mathrm{T}(x)\mathrm{MA} \tag{8}$$

It is remarked that this matrix is the same as the matrix provided by Sezer et al. [19] and Akyüz [20] for the usual first-order derivative. Thus, it is obvious that there is a correlation between the methods for the conformable fractional derivative and the usual derivative.

Finally, the corresponding matrix relation of the conditions in Equation 2 is formulated as

$$y(0) = \mathrm{T}(0)\mathrm{A} = c_0 \tag{9}$$

At this stage, the condition matrix $\mathrm{T}(0)$ is referred to as U where the matrix U is a row matrix with $(N+1)$ columns. Thus, Equation 9 transforms into $\mathrm{UA} = c_0$.

4. Solution Method

In this section, we maintain the approximate solution method, which can be specified as a collocation method since we use the collocation points at the end to solve the matrix equation. In other words, we determine the unknown coefficients a_i in Equation 3 to attain the solution of Equations 1 and 2 by a collocation method.

Before all, we interchange the formulated matrix forms given with Equations 4 and 8 into Equation 1, and thus we attain the matrix equation of the fractional integro-differential equation

$$\sum_{i=0}^{l} p_i(x)x^{1-\alpha_i}2\mathrm{T}(x)\mathrm{MA} = g(x) + \lambda \int_{-1}^{1} K(x,t)\mathrm{T}(t)\mathrm{A}dt \tag{10}$$

Secondly, we substitute the chosen collocation points $x_s > 0$, for $s \in \{0,1,\dots,N\}$, into the matrix Equation 10, we get a linear system of the $N+1$ equations

$$\left\{\sum_{i=0}^{l} p_i(x_s){x_s}^{1-\alpha_i}2\mathrm{T}(x_s)\mathrm{MA} - \lambda f(x_s)\right\}\mathrm{A} = g(x_s) \tag{11}$$

where $f(x_s) = \int_{-1}^{1} K(x_s,t)\mathrm{T}(t)dt$. This linear system can be expressed in compact forms:

$$\left\{\sum_{i=0}^{l} 2\mathrm{P}_i\mathrm{X}_{\alpha_i}\mathrm{LTM} - \lambda\mathrm{F}\right\}\mathrm{A} = \mathrm{G} \tag{12}$$

where

$$\mathrm{X}_{\alpha_i} = \begin{bmatrix} x_0^{1-\alpha_i} & 0 & \cdots & 0 \\ 0 & x_1^{1-\alpha_i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_N^{1-\alpha_i} \end{bmatrix}, \quad \mathrm{T} = \begin{bmatrix} \mathrm{T}(x_0) \\ \mathrm{T}(x_1) \\ \vdots \\ \mathrm{T}(x_N) \end{bmatrix}, \quad \mathrm{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}$$

$$\mathrm{P}_i = \begin{bmatrix} p_i(x_0) & 0 & \cdots & 0 \\ 0 & p_i(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_i(x_N) \end{bmatrix}, \quad \text{and} \quad \mathrm{F} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}$$

After that, when we denote the formulation in parenthesis of Equation 12 by W, the main matrix equation for Equation 1 is abbreviated into the equation $\mathrm{WA} = \mathrm{G}$ representing a system of $N + 1$ linear algebraic equations with $N + 1$ undetermined Chebyshev coefficients a_i's, for $i \in \{0,1, \dots, N\}$.

Eventually, we solve the obtained linear algebraic system to calculate the unknown coefficients. For that purpose, there are several ways to solve this system, but we primarily use it to replace or to stack up the n rows of the augmented matrix $[\mathrm{W}; \mathrm{G}]$ with the rows of the augmented matrix $[\mathrm{U}; c_0]$. We choose the best way to get the most accurate solutions for each problem. Therefore, since the unknown Chebyshev coefficients are discovered by resolving this system, we end up with the solution of Equation 1 under Condition 2.

5. Numerical Examples

In that part of the paper, we use the presented method in the previous section for two different examples. The collocation points that are used to transform the equations have been formalized as

$$x_s = \frac{\left[1 - \cos\left(\frac{(s+1)\pi}{N+1}\right)\right]}{2}, \quad s \in \{0,1, \dots N\}$$

for these two examples. All the numerical calculations have been executed with the program Mathcad 15.

Example 5.1. The fractional Fredholm IDE in the form of Equation 1

$$D^{\frac{1}{2}}y(x) = y(x) + 2x^{1.5} - x^2 - \frac{2}{3} + \int_{-1}^{1} y(t)dt$$

subject to initial condition $y(0) = 0$ in the form of Equation 2.

It can easily be confirmed that the exact solution to the above problem is the polynomial solution of degree two, $y(x) = x^2$. Implementing the methodology explained in Section 4, the expected fundamental matrix equation of the problem and its conditions can be presented as $\mathrm{P}_0 = \mathrm{I}$, I is the identity matrix, $\lambda = 1$,

$$\left\{2\mathrm{X}_{\frac{1}{2}}\mathrm{TM} - \mathrm{T} - \mathrm{F}\right\}\mathrm{A} = \mathrm{G}, \text{ and } \mathrm{UA} = 0$$

When we select $N = 2$, the formula gives us the points $x_0 = 0.25$, $x_1 = 0.75$, and $x_2 = 1$ as the collocation points. Then, the matrices mentioned above are

$$\mathrm{X}_{\frac{1}{2}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{T} = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{7}{8} \\ 1 & \frac{3}{4} & \frac{1}{8} \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathrm{F} = \begin{bmatrix} 2 & 0 & -\frac{2}{3} \\ 2 & 0 & -\frac{2}{3} \\ 2 & 0 & -\frac{2}{3} \end{bmatrix},$$

$$\mathrm{G} = \begin{bmatrix} -\frac{23}{48} \\ \frac{3\sqrt{3}}{4} - \frac{59}{48} \\ \frac{1}{3} \end{bmatrix}, \quad \text{and} \quad \mathrm{U} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

As a result of the solution of the above system, the unknown coefficients in Equation 3, for $N = 2$, can be calculated as $a_0 = \frac{1}{2}$, $a_1 = 0$, and $a_2 = \frac{1}{2}$. In the final step, we substitute these coefficients into approximate Equation 3. Then, we obtain the exact solution.

Example 5.2. The Fredholm fractional IDE in the form of Equation 1

$$D^{\frac{1}{3}}y(x) = x^{\frac{2}{3}}y(x) - 2e^x + \int_{-1}^{1} e^{x-t}y(t)dt\,, 0 \le x \le 1$$

subject to $y(0) = 1$ having the exponential function e^x as the exact solution.

The exact solution could not be attained by the introduced method in Section 4 since this problem does not have a polynomial solution. Therefore, we attain the approximate solutions with some insignificant errors. The absolute maximum errors between the approximate solution obtained by the proposed method and the exponential function e^x, the exact solution to the given problem is stated in Table 1. In Table 1, the maximum absolute errors are calculated by interchanging the row in the last place of the evaluated augmented matrix $[\mathrm{W};\mathrm{G}]$ with the augmented matrix $[\mathrm{U};1]$, for the values $N \in \{2,4,6,8,10\}$ and the values $N \in \{14,16\}$; by stacking up the rows of the computed augmented matrices for this problem.

Table 1. The maximum errors of Example 2 for different N values

$N=2$	$N=4$	$N=6$	$N=8$	$N=10$	$N=14$	$N=16$
0.36	1.9×10^{-2}	6.0×10^{-5}	1.2×10^{-7}	1.3×10^{-10}	6.4×10^{-14}	5.4×10^{-14}

6. Conclusion

This paper uses Chebyshev polynomials to construct an approximation method to attain the solutions of the linear fractional Fredholm integro-differential equations (IDEs). By this approximation method, the fractional IDE has been transformed into a linear algebraic equations system with the aid of the collocation points. There are numerous methods for obtaining the solutions of the fractional IDEs in the Caputo differential operator sense. However, investigating the solutions of the fractional IDEs in the conformable differential operator sense is a new field of study among mathematicians. Therefore, the relation for the matrix of the conformable fractional derivative of the Chebyshev polynomials is attained for the first time in fractional calculus literature. The fractional IDE has been turned into an algebraic equations system using suitable collocation points and the obtained matrix relations. The proposed approximation method's simplicity and efficiency have been strengthened by the results of the numerical examples provided in the previous section. For future research, the fractional derivatives of the Chebyshev polynomials and the related matrix relations can be obtained for the different types of fractional derivatives, such as the Caputo fractional derivative and fractional beta derivative.

Author Contributions

The author read and approved the final version of the paper.

Conflict of Interest

The author declares no conflict of interest.

References

[1] S. Abbas, M. Benchohra, G. M. N'Guerekata, Advanced Fractional Differential and Integral Equations, Nova Science Publishers, New York, 2015.

[2] A. A. Hamoud, K. P. Ghadle, *The Approximate Solutions of Fractional Volterra-Fredholm Integrodifferential Equations by Using Analytical Techniques*, Problemy Analiza Issues of Analyses 7 (25) (2018) 41–58.

[3] A. Hamoud, K. Ghadle, *Usage of the Homotopy Analysis Method for Solving Fractional Volterra-Fredholm Integrodifferential Equation of the Second Kind*, Tamkang Journal of Mathematics 49 (4) (2018) 301–315.

[4] L. Zhu, Q. Fan, *Solving Fractional Nonlinear Fredholm Integro-Differential Equations by the Second Kind Chebyshev Wavelet*, Communications in Nonlinear Science and Numerical Simulation 17 (6) (2012) 2333–2341.

[5] A. Setia, Y. Liu, A. S. Vatsala, *Solution of Linear Fractional Fredholm Integro-differential Equation by Using Second Kind Chebyshev Wavelet*, in: S. Latifi (Ed.), 11th International Conference on Information Technology: New Generations, Las Vegas, 2014, pp. 465–469.

[6] A. M. S. Mahdy, E. M. H. Mohamed, G. M. A. Marai, *Numerical Solution of Fractional Integro-Differential Equations by Least Squares Method and Shifted Chebyshev Polynomials of the Third Kind Method*, Theoretical Mathematics & Applications 6 (4) (2016) 87–101.

[7] A. M. Mahdy, E. M. Mohamed, *Numerical Studies for Solving System of Linear Fractional Integro-Differential Equations by Using Least Squares Method and Shifted Chebyshev Polynomials*, Journal of Abstract and Computational Mathematics 1 (1) (2016) 24–32.

[8] Y. Wang, L. Zhu, *SCW Method for Solving the Fractional Integro-Differential Equations with a Weakly Singular Kernel*, Applied Mathematics and Computation 275 (2016) 72–80.

[9] A. Setia, Y. Liu, A. S. Vatsala, *Numerical Solution of Fredholm-Volterra Fractional Integro-Differential Equations with Nonlocal Boundary Conditions*, Journal of Fractional Calculus and Applications 5 (2) (2014) 155–165.

[10] S. T. Mohyud-Din, H. Khan, M. Arif, M. Rafiq, *Chebyshev Wavelet Method to Nonlinear Fractional Volterra–Fredholm Integro-Differential Equations with Mixed Boundary Conditions*, Advances in Mechanical Engineering 9 (3) (2017) 1–8.

[11] F. Zhou, X. Xu, *Numerical Solution of Fractional Volterra-Fredholm Integro-Differential Equations with Mixed Boundary Conditions via Chebyshev Wavelet Method*, International Journal of Computer Mathematics 96 (2) (2019) 436–456.

[12] M. Bayram, V. F. Hatipoğlu, S. Alkan, S. E. Das, *A Solution Method for Integro-Differential Equations of Conformable Fractional Derivative*, Thermal Science 22 (1) (2018) 7–14.

[13] A. Daşcıoğlu, D. Varol, *Laguerre Polynomial Solutions of Linear Fractional Integro-Differential Equations*, Mathematical Sciences 15 (1) (2021) 47–54.

[14] D. Varol Bayram, A. Daşcıoğlu, *A Method for Fractional Volterra Integro-Differential Equations by Laguerre Polynomials*, Advances in Difference Equations 2018 (2018) Article Number 466 11 pages.

[15] A. Daşcıoğlu, D. Varol Bayram, *Solving Fractional Fredholm Integro-Differential Equations by Laguerre Polynomials*, Sains Malaysiana 48 (1) (2019) 251–257.

[16] R. Khalil, M. A. Horani, A. Yousef, M. Sababheh, *A New Definition of Fractional Derivative*, The Journal of Computational and Applied Mathematics 264 (2014) 65–70.

[17] T. Abdeljawad, *On Conformable Fractional Calculus*, The Journal of Computational and Applied Mathematics 279 (2015) 57–66.

[18] J. C. Mason, D. C. Handscomb, Chebyshev Polynomials, Chapman & Hall /CRC, New York, 2002.

[19] M. Sezer, M. Kaynak, *Chebyshev Polynomial Solutions of Linear Differential Equations*, The International Journal of Mathematical Education in Science and Technology 27 (4) (1996) 607–618.

[20] A. Akyüz-Daşcıoğlu, *A Chebyshev Polynomial Approach for Linear Fredholm–Volterra Integro-Differential Equations in the Most General Form*, Applied Mathematics and Computation 181 (1) (2006) 103–112.