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Beam Diffraction by a Conductive Half Plane

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Keywords

Laser Beam,
Scattering,
Diffraction,
Scattering theory.

Abstract

In this paper scattering and diffraction of Gaussian beam by a conductive half-plane is studied. To generate Gaussian beam complex point source method is used. To evaluate geometrical optics and diffracted fields far-field approximation is used. Obtained diffracted and scattered fields plotted and examined numerically by the help of MATLAB.

1. Introduction

Diffraction is a phenomenon that takes place when waves encounter an aperture [1] or obstacle. An obstacle can be a wedge [2], disk [3], or a half-plane [4], etc. Examining the diffraction of waves by a half-plane is a fundamental task for those who are willing to investigate the scattering process of more complicated geometries [5]. The milestone of solving the diffraction of waves by a half-plane was introduced by Sommerfeld in 1896 for a perfectly conducting half-plane [6]. In 1927, Raman and Krishnan modified the Sommerfeld formulation and experimentally confirmed the results for the diffraction of light by metallic screen [7]. In 1975, Senior studied half-plane edge diffraction [8] and then proceeded to diffraction by a resistive half-plane [9]. In 1971, Deschamps introduced the Gaussian beam as a bundle of complex rays [10]. Then in 1979, Arthur C. Green et al. introduced properties of the shadow cast by a half-screen when illuminated by a Gaussian beam [11]. After these two studies of Gaussian beam, researches on beam diffraction by a half-plane, attracted attention as well as the diffraction of waves by a half plane. In 1987, Suedan and Jull studied two-dimensional beam diffraction by a half-plane and wide slit [12], and they continued their study in 1989, with beam diffraction by half-planes and wedges for uniform and asymptotic solutions [13]. In 2015, Umul studied beam diffraction by a resistive half-plane [14], and in 2017, diffraction of waves by a conductive half-plane [15], has been studied with more satisfactory results for conductive boundary conditions.

To the best of our knowledge, beam diffraction by a conductive half-plane has not been studied yet. In this study, Gaussian beam diffraction by a conductive half-plane will be investigated and time factor $e^{j\omega t}$, where ω is the angular frequency, is considered and suppressed throughout this paper.

2. Theory

2.1. Scattered Fields by a Conductive Half Plane

The geometry of experiment is given in Figure 1. A conductive half plane is located at $x \in [0, \infty), y = 0$ and $z \in (-\infty, \infty)$. It is illuminated by electrical line source located at (ρ_0, ϕ_0). Q_R and Q_E are the reflection and diffraction points respectively. P is the observation point. The distance between diffraction point and observer is ρ.

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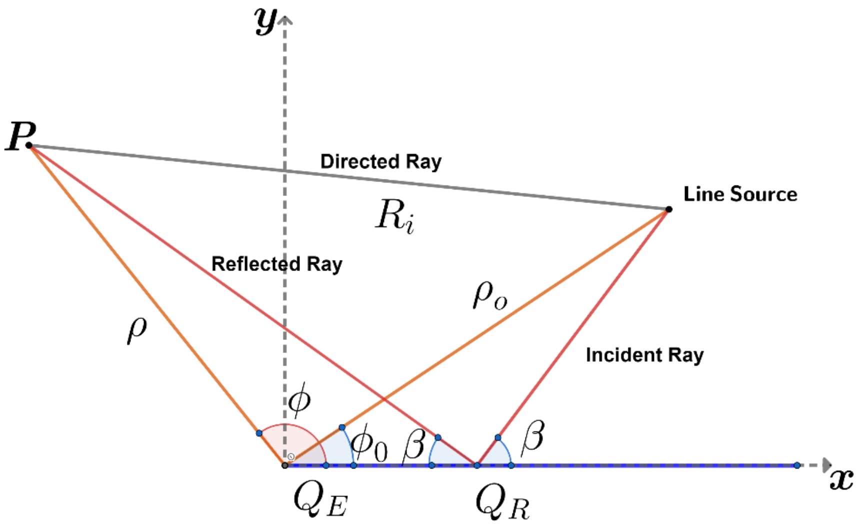

Figure 1. Geometry of the cylindrical wave diffraction by a conductive half-plane

Conductive half-plane supports only magnetic surface current density [16]. The conductive half-plane's reflection and transmission coefficients can be named as Γ and T respectively and introduced as [17]

$$\Gamma = \frac{\sin\beta}{\sin\beta\ + \sin\theta}, \tag{1}$$

and

$$\mathrm{T} = \frac{\sin\theta}{\sin\beta + \sin\theta}. \tag{2}$$

Where $\sin\theta$ is equal to $2Z_0R_m$ in which Z_0 is the impedance of free space, and R_m is the conductivity parameter of the surface. β is the reflection angle and it can be defined as,

$$\beta = \sin^{-1}\left[\frac{\rho\sin\phi_0 + \rho_0\sin\phi_0}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi+\phi_0)}}\right]. \tag{3}$$

Since the electrical line source lies along the z-axis, electric field is polarized in the z-direction, and this makes the problem two-dimensional. The radiated cylindrical wave by the electric line source can be defined as,

$$u_i = u_0\frac{e^{-jkR_i}}{\sqrt{kR_i}} \tag{4}$$

where u_i is the incident electric field intensity. u_0 is the complex amplitude, k is the wavenumber and denoted by $k = 2\pi/\lambda$, and R_i is the distance between observer and line source and R_i can be expressed as,

$$R_i = \sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\emptyset - \emptyset_0)}. \tag{5}$$

The geometric optic (GO) field is the summation of incident, transmitted and reflected fields. Then GO field can be written as

$$u_{GO} = u_{iGO} + u_{tGO} + u_{rGO}\,, \tag{6}$$

In our case when the conductive half-plane is illuminated by cylindrical wave given in Eq. (4). Then GO field can be defined as

$$u_{GO} = u_i U(-\xi_i) + \mathrm{T} u_i U(\xi_i) + \Gamma u_r U(-\xi_r)\,. \tag{7}$$

Where $U(x)$ is the unit step function which is equal to one if $x > 0$ and zero, otherwise. ξ_i and ξ_r are detour parameters for incident and reflected fields respectively and can be defined as

$$\xi_i = -2\sqrt{\frac{k\rho\rho_0}{\rho + \rho_0 + R_i}}\cos\frac{\phi - \phi_0}{2}\ , \tag{8}$$

$$\xi_r = -2\sqrt{\frac{k\rho\rho_0}{\rho + \rho_0 + R_r}}\cos\frac{\phi + \phi_0}{2}\,, \tag{9}$$

and u_r is the reflected wave and can be described as

$$u_r = u_0\frac{e^{-jkR_r}}{\sqrt{kR_r}}\,, \tag{10}$$

where R_r is the distance between an image source which located at $(\rho_0, 2\pi - \phi_0)$ and the observation point P. R_r can be defined as follows

$$R_r = \sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\emptyset + \emptyset_0)} \tag{11}$$

According to Senior [8,9] the high-frequency asymptotic expression of the edge-diffracted wave by conductive half-plane can be written as

$$u_d = \frac{e^{-j\frac{\pi}{4}}}{\sqrt{2\pi}}\frac{2\cos\frac{\phi}{2}\cos\frac{\phi_0}{2}}{\sin\theta}\frac{K(\phi,\theta)K(\phi_0,\theta)}{\cos\phi + \cos\phi_0}\frac{e^{-jk\rho_0}}{\sqrt{k\rho_0}}\frac{e^{-jk\rho}}{\sqrt{k\rho}} \tag{12}$$

Where the K function is defined to be

$$K(\gamma,\theta) = \frac{4\sqrt{\sin\theta}\sin\left(\frac{\gamma}{2}\right)}{\left[1 + \sqrt{2}\cos\frac{\left(\frac{\pi}{2}\right) - \gamma + \theta}{2}\right]\left[1 + \sqrt{2}\cos\frac{\left(\frac{3\pi}{2}\right) - \gamma - \theta}{2}\right]}$$
$$\times\left\{\frac{\psi_\pi\left(\frac{\left(\frac{3\pi}{2}\right) - \gamma - \theta}{2}\right)\psi_\pi\left(\frac{\left(\frac{\pi}{2}\right) - \gamma + \theta}{2}\right)}{\left[\psi_\pi\left(\frac{\pi}{2}\right)\right]^2}\right\}^2 \tag{13}$$

and ψ_π is the Malyughinetz function [18], which can be defined as

$$\psi_\pi = e^{\frac{-1}{8\pi}} \int_0^x \frac{\pi \sin v - 2\sqrt{2}\pi \sin\left(\frac{v}{2}\right) + 2v}{\cos v} dv \tag{14}$$

The uniform diffracted wave can be written as

$$u_d = \frac{\cos\frac{\phi}{2} K(\phi,\theta) K(\phi_0,\theta)}{\sin\theta \sin\frac{\phi_0}{2}} \times \left\{ \sin\frac{\phi-\phi_0}{2} \sqrt{\frac{2R_i}{\rho+\rho_0+R_i}} \frac{e^{-jkR_i}}{\sqrt{kR_i}} sign(\xi_i) F[|\xi_i|] - \sin\frac{\phi+\phi_0}{2} \sqrt{\frac{2R_r}{\rho+\rho_0+R_r}} \frac{e^{-jkR_r}}{\sqrt{kR_r}} sign(\xi_r) F[|\xi_r|] \right\} \tag{15}$$

where sign(x) is signum function and defined as

$$sign(x) = \begin{cases} 1, & if\ x > 0 \\ 0, & if\ x = 0 \\ -1, & if\ x < 0 \end{cases} , \tag{16}$$

And $F[x]$ is the Fresnel function [19] given by

$$F[x] = \frac{e^{-j\frac{\pi}{4}}}{\sqrt{\pi}} \int_x^\infty e^{-jv^2} dv \tag{17}$$

In our study, it is accepted that $k\rho \gg 1$ and this makes the observer at far field region [16]. And also, it is accepted that ρ is adequately greater than ρ_0. So in the far field, equations (5,8,9,11) can be simplified and approximated by,

$$R_i \approx \rho - \rho_0 \cos(\phi - \phi_0) , \tag{18}$$

$$R_r \approx \rho - \rho_0 \cos(\phi + \phi_0) , \tag{19}$$

$$\xi_i \approx -\sqrt{2k\rho_0} \cos\frac{\phi-\phi_0}{2} , \tag{20}$$

and

$$\xi_r \approx -\sqrt{2k\rho_0} \cos\frac{\phi+\phi_0}{2} \tag{21}$$

so the uniform diffracted fields of Senior becomes

$$u_d \approx \frac{\cos\frac{\phi}{2} K(\phi,\theta) K(\phi_0,\theta)}{\sin(\theta)\sin\frac{\phi_0}{2}} \times \left\{\sin\frac{\phi-\phi_0}{2} e^{jk\rho_0\cos(\phi-\phi_0)} sign(\xi_i) F[|\xi_i|] - \sin\frac{\phi+\phi_0}{2} e^{jk\rho_0\cos(\phi+\phi_0)} sign(\xi_r) F[|\xi_r|]\right\} \frac{e^{-jk\rho}}{\sqrt{k\rho}} \tag{22}$$

in far-field approximation The GO waves can be written as

$$u_{GO} \approx u_0 \left[\frac{e^{-jkR_i}}{\sqrt{kR_i}} U(-\xi_i) + \mathrm{T}\frac{e^{-jkR_i}}{\sqrt{kR_i}}(\xi_i) - \Gamma\frac{e^{-jkR_r}}{\sqrt{kR_r}} U(-\xi_r)\right], \tag{23}$$

which yields

$$u_{GO} \approx u_0 \frac{e^{-jk\rho}}{\sqrt{k\rho}} \left[e^{jk\rho_0\cos(\phi-\phi_0)} U(-\xi_i) + \frac{\sin\theta}{\sin\beta + \sin\theta} e^{jk\rho_0\cos(\phi-\phi_0)} U(\xi_i) - \frac{\sin\beta}{\sin\beta\ + \sin\theta} e^{jk\rho_0\cos(\phi+\phi_0)} U(-\xi_r)\right]. \tag{23}$$

2.2. Beam Diffraction

To evaluate Gaussian beam diffraction from conductive half-plane, cylindrical wave radiating electric line source needs to be converted to Gaussian beam source. For this propose, complex point source method [15] is going to be used. In this method, the location of the line source is going to be defined in the complex coordinates as shown in Figure 2. The beam half waist is equal to b, intersection angle of the x axis and main axis of the beam is equal to α. In complex coordinates, reflection angle is denoted by β_c and defined as

$$\beta_c = \sin^{-1}\left[\frac{\rho\sin\phi_0 + \rho_0\sin\phi_0 + jb\sin\alpha}{\sqrt{\rho^2+\rho_0^2-b^2-2\rho\rho_0\cos(\phi+\phi_0) - j2\rho b\cos(\phi+\alpha) + j2\rho_0 b\cos(\phi_0-\alpha)}}\right], \tag{24}$$

angle of incidence is denoted by ϕ_{ci} and defined as

$$\phi_{ci} = \tan^{-1}\frac{\rho\sin\phi_0 + jb\sin\alpha}{\rho\cos\phi_0 + jb\cos\alpha} \tag{25}$$

and distance between complex source and diffraction point is ρ_{ci} and defined as

$$\rho_{ci} = \sqrt{\rho_0^2 - b^2 + 2jb\rho_0\cos(\emptyset_0 - \alpha)}. \tag{26}$$

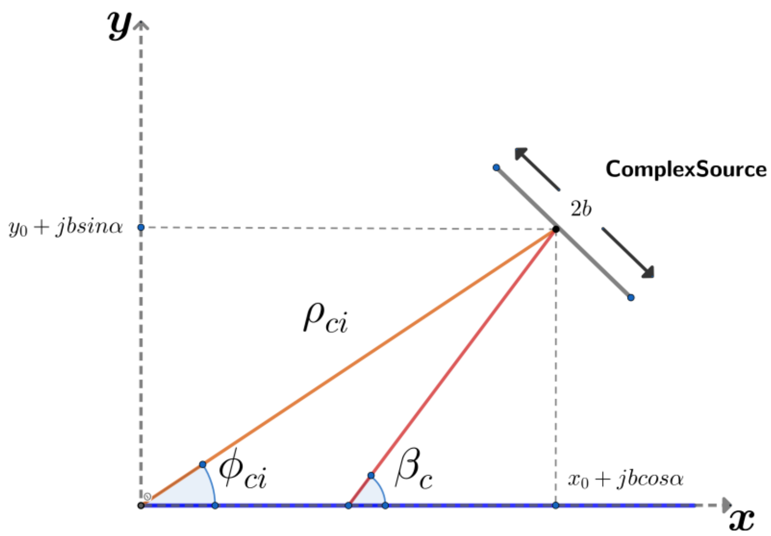

Figure 2. Geometry of the electric line source in complex coordinates

The distances between the observer and source R_i, and the distance between the image source and observer R_r in complex coordinates are defined as

$$R_i = \sqrt{\rho^2 + \rho_0^2 - b^2 - 2\rho\rho_0 \cos(\emptyset - \emptyset_0) - j2\rho\cos(\phi - \alpha) + j2\rho\cos(\phi_0 - \alpha)} \tag{27}$$

and

$$R_r = \sqrt{\rho^2 + \rho_0^2 - b^2 - 2\rho\rho_0 \cos(\emptyset + \emptyset_0) - j2\rho\cos(\phi + \alpha) + j2\rho\cos(\phi_0 + \alpha)}. \tag{28}$$

By far field approximation R_i and R_r becomes

$$R_i \approx \rho - \rho_0 \cos(\phi - \phi_0) - jb\cos(\phi - \alpha) \tag{29}$$

and

$$R_r \approx \rho - \rho_0 \cos(\phi + \phi_0) - jb\cos(\phi + \alpha) \tag{30}$$

respectively, and the GO waves can be introduced in the complex coordinates.

$$\begin{aligned} u_{GO} \approx u_0 \frac{e^{-jk\rho}}{\sqrt{k\rho}} \Big[& e^{-kb\cos(\phi-\alpha)} e^{jk\rho_0\cos(\phi-\phi_0)} U(-s_i) \\ & + \frac{\sin\theta}{\sin\beta_c + \sin\theta} e^{-kb\cos(\phi-\alpha)} e^{jk\rho_0\cos(\phi-\phi_0)} U(s_i) \\ & - \frac{\sin\beta_c}{\sin\beta_c + \sin\theta} e^{-kb\cos(\phi+\alpha)} e^{jk\rho_0\cos(\phi+\phi_0)} U(-s_r) \Big], \end{aligned} \tag{31}$$

$\sin\beta$ at reflection and transmission coefficients given in Eq.(1) and Eq.(2) is replaced with $\sin\beta_c$ which is the sine function of the same angle in complex coordinates. The function $s_{i,r}$ is defined by

$$s_{i,r} = Re(\xi_{i,r}) - Im(\xi_{i,r}) \tag{32}$$

where $Re(z)$ and $Im(z)$ takes only real and imaginary parts of *z* respectively. The detour parameters in complex coordinates defined by.

$$\xi_i \approx -\sqrt{2k\rho_c}\cos\frac{\phi-\phi_c}{2}\ , \tag{33}$$

and

$$\xi_r \approx -\sqrt{2k\rho_c}\cos\frac{\phi+\phi_c}{2}\,. \tag{34}$$

The diffracted beam field can be expressed by summation of the incident and reflected diffracted fields as

$$u_d = u_{id} + u_{rd} \tag{35}$$

we can get the uniform expressions of the diffracted fields with the help of given equations (19a) and (19b) in [20], which are given in a combined form of,

$$u_{id,rd} = u_{i,r}\, sign(s_{i,r})F\left[|s_{i,r}| + sign(s_{i,r})\gamma_{i,r}\right]. \tag{35}$$

The uniform expressions of incident diffracted and reflected diffracted fields could be defined respectively,

$$\begin{aligned} u_{id} \approx & \frac{2\cos\left(\frac{\beta_c}{2}\right)\cos\left(\frac{\phi}{2}\right)K(\phi,\theta)K(\beta_c,\theta)}{\sin(\theta)\sin(\beta_c)}\frac{e^{-jk\rho}}{\sqrt{k\rho}} \\ & \times\left\{\sin\frac{\phi-\phi_{ci}}{2}e^{jk\rho_0\cos(\phi-\phi_0)}e^{-kb\cos(\phi-\alpha)}\, sign(s_i)F[|s_i| + sign(s_i)\gamma_i]\right\} \end{aligned} \tag{37}$$

and

$$\begin{aligned} u_{rd} \approx & -\frac{2\cos\left(\frac{\beta_c}{2}\right)\cos\left(\frac{\phi}{2}\right)K(\phi,\theta)K(\beta_c,\theta)}{\sin(\theta)\sin(\beta_c)}\frac{e^{-jk\rho}}{\sqrt{k\rho}} \\ & \times\left\{\sin\frac{\phi+\phi_{ci}}{2}e^{jk\rho_0\cos(\phi+\phi_0)}e^{-kb\cos(\phi+\alpha)}\, sign(s_r)F[|s_r| + sign(s_r)\gamma_r]\right\}, \end{aligned} \tag{38}$$

and $\gamma_{i,r}$ can be introduced as,

$$\gamma_{i,r} = \sqrt{2}Im(\xi_{i,r})e^{j\frac{\pi}{4}}\,. \tag{39}$$

The total incident scattered field can be defined as

$$u_{is} = u_{iGO} + u_{tGO} + u_{id} \tag{40}$$

and total reflected scattered field can be defined as

$$u_{rs} = u_{rGO} + u_{rd} \tag{41}$$

3. Numerical Results

In this part, behavior of the GO, diffracted and scattered fields are examined numerically by MATLAB software. For numerical calculations, ρ_0 is taken 2λ where λ is wavelength, ρ is taken 20λ to satisfy far-field approximation. The angle of incidence ϕ_0 is taken $60°$, the beam half waist b is taken $\lambda/2$, the intersection angle α is taken $30°$, and $\sin\theta$ is going to be taken 4. On plots, the observation angle ϕ varies from $0°$ to $360°$.

Figure 3. shows the variation of incident GO, diffracted and scattered fields according to observation angle. The incident geometric optic field is cut around $247°$ as a result of multiplication with $U(-s_i)$. The incident diffracted field has shadow boundary around $247°$. If the conductive half plane were illuminated by plane wave[15] the shadow boundary would be located at $\pi + \phi_0$ which is $240°$. The $7°$ shift is the result of complex location of the source, as expected result. When incident scattered field is taken into account, it is seen that the field is continuous among all angles of observation. Incident scattered field is summation of incident GO, transmitted GO, and incident diffraction fields, and the continuity is provided by incident diffracted field.

Figure 4. shows the variation of reflected GO, diffracted, and scattered fields according to observation angle. The reflected GO field is cut around $112.9°$. This cut is also resulted from multiplier $U(-s_r)$. The reflected diffracted field has shadow boundary around 112.9 which is also shifted $7.1°$ when it is compared to plane wave illuminated conductive half-plane [15]. As it can be seen in the figure the reflected scattered field has continuity all around the observer angles, and it's the validation of the calculations and far-field approximation.

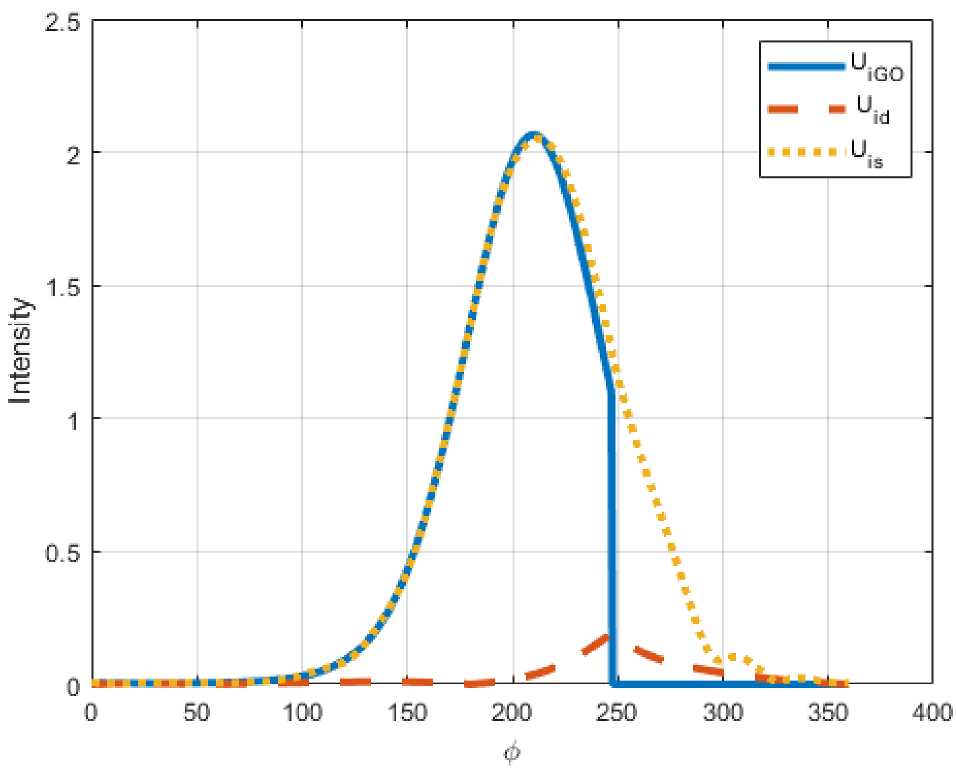

Figure 3. Incident GO diffracted and scattered fields.

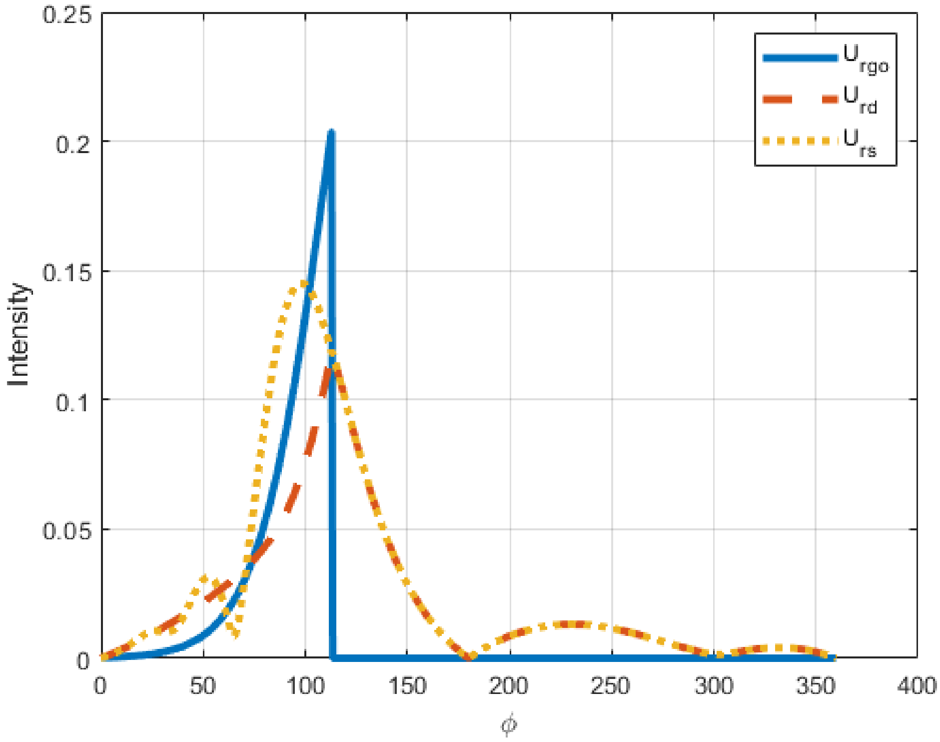

Figure 4. Reflected GO, diffracted and scattered fields.

In Figure 5. Incident reflected and total diffracted fields are shown. Total diffracted field is harmonious with incident and reflected diffracted fields. When the observation angle is equal to 0°, 180°, and 360°, total fields go to zero which is very well expected result when Senior's solution taken into account. Left hand side of the total diffracted field is mostly influenced by reflected diffracted fields while right hand side mostly effected by incident diffracted fields. The shadow boundaries of total diffracted field is observed ad 113.5° and 246°5 respectively. The reason of shadow boundaries differing slightly from previously given results is interference of the reflected and incident diffracted fields.

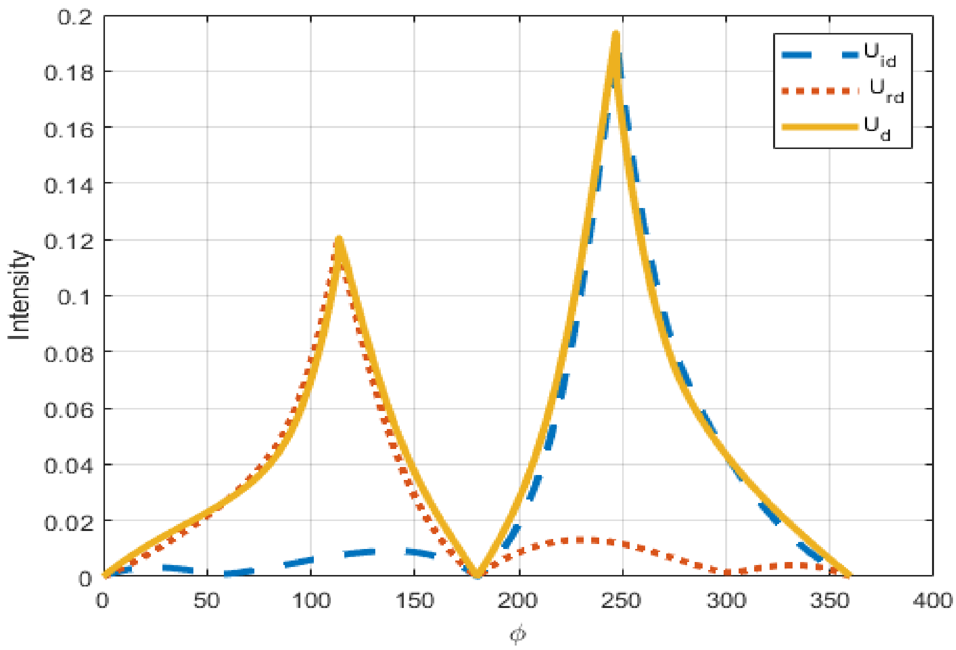

Figure 5. Incident reflected and total diffracted fields.

4. Comparisons with Resistive Half Plane

In Figure 5. reflected GO, diffracted, and scattered fields of resistive and conductive half planes are plotted by MATLAB, respectively. For simulations, parameters are taken as given in section 3, except for ρ , which is taken 9λ instead of 20λ to obtain the same results presented in [14]. For comparisons, the resistive half plane is chosen because resistive and conductive half planes are electromagnetic duals of each other. The one's reflection coefficient is the other's transmission coefficient. Both reflected geometric optical fields took their maximum value at 112.9°, as same as reflected diffracted fields. Moreover, reflected scattered field took its maximum value at 98.5°, just as in [14]. The results show that the geometry of the experiment is set and simulated correctly when compared to the results in the literature. The only difference in the plots is intensity levels, resulting from the different reflection coefficients of the surfaces.

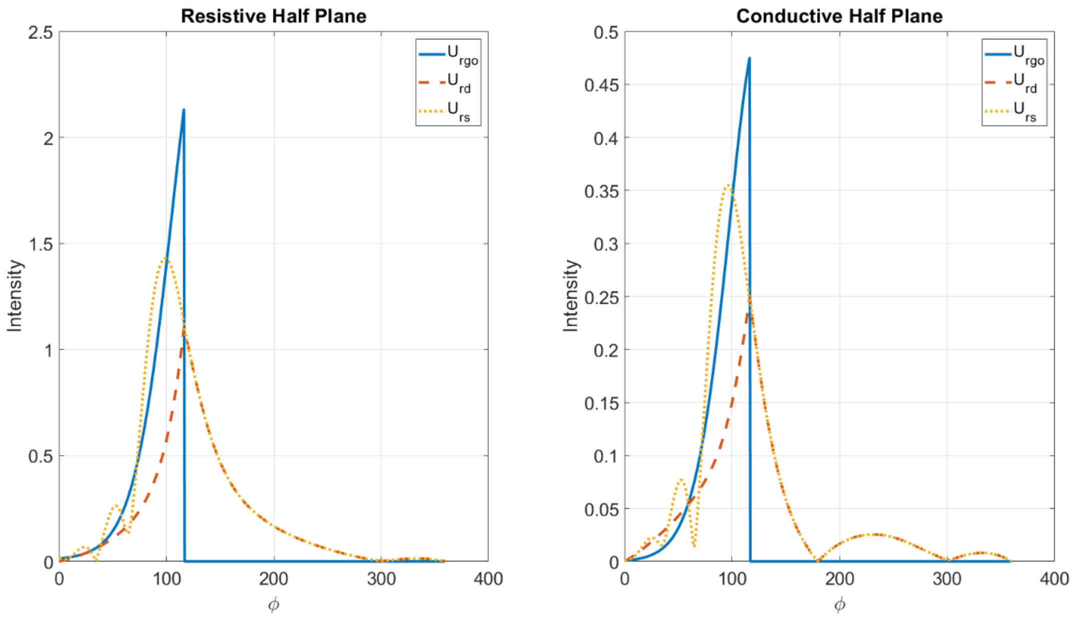

Figure 6. Comparison of, Reflected GO, diffracted and scattered fields from Resistive and Conductive Half Planes

5. Conclusions

In this study, the diffraction and scattering process of a Gaussian beam by a conductive half plane is studied. To accomplish this process, the solution of Senior is considered. For the mathematical modelling of the Gaussian beam, the complex point source method is used. With the aid of [20], uniform expressions of the diffracted fields are obtained. Numerical results for GO, diffracted and scattered waves are calculated and plotted by means of MATLAB. The shadow boundary shifts which caused by complex source are observed. Also, the changes on incident and reflected diffracted fields are observed according to conductive of the surface.

Declaration of Competing Interest

No conflict of interest was declared by the authors.

Authorship Contribution Statement

Ömer Kemal ÇATMAKAŞ: Writing, Reviewing, Editing, Analysis & Interpretation of Results and Manuscript Preparation

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