On the Depth of Independence Complexes

ALPER ÜLKER

Ağrı İbrahim Çeçen University, Faculty of Science and Letters, Department of Mathematics, Ağrı, TURKEY.

Abstract
Let $G$ be a graph and $I(G)$ be its edge ideal so we call $k[\text{Ind}(G)] = k[x_1, \ldots, x_n]/I(G)$ Stanley-Reisner ring of $G$. The depth of a ring is a well-studied and important algebraic invariant in commutative algebra. In this paper we give some results on the depth of Stanley-Reisner rings of graphs and simplicial complexes. By depth Lemma we reduce the computing depth of a codismantlable graph into its induced subgraphs.

Introduction
Let $G$ be a simple undirected graph on the vertex set $V(G) = \{x_1, \ldots, x_n\}$. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring on $n$ variables corresponding to $V(G)$. If we define set $I(G) = \{x_i x_j : \{x_i, x_j\} \in E(G)\}$ such that $E(G)$ is the edge set of $G$ and indeterminates are from $V(G) = \{x_1, \ldots, x_n\}$, then this $I(G)$ is called the edge ideal of graph $G$ VILLARREAL (1990). The independence complex of $G$ is a simplicial complex with vertex set $V(G)$ and with faces are independent sets of $G$ and denoted by $\text{Ind}(G)$. The Stanley-Reisner ring of a simplicial complex over a field $k$ provides a link between commutative algebra and combinatorial structures such as graphs and simplicial complexes. For a field $k$, the Stanley-Reisner ring of a simplicial complex $\Delta$ is denoted by $k[\Delta]$.

If our complex is an independence complex of a graph $G$ on $V(G)$, then it is denoted by $k[\text{Ind}(G)]$ and equals to quotient ring $R/I(G)$ with $R = k[x_1, \ldots, x_n]$. Krull dimension of $k[\text{Ind}(G)]$ is the supremum of the longest chain of the strict inclusions of prime ideals of $k[\text{Ind}(G)]$ and denoted by $\dim(\text{Ind}(G))$. depth$(\text{Ind}(G))$ is the longest homogeneous sequence $f_1, f_2, \ldots, f_k$ such that $f_i$ is not a zero-divisor of $k[x_1, \ldots, x_n]/(f_1, f_2, \ldots, f_k)$ for all $1 \leq i \leq k$.


In this paper we give some results about depth of Stanley-Reisner rings of complexes. And we determine depth of codismantlable graphs in terms of its induced subgraphs.

Preliminaries
Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $x \in V(G)$ the open and closed neighborhoods of $x$ are denoted by $N_o(x)$ and $N_c[x]$ respectively. For further notions and definitions see

**Definition 1.** Let $\Delta$ be simplicial complex and $\sigma$ a face of $\Delta$. Then we have that,

$$\text{lk}_x(\sigma) = \{ t \in \Delta : t \cap \sigma = \emptyset, t \cup \sigma \in \Delta \}$$

$$\text{del}_x(\sigma) = \{ t \in \Delta : t \cap \sigma = \emptyset \}$$

If $\Delta$ is an independence complex of a graph $G$ and $x$ is a vertex of $G$ then,

$$\text{lk}_{\text{del}(G)}(x) = \text{Ind}(G - N_x[x])$$

and

$$\text{del}_{\text{del}(G)}(x) = \text{Ind}(G - x).$$

**Theorem 2** (See REISNER (1976)) Let $\Delta$ be a simplicial complex and $\sigma$ be a face of $\Delta$. If $k$ is a field, then the following conditions are equivalent:

1. $\Delta$ is a Cohen-Macaulay over $k$.
2. $H_i(\text{lk}_x(\sigma); k) = 0$ for all $\sigma \in \Delta$ and $i < \dim(\text{lk}_x(\sigma))$.

**Definition 3.** The $i$-skeleton of a simplicial complex $\Delta$ is the simplicial complex consists of all $j$-simplices of $\Delta$ with $i < j$ and denoted by $\Delta^i$.

**Theorem 4.** [*] Let $\Delta$ be a simplicial complex. Then $\text{depth}(k[\Delta]) = \max \{ i : k[\Delta^i] \text{ is Cohen-Macaulay} \} + 1$.

**Lemma 5.** (Depth Lemma) If $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated $R$ modules with $R$ a local ring then,

(a) If $\text{depth}(B) \geq \text{depth}(C)$ then $\text{depth}(B) = \text{depth}(C)$.

(b) If $\text{depth}(B) \geq \text{depth}(A)$ then $\text{depth}(A) = \text{depth}(C) + 1$.

(c) If $\text{depth}(C) \geq \text{depth}(A)$ then $\text{depth}(A) = \text{depth}(B)$.

**Lemma 6.** (See VILLARREAL (2015)) Let $R$ be a ring and $I$ is an ideal of $R$. If $x$ is an element of $R$ then,

$$0 \to R/I(x) \to R/I \to R/I(x) \to 0$$

is a short exact sequence.

**Remark 7.** (See DAO AND SCHWEIG (2013)) Let $G$ be a graph on $V(G) = \{x_1, \ldots, x_n\}$. If we assume $R = k[x_1, \ldots, x_n]$ as a polynomial ring over $V(G)$ and $I(G)$ as its edge ideal. Then $(I(G), x_i) = (I(G - x_i), x_i)$ and $(I(G): x_i) = (I(G - N_x[x]), N(x))$.

**Lemma 8.** (See MOREY (2010)) Let $I$ be an ideal in a polynomial ring $R$, let $x$ be an indeterminate over $R$, and let $S = R[x]$. Then $\text{depth}(S/I[S]) = \text{depth}(R/I) + 1$.

In their paper, authors BIYIKOGLU AND CIVAN (2014) introduced a new graph class called codismantlable graphs as follows:

**Definition 9.** (See BIYIKOGLU AND CIVAN (2014)) A vertex $x$ of $G$ is called codominated if there exists a vertex $y \in N(x)$ such that $N_y[y] \subseteq N_x[x]$.

**Definition 10.** (See BIYIKOGLU AND CIVAN (2014)) Let $G$ and $H$ be graphs. If there exist graphs $G_0, G_1, \ldots, G_{i+1}$ satisfying $G \cong G_0$, $H \cong G_{i+1}$ and $G_{i+1} = G_i - x_i$ for each $0 \leq i \leq k$, where $x_i$ is codominated in $G_i$. A graph $G$ is called codismantlable if either it is an edgeless graph or it is codismantlable to an edgeless graph.

**Main Results**

**Lemma 11.** Let $G$ be a graph and $x$ be its vertex. If $N'_G(x)$ is a set of degree one neighbors of $x$, then

$$\text{depth}(R/(I(G), x)) = \text{depth}(k[\text{del}_{\text{del}(G)}(x)]) + [N'_G(x)].$$

**Proof.** By Remark 7 we have

$$\text{depth}(R/(I(G), x)) = \text{depth}(R/(I(G - x), x))$$

and from Lemma 8 one can derive that

$$\text{depth}(R/(I(G), x)) = \text{depth}(k[\text{del}_{\text{del}(G)}(x)]) + [N'_G(x)].$$

Since the quotient ring $R/(I(G - x))$ is exactly $k[\text{del}_{\text{del}(G)}(x)]$. And with the equality

$$\text{depth}(R/(I(G - x))) \otimes (N'_G(x) = \text{depth}(R/(I(G - x))) + [N'_G(x)]),$$

the argument gives us that

$$\text{depth}(R/(I(G), x)) = \text{depth}(k[\text{del}_{\text{del}(G)}(x)]) + [N'_G(x)].$$

**Lemma 12.** Let $G$ be a graph and $x$ be its vertex. Then

$$\text{depth}(R/(I(G), x)) = \text{depth}(k[\text{lk}_{\text{del}(G)}(x)]) + 1.$$

**Proof.** Since from Remark 7 we have the equality

$$\text{depth}(R/(I(G), x)) = \text{depth}(R/(I(G - N'_G(x)), N(x)))$$

by using Lemma 8 one can conclude that

$$\text{depth}(R/(I(G), x)) = \text{depth}(R/(I(G - N'_G(x)))) \otimes k[x].$$

Since the quotient ring $R/(I(G - N'_G(x)))$ is exactly $k[\text{lk}_{\text{del}(G)}(x)]$, we have that

$$\text{depth}(R/(I(G), x)) = \text{depth}(k[\text{lk}_{\text{del}(G)}(x)]) + 1.$$
Lemma 13. Let $\Delta$ be a simplicial complex. If $\text{depth}(k[\Delta]) = d$ then $\tilde{H}_i(\Delta, k) = 0$ for all $i < d - 1$.

Proof. By Reisner criterion if a complex Cohen-Macaulay all homology groups under top dimension vanish. If depth of complex is $d$ then complex has $d$ dimensional Cohen-Macaulay skeleton. This concludes the proof.

Proposition 14. Let $\Delta$ be simplicial complex and $\sigma \in \Delta$ be its face. If $\text{depth}(k[\Delta]) = \text{depth}(k[\operatorname{lk}_i(\sigma)])$ then $\text{depth}(k[\operatorname{lk}_i(\sigma)]) + 1 \leq \text{depth}(k[\operatorname{del}_i(\sigma)])$.

Proof. If we assume that $j$-skeletons of $\Delta$ and $\operatorname{del}_i(\sigma)$ are Cohen-Macaulay, then for all $i < j - 1$, $\tilde{H}_i(\Delta)$ and $\tilde{H}_i(\operatorname{del}_i(\sigma))$ vanish. The exactness of the sequence, $\ldots \rightarrow \tilde{H}_{j-1}(\Delta) \rightarrow \tilde{H}_j(\operatorname{lk}_i(\sigma)) \rightarrow \tilde{H}_j(\operatorname{del}_i(\sigma)) \rightarrow \tilde{H}_j(\Delta) \rightarrow \tilde{H}_{j+1}(\operatorname{lk}_i(\sigma)) \rightarrow \ldots$ gives us, $\tilde{H}_j(\operatorname{lk}_i(\sigma)) \neq 0$, and for all $i < j - 2$ the groups $\tilde{H}_i(\operatorname{del}_i(\sigma)) = 0$. This concludes the proof.

Theorem 15 Let $G$ be a codismantlable graph and $\text{Ind}(G)$ be its independence complex. If $x$ is a codominated vertex, then $\text{depth}(k[\text{Ind}(G)]) = \text{depth}(k[\text{Ind}_{\text{bol}}(G)]) + 1$.

Proof. Let $G$ be codismantlable graph. If $x$ codominated vertex then there exist some $y \in N_0(x)$ such that $N_0(y) \subseteq N_0(x)$. If $y \in N_0(x)$ then by Lemma 11 and Lemma 12 we can say that $\text{depth}(R/I(G), x) \geq \text{depth}(R/I(G), x)$. Otherwise by considering Proposition 14 we still have $\text{depth}(R/I(G), x) \geq \text{depth}(R/I(G), x)$. If we combine this argument with depth lemma and the exactness of the sequence: $0 \rightarrow R/I(G) \rightarrow R/I(G) \rightarrow R/I(G) \rightarrow 0$ then we conclude that $\text{depth}(R/I(G)) = \text{depth}(R/I(G), x)$. Therefore, considering Lemma 12, we get that, $\text{depth}(R/I(G)) = \text{depth}(R/I(G), x) = \text{depth}(k[\text{Ind}_{\text{bol}}(G)]) + 1$.

References


