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On Neutrosophic Square Matrices and Solutions of Systems of Linear Equations

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Abstract: We started this work with a theorem that shows in which case the abbreviation rule for neutrosophic real numbers is true. We then detail in which cases the division of two neutrosophic real numbers yields a new neutrosophic number. Then, the solution cases of a neutrosophic linear equation with one unknown were examined. After calculating the determinant of a square matrix and giving the necessary and sufficient conditions for a square matrix to be invertible, the solution conditions of the systems of equations with the number of unknowns equal to the number of equations were examined.

Key words: Neutrosophic matrices, Neutrosophic systems of linear equations, Determinant of a neutrosophic square matrix, Inverse of neutrosophic square matrix

1. Introduction

Smarandache firstly studied the concept of neutrosophy to address the uncertainty in nature and science. [1]. Neutrosophy has important applications in a lot of fields and researchers done many studies on the subject. Neutrophic number theory and neutrophic linear algebra are just two of these fields. Some of the studies carried out in these areas are given in the references [2-10].

We firstly started this work with a theorem that shows in which case the abbreviation rule for neutrosophic real numbers is true. We then detail in which cases the division of two neutrosophic real numbers yields a new neutrosophic number. Then, the solution cases of a neutrosophic linear equation with one unknown were examined. After calculating the determinant of a square matrix and giving the necessary and sufficient conditions for a square matrix to be invertible, the solution conditions of the systems of equations with the number of unknowns equal to the number of equations were examined. Also, we gave some important examples to clarify the theory.

2. Material and Method

In this section, some definitions and theorems of neutrosophic numbers and matrices will be given, which we will use in later chapters. As known, the set of neutrosophic real numbers $is\ R[I] = \{\alpha + \beta I | \alpha, \beta \in R, I^2 = I\}$ and the I used here is called the unknown.

The following definition is given for neutrosophic rational numbers in [3]. But it would not be wrong to define it for real numbers as well.

Definition 2.1 [3] Let $\alpha+\beta I \in R[I]$. The norm and the conjugate of $\alpha+\beta I$ are defined by $N(\alpha+\beta I)=\alpha(\alpha+\beta)$ and $\overline{\alpha+\beta I}=\alpha+\beta-\beta I$.

It is seen that $(\alpha+\beta I)\left(\overline{\alpha+\beta I}\right)=N(\alpha+\beta I)$. Also, it is seen that $N(\alpha+\beta I)=0$ if and only if $\alpha=0$ or $\alpha+\beta=0$.

Also, it is true that $N(x.y)=N(x).N(y)$ for $x,y\in R[I]$ by Proposition 3.5 (vi) in [3].

Definition 2.2 [8] Let $A=N+MI$ be a $n\times n$ neutrosophic matrix where N and M are $n\times n$ real matrices. The determinant of A is defined as

$$detA=detN+(\det(N+M)-detN)I.$$

Theorem 2.3 [8] Let $A=N+MI$ be a $n\times n$ neutrosophic matrix where N and M are $n\times n$ real matrices. Then a necessary and sufficient condition for the inverse of A to exist N and $N+M$ invertible matrices and

$$A^{-1}=N^{-1}+((N+M)^{-1}-N^{-1})I.$$

3. Results

Theorem 3.1 Let $\alpha,\beta,\gamma\in R[I]$. If $\alpha\beta=\alpha\gamma$ and $N(\alpha)\neq 0$, then $\beta=\gamma$.

Proof. Let $\alpha\beta=\alpha\gamma$ and $N(\alpha)\neq 0$ where $\alpha=\alpha_1+\alpha_2 I,\beta=\beta_1+\beta_2 I$ and $\gamma=\gamma_1+\gamma_2 I$. Since $N(\alpha)=\alpha(\alpha+\beta)\neq 0$, we have $\alpha\neq 0$ and $\alpha+\beta\neq 0$. Since $\alpha\beta=\alpha\gamma$, we have $(\alpha_1+\alpha_2 I)(\beta_1+\beta_2 I)=(\alpha_1+\alpha_2 I)(\gamma_1+\gamma_2 I)\Rightarrow \alpha_1\beta_1+\left((\alpha_1+\alpha_2)(\beta_1+\beta_2)-\alpha_1\beta_1\right)I=\alpha_1\gamma_1+((\alpha_1+\alpha_2)(\gamma_1+\gamma_2)-\alpha_1\gamma_1)I$. Hence, we get $\alpha_1\beta_1=\alpha_1\gamma_1$ and since $\alpha_1\neq 0$, we have $\beta_1=\gamma_1$. Also, since $(\alpha_1+\alpha_2)(\beta_1+\beta_2)=(\alpha_1+\alpha_2)(\gamma_1+\gamma_2)$, $\alpha+\beta\neq 0$ and $\beta_1=\gamma_1$, we have $\beta_2=\gamma_2$. Consequently, it is seen that $\beta=\gamma$.

Definition 3.2 Let $0\neq a+bI, c+dI\in R[I]$. If there exists a neutrosophic real number $k+tI\in R[I]$ such that $c+dI=(k+tI)(a+bI)$, then we say $a+bI$ divides c$+dI$ and denote $a+bI|c+dI$. In this case $\frac{c+dI}{a+bI}=k+tI\in R[I]$.

Note that the set $R[I]$ is not closed according to the division. The quotient of two neutrosophic numbers may not be a neutrosophic number.

Example 3.3 Since $10+5I=(2+3I)(5-2I)$, we have $2+3I|10+5I$. But there do not exist any neutrosophic real number $k+tI$ such that $2+4I=(k+tI)(1-I)$, we have $1-I\nmid 2+4I$.

Teorem 3.4 Let $0\neq\alpha+\beta I,\ \gamma+\delta I\in R[I]$ and $x=\frac{\gamma+\delta I}{\alpha+\beta I}$. Then

$i)$ if $N(\alpha+\beta I)\neq 0$, then $x=\frac{\gamma}{\alpha}+\frac{\alpha\delta-\beta\gamma}{\alpha(\alpha+\beta)}I\in R[I]$,

$ii)$ in case $N(\alpha+\beta I)=0$,

a) if $\alpha = 0,\ \gamma \neq 0$, then $x = \dfrac{\gamma+\delta I}{\alpha+\beta I} = \dfrac{\gamma+\delta I}{\beta I} \notin R[I]$,

b) if $\alpha = 0,\ \gamma = 0$, then $x = \dfrac{\gamma+\delta I}{\alpha+\beta I} = \dfrac{\delta I}{\beta I} = m + nI \in R[I]$ where $m + n = \dfrac{\delta}{\beta}$,

c) if $\beta = -\alpha \neq 0, \gamma + \delta \neq 0$, then $x = \dfrac{\gamma+\delta I}{\alpha+\beta I} = \dfrac{\gamma+\delta I}{\alpha-\alpha I} \notin R[I]$,

d) if $\beta = -\alpha \neq 0, \gamma = -\delta \neq 0$, then $x = \dfrac{\gamma+\delta I}{\alpha+\beta I} = \dfrac{\gamma-\gamma I}{\alpha-\alpha I} = \dfrac{\gamma}{\alpha} + nI \in R[I]$ where $n \in R$.

Proof. $i)$ Let $N(\alpha+\beta I) = \alpha(\alpha+\beta) \neq 0$. Then $\alpha \neq 0$ and $\alpha+\beta \neq 0$. Hence, we get

$$
\begin{aligned}
x &= \frac{\gamma+\delta I}{\alpha+\beta I} \\
&= \frac{(\gamma+\delta I)(\alpha+\beta-\beta I)}{(\alpha+\beta I)(\alpha+\beta-\beta I)} \\
&= \frac{\gamma(\alpha+\beta)+(\alpha\delta-\beta\gamma)}{N(\alpha+\beta I)} \\
&= \frac{\gamma}{\alpha} + \frac{\alpha\delta-\beta\gamma}{\alpha(\alpha+\beta)} I \in R[I].
\end{aligned}
$$

$ii)$ Let $N(\alpha+\beta I) = \alpha(\alpha+\beta) = 0$. Then $\alpha = 0$ or $\alpha+\beta = 0$. (since $\alpha+\beta I \neq 0, \alpha$ and β can not both be zero) Firstly, let $\alpha = 0$ and $\alpha+\beta \neq 0$. Then if $x = \dfrac{\gamma+\delta I}{\alpha+\beta I} = m + nI$ for any $m,\ n \in R$, we have $m\alpha = \gamma$ and $(m+n)(\alpha+\beta) = \gamma+\delta$. **(a)** Since $\alpha = 0$, if $\gamma \neq 0$, there do not exist any $m \in R$ such that $m\alpha = \gamma$. That is, $x = \dfrac{\gamma+\delta I}{\beta I} \notin R[I]$ for $\gamma \neq 0$. **(b)** If $\alpha = 0, \gamma = 0$, the equality $m\alpha = \gamma$ is true for all $m \in R$. From the equality $(m+n)(\alpha+\beta) = \gamma+\delta$, we have $m+n = \dfrac{\delta}{\beta}$. So $x = \dfrac{\gamma+\delta I}{\alpha+\beta I} = \dfrac{\delta I}{\beta I} = m + nI \in R[I]$ where $m+n = \dfrac{\delta}{\beta}$. **(c)** Let $\alpha \neq 0$ and $\alpha+\beta = 0$. Then we have $\beta = -\alpha$. From the equality $m\alpha = \gamma$, we have $m = \dfrac{\gamma}{\alpha}$ and from the equality $(m+n)(\alpha+\beta) = \gamma+\delta$, we have $\left(\dfrac{\gamma}{\alpha}+n\right).0 = \gamma+\delta$. Then if $\gamma+\delta \neq 0$, there are not any $n \in R$ such that $\left(\dfrac{\gamma}{\alpha}+n\right).0 = \gamma+\delta$. Hence $x = \dfrac{\gamma+\delta I}{\alpha+\beta I} = \dfrac{\gamma+\delta I}{\alpha-\alpha I} \notin R[I]$ where $\gamma+\delta \neq 0$. **(d)** If $\gamma+\delta = 0$, it is true the equality $\left(\dfrac{\gamma}{\alpha}+n\right).0 = \gamma+\delta$ for all $n \in R$. In this case $x = \dfrac{\gamma+\delta I}{\alpha+\beta I} = \dfrac{\gamma-\gamma I}{\alpha-\alpha I} = \dfrac{\gamma}{\alpha} + nI \in R[I]$ for all $n \in R$.

Example 3.5 $\frac{2+I}{1+I} = 2 - \frac{1}{2}I \in R$, $\frac{2+I}{I} \notin R[I]$, $\frac{4I}{2I} = m + nI \in R[I]$ where $m + n = 2$, $\frac{2+I}{1-I} \notin R[I]$, $\frac{2-2I}{1-I} = 2 + nI \in R[I]$ where $n \in R$.

Theorem 3.6 Let $\alpha x = \beta$ be a neutrosophic liner equation where $0 \neq \alpha, \beta \in R[I]$.
i) If $N(\alpha) \neq 0$, then $\alpha x = \beta$ has unique solution in $R[I]$ and the solution is $x = \frac{\bar{\alpha}.\beta}{N(\alpha)}$
ii) If $N(\alpha) = 0$ and $\alpha|\beta$, then $\alpha x = \beta$ has an infinite number of solutions.
iii) If $N(\alpha) = 0$ and $\alpha \nmid \beta$, then $\alpha x = \beta$ has no solutions in $R[I]$.

Proof. It is clear by Theorem 3.4.

Example 3.7 i) Consider the neutrosophic linear equation $(2 + 3I)x = 4 - I$. Since $N(2 + 3I) = 10 \neq 0$ and $\overline{2 + 3I} = 5 - 3I$, we have $x = \frac{\overline{(2+3I)}(4-I)}{N(2+3I)} = 2 - \frac{7}{5}I$.
ii) For $(1 - I)x = 3 - 3I$, since $1 - I \neq 0, N(1 - I) = 0$ and $1 - I|3 - 3I$, the equation has an infinite number of solutions: Let $x = a + bI$. Then since $(1 - I)(a + bI) = 3 - 3I$, we have $a - aI = 3 - 3I$. Hence, we see that $a = 3, b \in R$. Then the solution set is $\{3 + bI : b \in R\}$.
iii) Consider the neutrosophic linear equation $(1 - I)x = 2 + I$. We have $1 - I \neq 0, N(1 - I) = 0$ and $1 - I \nmid 2 + I$. Since there are no neutrosophic number $a + bI$ such that $(1 - I)(a + bI) = 2 + I$, the equation has no solutions.
iv) The solution set of the equation $2Ix = 4I$ is $\{m + nI : m + n = 2, m, n \in R\}$.

Consider the equation $ax + by = c$ ($a \neq 0$ or $b \neq 0$) in R. It is known that

i) if $b \neq 0$, then the solution set is $\left\{\left(x, \frac{c-ax}{b}\right) : x \in R\right\}$,

ii) if $a \neq 0$, then the solution set is $\left\{\left(\frac{c-by}{a}, y\right) : y \in R\right\}$.

Now we investigate the solutions of a neutrosophic liner equation with two variables.

Theorem 3.8 Let $\alpha x + \beta y = \gamma$ be a neutrosophic liner equation with two variables where $\alpha, \beta, \gamma \in R[I]$ and $\alpha \neq 0, \beta \neq 0$.
i) If $N(\alpha) \neq 0$, then the solution set is $\left\{\left(\frac{(\gamma-\beta y)\bar{\alpha}}{N(\alpha)}, y\right) \middle| y \in R[I]\right\}$,
ii) If $N(\beta) \neq 0$, then the solution set is $\left\{\left(x, \frac{(\gamma-\alpha x)\bar{\beta}}{N(\beta)}\right) \middle| x \in R[I]\right\}$,
iii) If $N(\alpha) = 0$ and $N(\beta) = 0$, then
a) there exist infinitely many $y \in R[I]$ for all x that satisfies the property $\beta|\gamma - \alpha x$,
b) there do not exist any $y \in R[I]$ for an x that satisfies the property $\beta \nmid \gamma - \alpha x$,
or
c) there exist infinitely many $x \in R[I]$ for all y that satisfies the property $\alpha|\gamma - \beta y$,
d) there do not exist any $x \in R[I]$ for an y that satisfies the property $\alpha \nmid \gamma - \beta y$

Proof. i) If $N(\alpha) \neq 0$, then we have $x = \frac{\gamma-\beta y}{\alpha} = \frac{(\gamma-\beta y)\bar{\alpha}}{\alpha\bar{\alpha}} = \frac{(\gamma-\beta y)\bar{\alpha}}{N(\alpha)} \in R[I]$.
Then the solution set is $\left\{\left(\frac{(\gamma-\beta y)\bar{\alpha}}{N(\alpha)}, y\right) \middle| y \in R[I]\right\}$ $y \in R[I]$.
ii) $N(\beta) \neq 0$, the proof is similar (i).

iii) Let $N(\alpha)=0$ and $N(\beta)=0$. From the equation $\alpha x+\beta y=\gamma$, we have $y=\frac{\gamma-\alpha x}{\beta}$. In this case, by Theorem 3.6, if $\beta|\gamma-\alpha x$ for any $x\in R[I]$, then there exist infinitely many $y=\frac{\gamma-\alpha x}{\beta}\in R[I]$. But if $\beta\nmid\gamma-\alpha x$ for any $x\in R[I]$, then there do not exist an $y=\frac{\gamma-\alpha x}{\beta}$ in R[I]. Hence (a) and (b) are true. Similarly (c) and (d) are true.

In [9], according to Alhasan's analysis in part 3.1, every neutrosophic linear equation with two variables is solvable. But as seen from the Theorem 3.8, some equations may be unsolvable.

Example 3.9 i) Consider the equation $(1+I)x+(2-I)y=1+2I$. Since $N(2-I)=2\neq 0$ and $\overline{2-I}=1+I$, we see that, the solution is

$$y=\frac{1+2I}{2-I}-\frac{1+I}{2-I}x$$

$$=\frac{1}{2}-\frac{5}{2}\mathrm{I}-\left(\frac{1}{2}+\frac{3}{2}\mathrm{I}\right)x$$

for all $x\in R[I]$.

ii) Consider the equation $2Ix+3Iy=4I$. We see that $N(2I)=0$ and $N(3I)=0$. In this case, since $y=\frac{4I-2Ix}{3I}$ and $3I|4I-2Ix$ for all $x\in R[I]$, there exist infinitely many solutions. For example, for $x=0$, $y=a+bI$ $(a+b=\frac{4}{3})$ are the solutions since $\frac{4\mathrm{I}}{3\mathrm{I}}=\left\{a+bI\in R[I]\middle|a+b=\frac{4}{3}\right\}$.

iii) Consider the equation $2Ix+3Iy=1+4I$. We see that $N(2I)=0$ and $N(3I)=0$. In this case, since $y=\frac{1+4I-2Ix}{3I}$ and $3I\nmid 1+4I-2Ix$ for all $x\in R[I]$, So this equation has no solution.

In [8, Definition 3.2], the determinant of the matrix $M=A+BI$ is given as a definition in terms of A and B. In the following theorem, we give this property as a theorem.

Theorem 3.10 Let $\mathcal{A}$ and $\mathcal{B}$ be $n\times n$ real matrix and $\mathcal{M}=\mathcal{A}+\mathcal{B}I$. Then the determinant of $\mathcal{M}$ is

$$det\mathcal{M}=det\mathcal{A}+(\det(\mathcal{A}+\mathcal{B})-det\mathcal{A})I.$$

Proof. Let $\mathcal{M}=\mathcal{A}+\mathcal{B}I=[m_{ij}]_{2\times 2}$, $\mathcal{A}=[a_{ij}]_{2\times 2}$ and $\mathcal{B}=[b_{ij}]_{2\times 2}$. Then $\det(\mathcal{M})=m_{11}m_{22}-m_{12}m_{21}$

$$=(a_{11}+b_{11}I)(a_{22}+b_{22}I)-(a_{12}+b_{12}I)(a_{21}+b_{21}I)$$

$$=a_{11}a_{22}-a_{12}a_{21}+(a_{11}b_{22}+b_{11}a_{22}+b_{11}b_{22}-a_{12}b_{21}-b_{12}a_{21}-b_{12}b_{21})I$$

$$=a_{11}a_{22}-a_{12}a_{21}+(a_{11}b_{22}+b_{11}a_{22}+b_{11}b_{22}+a_{11}a_{22}-a_{12}a_{21}-a_{11}a_{22}-a_{12}a_{21}-a_{12}b_{21}-b_{12}a_{21}-b_{12}b_{21})I$$

$$= a_{11}a_{22} - a_{12}a_{21} - \big((a_{11} + b_{11})(a_{22} + b_{22}) - (a_{21} + b_{21})(a_{12} + b_{12}) - (a_{11}a_{22} - a_{12}a_{21})\big)I$$

$$= det\mathcal{A} + (\det(\mathcal{A} + \mathcal{B}) - det\mathcal{A})I.$$

Hence the claim is true for case n=2. Now suppose that the assertion is true for case n-1. Then, by the cofactor expansion about the first row, we have

$$\det(\mathcal{M}) = m_{11}M_{11} + m_{12}M_{12} + \cdots + m_{1n}M_{1n}$$

where M_{1j} is the cofactor of $m_{1j} = a_{1j} + b_{1j}I$ for $1 \leq j \leq n$. Let M'_{1j}, A'_{1j} and B'_{1j} be the $(n-1) \times (n-1)$ submatrices of $\mathcal{M}, \mathcal{A}$ and $\mathcal{B}$ obtained by deleting the first row and jth column respectively. Then since $M'_{1j} = A'_{1j} + B'_{1j}I$, we have by induction hypothesis, and $M_{1j} = (-1)^{1+j}detM'_{1j} = (-1)^{1+j}\big(detA'_{1j} + \big(\det(A'_{1j} + B'_{1j}) - \det A'_{1j}\big)I\big)$. Hence, we get that

$$\begin{aligned}\det(\mathcal{M}) &= m_{11}(detA'_{11} + (\det(A'_{11} + B'_{11}) - \det A'_{11})I) \\ &\quad - m_{12}(detA'_{12} + (\det(A'_{12} + B'_{12}) - \det A'_{12})I) \\ &\quad + \cdots + m_{1n}(-1)^{1+n}(detA'_{1n} + (\det(A'_{1n} + B'_{1n}) - \det A'_{1n})I) \\ &= m_{11}detA'_{11} - m_{12}detA'_{12} + \cdots + m_{1n}(-1)^{1+n}detA'_{1n} \\ &\quad +(m_{11}\det(A'_{11} + B'_{11}) - m_{12}\det(A'_{12} + B'_{12}) + \cdots + m_{1n}(-1)^{1+n}\det(A'_{1n} + B'_{1n}) \\ &\quad -(m_{11}detA'_{11} - m_{12}detA'_{12} + \cdots + m_{1n}(-1)^{1+n}detA'_{1n}))I \\ &= det\mathcal{A} + (\det(\mathcal{A} + \mathcal{B}) - det\mathcal{A})I.\end{aligned}$$

So, theorem is true for all $n \in Z^+$.

We can write the following theorem examining the existence of the matrix M^{-1}. Note that if $N(a + bI) = 0$ for any $a + bI \in R[I]$, we have $a(a + b) = 0$. So, we see that $a = 0$ or $a + b = 0$. Then $a + bI$ is a neutrosophic number such that bI or $a - aI$. Also, we see that $N(detM) = detA.det(A + B)$ where $M = A + BI$ by Theorem 3.10, Definition 2.1 and Definition 2.2.

Theorem 3.11 Let $\mathcal{A}$ and $\mathcal{B}$ be $n \times n$ real matrix and $\mathcal{M} = \mathcal{A} + \mathcal{B}I$. Then

$N(det\mathcal{M}) \neq 0$ if and only if $\mathcal{M}$ is invertible.

Proof. Let $N(det\mathcal{M}) \neq 0$. Then we have $det\mathcal{A} \neq 0$ and $det(\mathcal{A} + \mathcal{B}) \neq 0$. Hence $det\mathcal{M} \neq 0$. We know that $\mathcal{M}.Adj(\mathcal{M}) = det\mathcal{M}.I_n$. Hence it is seen that $\mathcal{M}\frac{1}{det\mathcal{M}}Adj(\mathcal{M}) = I_n$. Say $K = \frac{1}{det\mathcal{M}}Adj(\mathcal{M})$. Since $\frac{1}{det\mathcal{M}} = \frac{\overline{det\mathcal{M}}}{det\mathcal{M}.\overline{det\mathcal{M}}} = \frac{\overline{det\mathcal{M}}}{N(det\mathcal{M})} \in R[I]$, all entries of the matrix K are neutrosophic real numbers and $K = \mathcal{M}^{-1}$. So $\mathcal{M}$ is invertible matrix. Conversely, let $\mathcal{M}$ is invertible matrix. Then there exists a neutrosophic matrix $N = C + DI$ such that $\mathcal{M}N = N\mathcal{M} = I_n$. Hence since $(\mathcal{A} + \mathcal{B}I)(C + DI) = I_n$ and $(C + DI)(\mathcal{A} + \mathcal{B}I) = I_n$, we have $\mathcal{A}C = C\mathcal{A} = I_n$ and $(\mathcal{A} + \mathcal{B})(C + D) = (C + D)(\mathcal{A} + \mathcal{B}) = I_n$. So $\mathcal{A}$ and $\mathcal{A} + \mathcal{B}$ are invertible real matrices. In this case, since $det\mathcal{A} \neq 0$ and $\det(\mathcal{A} + \mathcal{B}) \neq 0$, we obtain $N(\mathcal{M}) = det\mathcal{A}.\det(\mathcal{A} + \mathcal{B}) \neq 0$. Note that, in case $N(\det\mathcal{M}) = 0$ (this includes $det\mathcal{M} = 0$), suppose that $\mathcal{M}$ is invertible. Then since $\mathcal{M}.\mathcal{M}^{-1} = I_n$, we have $det(\mathcal{M}.\mathcal{M}^{-1}) = 1$. Hence $\det(\mathcal{M}) \cdot \det(\mathcal{M}^{-1}) = 1$. Then the equality $\mathrm{N}(\det(\mathcal{M}) \cdot$

$\det(\mathcal{M}^{-1})) = \underbrace{N(det\mathcal{M})}_{0}.N(det\mathcal{M}^{-1}) = N(1) = 1$ is not true. So $\mathcal{M}$ is not an invertible matrix.

Example 3.12 i) Let $M = \begin{bmatrix} 1+I & 3-I \\ 0 & 0 \end{bmatrix}$. Then since $N(detM) = N(0) = 0$, M is not invertible.

ii) Let $M = \begin{bmatrix} 2-I & 1+I \\ 3 & 4I \end{bmatrix}$**.** Then $detM = -3 + I \neq 0$ and $N(detM) = 6 \neq 0$. Hence M is an invertible matrix and

$$M^{-1} = \frac{1}{-3+I} \cdot \begin{bmatrix} 4I & -1-I \\ -3 & 2-I \end{bmatrix}$$

$$= \frac{-2-I}{(-3+I)(-2-I)} \cdot \begin{bmatrix} 4I & -1-I \\ -3 & 2-I \end{bmatrix}$$

$$= \frac{1}{6} \cdot (-2-I). \begin{bmatrix} 4I & -1-I \\ -3 & 2-I \end{bmatrix}$$

$$= \frac{1}{6} \cdot \begin{bmatrix} -12I & 2+4I \\ 6+3I & -4+I \end{bmatrix}$$

iii) Let $M = \begin{bmatrix} 3I & 0 \\ 0 & 2I \end{bmatrix}$. Then $detM = 6I \neq 0, N(detM) = 0$. There do not exist any inverse of M by Theorem 3.11. As a second way, if there exists an inverse of the matrix M such that $M^{-1} = \begin{bmatrix} a+bI & c+dI \\ e+fI & g+hI \end{bmatrix}$, since $M.M^{-1} = I$, we get $3I(a+bI) = 1$ and $2I(g+hI) = 1$. But there do not exist $a, b \in R$ and $g, h \in R$ satisfying the above equations by Theorem 3.4. So, the matrix M do not have any inverse.

Remark 1. By Theorem 3.11 and Example 3.12 (iii), we see that the condition $detM \neq 0$ is not sufficient for M to be an invertible matrix. Therefore, Theorem 3.4 in [8] is not entirely correct.

Now, let $\mathcal{A}$ and $\mathcal{B}$ be $n \times n$ real matrix and $\mathrm{C} = D + EI$ be $n \times 1$ be column vector and $\mathcal{M} = \mathcal{A} + \mathcal{B}I$. Consider the systems of neutrosophic linear equations $\mathcal{M}Z = C$.

Theorem 3.13 If $N(detM) \neq 0$, then the systems of neutrosophic linear equation $MZ = C$ has unique solution and this solution is $Z = M^{-1}C$.

Proof. By Theorem 3.11, M is an invertible matrix. Multiplying $MZ = C$ by M^{-1} from the left, we get $Z = M^{-1}C$. If Z_1 and Z_2 are two solutions of $MZ = C$, then we have $MZ_1 = MZ_2$. Multiplying by M^{-1} from the left, we get $Z_1 = Z_2$.

The following Corollary states the solution vector $Z = X + YI$ of the systems of neutrosophic linear equation $MZ = C$ in terms of A, B, C and D where $M = A + BI$ and $C = D + EI$.

Corollary 3.14 Let $\mathcal{A}$ and $\mathcal{B}$ are $n \times n$ real matrices and C and D are $n \times 1$ real column vector. Let $\mathcal{M} = \mathcal{A} + \mathcal{B}I$ be an $n \times n$ matrix and $\mathrm{C} = D + EI$ be $n \times 1$ be column vector. If $N(detM) \neq 0$, then the solution of the systems of neutrosophic linear equations $MZ = C$ is the vector $Z = X + YI$ where $X = \mathcal{A}^{-1}D$ and $Y = (\mathcal{A} + \mathcal{B})^{-1}(D + E) - \mathcal{A}^{-1}D$.

Proof. By Theorem 3.13, we have $Z = M^{-1}C$. Hence using Theorem 2.3, we obtain that

$$\begin{aligned} Z &= X + YI \\ &= M^{-1}C \\ &= (\mathcal{A}^{-1} + ((\mathcal{A}+\mathcal{B})^{-1} - \mathcal{A}^{-1})I)(D + EI) \\ &= \mathcal{A}^{-1}D + (\mathcal{A}^{-1}E + (\mathcal{A}+\mathcal{B})^{-1}D - \mathcal{A}^{-1}D + (\mathcal{A}+\mathcal{B})^{-1}E - \mathcal{A}^{-1}E)I \\ &= \mathcal{A}^{-1}D + ((\mathcal{A}+\mathcal{B})^{-1}(D+E) - \mathcal{A}^{-1}D)I. \end{aligned}$$

Example 3.15 Consider the systems of neutrosophic equations

$$(2 - I)Z_1 + (1 + I)Z_2 = 1 + 2I$$

$$3Z_1 + 4IZ_2 = 3 + 4I.$$

Then $M = \begin{bmatrix} 2-I & 1+I \\ 3 & 4I \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 4 \end{bmatrix}}_{B} I, C = \begin{bmatrix} 1+2I \\ 3+4I \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}_{E} I,$
$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$. If we use Theorem 3.13, since $M^{-1} = \frac{1}{6}\begin{bmatrix} -12I & 2+4I \\ 6+3I & -4+I \end{bmatrix}$, we have $Z = M^{-1}C = \begin{bmatrix} 1 \\ -1+2I \end{bmatrix}$. If we use the Corollary 3.14, since $A^{-1} = \frac{1}{3}\begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix}$, $(A+B)^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$, we have

$$X = A^{-1}D = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, Y = (A+B)^{-1}(D+E) - A^{-1}D = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Hence $Z = X + YI = \begin{bmatrix} 1 \\ -1+2I \end{bmatrix}$.

In Theorem 3.13 and Corollary 3.14, we investigate the solutions of the systems of neutrosophic linear equations $MZ = C$ where $N(detM) \neq 0$. If $N(detM) = 0$, since M has not an inverse, we can not find a solution using the matrix M^{-1}. In case $N(detM) = 0$, we can write the following Theorem:

Theorem 3.16 If $detM \neq 0$ but $N(detM) = 0$, then the systems of neutrosophic linear equations $MZ = C$ has either more than one solution or no solution.

Proof. Since $detM \neq 0$ and $N(detM) = 0$, we can use Cramer's rule. We know that ith component of the solution Z is $Z_i = \frac{\det M_i}{\det M}$ for $i = 1,2,\dots,n$ where M_i is the matrix obtained from M by replacing the ith column of M by the vector C. If $detM | detM_i$, then $Z_i \in R[I]$ for all i by Theorem 3.4. Hence $MZ = C$ has more than one solution. If $detM \nmid detM_i$ for some i, then $Z_i \notin R[I]$. Hence $MZ = C$ has no solution.

Example 3.17 For the system

$$\begin{aligned} 3IX + (1+I)Y &= 6I \\ 2IY &= 4I, \end{aligned}$$

$M = \begin{bmatrix} 3I & 1+I \\ 0 & 2I \end{bmatrix}$, $C = \begin{bmatrix} 6I \\ 4I \end{bmatrix}$, $detM = 6I \neq 0, N(detM) = 0$. Then, by the second equation, we have $Y = \frac{4I}{2I} = p + qI$ $(p + q = 2, p, q \in R)$. Substituting it in the first equation, we see that $X = \frac{-p+(4-q)I}{3I}$. In this case, if $p=0$, we obtain $3I|(4-q)I$ and $X = \frac{(4-q)I}{3I} \in R[I]$. (For $p \neq 0$, since $3I \nmid -p + (4-q)I$, there are no solution) Hence since $p + q = 2$, we have $q = 2$ and $Y = 2I$. So, the solutions of the given systems of the neutrosophic linear equations are

$$X = \frac{2I}{3I} = \left\{u + vI: u, v \in R, u + v = \frac{2}{3}\right\}$$
$$Y = 2I.$$

Note that, if $detA \neq 0$, then the system $AX = B$ has only one solution in real linear algebra.

Example 3.18 The system

$$3IX + (1 + I)Y = 6I$$
$$2IY = 1 + 4I$$

has no solutions since $2I \nmid 1 + 4I$.

Corollary 3.19 Consider the system $MZ = C$ where M is a neutrosophic $n \times n$ square matrix and C is an $n \times 1$ be neutrosophic column vector.
i) If $N(detM) \neq 0$, then the systems of neutrosophic linear equation $MZ = C$ has unique solution. (Theorem 3.13)
ii) If $detM \neq 0$ but $N(detM) = 0$, then the systems of neutrosophic linear equations $MZ = C$ has either more than one solution or no solution. (Theorem 3.16)
iii) If $detM = 0$, the systems of equations $MZ = C$ has either more than one solution or no solution.

Remark 2. Considering Corollary 3.18 and the examples above, in Alhasan's article ([9]), it can be seen that there are some errors in the results and some examples in Section 4.2. The system in Example 4.2.2 in Alhasan's article has unlimited number solutions:

$$x = \frac{13\mathrm{I}}{19\mathrm{I}} = \left\{u + vI \,\middle|\, u, v \in R, u + v = \frac{13}{19}\right\}$$
$$y = -\frac{1}{19}\mathrm{I}$$

are the solutions of the equation of the systems:

$$2Ix + 7y = I$$

$$3Ix + y = 2I.$$

4. Conclusion

In this paper, we firstly researched the subject in which cases the division of two neutrosophic real numbers yields a new neutrosophic number. Then, from a different perspective, the solution cases of a neutrosophic linear equation with one unknown were examined. After calculating the determinant of a square matrix and giving the necessary and sufficient conditions for a square matrix to be invertible, the solution conditions of the systems of equations with the number of unknowns equal to the number of equations were examined. In doing so, we used the real and neutrophic parts of a neutrophisophic matrix. Also, we gave some important examples to clarify the theory.

Authorship contribution statement

Y. Çeven: Supervision/Observation/Advice, Conceptualization;

A. I. Sekmen: Methodology, Original Draft Writing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Ethics Committee Approval and/or Informed Consent Information

As the authors of this study, we declare that we do not have any ethics committee approval and/or informed consent statement.

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