Explicit Exponential Finite Difference Methods for the Numerical Solution of Modified Burgers' Equation

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Abstract

In this study, explicit exponential finite difference schemes based on four different linearization techniques are given for the numerical solutions of the Modified Burgers' equation. A model problem is used to verify the efficiency and accuracy of the methods that we proposed. Also comparisons are made with the relevant ones in the literature. It is shown that all results that are found to be in good agreement with those available in the literature. L_2 and L_∞ error norms are calculated. The obtained error norms are suciently small in all computer runs. The results show that the present method is a successful numerical scheme for solving the Modified Burgers' equation.

Keywords: Burgers' equation, Modified Burgers' equation,
Explicit exponential finite difference method

Introduction

Burgers' equation was first given by Bateman (BATEMAN (1915)) and later was studied by Burgers (BURGERS (1939), BURGERS (1948)) as a mathematical model for turbulence. Since it has an extensive usage in engineering and other scientific fields, the Burgers' equation has found applications in various fields such as convection and diffusion, number theory, gas dynamics, heat conduction, elasticity etc (ROSHAN and BHAMRA (2011)). The one-dimensional generalized Burgers' equation is in the form

$$u_t + u^p u_x - v u_{xx} = 0 \qquad a \le x \le b, \quad p = 1, 2$$

in which u denotes the velocity for space x and time t and $v > 0$ is a constant representing the knematics viscosity of the fluid. It is known as Burgers' equation and modified Burgers' equation for $p = 1$ and $p = 2$, respectively.

Many analytical and numerical solutions of equation have been published by a number of authors using different methods and techniques. An analytical solution of Burgers' equation was given by Benton and Platzman (BENTON and PLATZMAN (1972)). Infinite series solutions of the equation have been introduced by Miller (MILLER (1966)) Kutluay et al. (KUTLUAY *et al.* (1999)) have obtained numerical solutions of one-dimensional Burgers' equations by using explicit and exact-explicit finite difference methods. A finite element approach was used to obtain the numerical solution of Burgers' equations by Öziş et al. (ÖZİŞ *et al.* (2003)). Dag *et al.* (DAG *et al.* (2005)) have obtained numerical solutions of the Burgers' equations by using B-spline collocation methods. Variational iteration method has been used to solve Burgers and coupled Burgers equations by Abdou and Soliman (ABDOU and SOLIMAN (2005)). The problem we deal with is in general form:

$$u_t + u^2 u_x - v u_{xx} = 0 \qquad a \le x \le b, \quad t \ge t_0$$

with initial condition

$$u(x, t_0) = f(x)$$

and boundary conditions

$$u(a, t) = g_1(t), \qquad u(b, t) = g_2(t)$$

where x and t are independent variables, $u = u(x, t)$, $[a, b]$ is the solution region $(a, b \in R)$, v is the viscosity parameter, $f(x)$, $g_1(t)$ and $g_2(t)$ are known functions.

In the present work, main aim is to apply the explicit exponential finite difference methods to improve a numerical method for the modified Burgers' equation. In the literature many numerical method was applied to approximate the solution of the modified Burgers equation by several authors. Ramadan and El-Danaf (RAMADAN and EL-DANAF (2005)) used the collocation method with quintic splines for the solution of equation. Then, the equation has been solved by Ramadan et al. (RAMADAN *et al.* (2005)) using the colocation method with septic splines. Burgers and the

modified burgers equations are solved numerically by using the time and space splitting techniques to equations, and then using B-spline collocation procedure to approximate the resulting systems by Saka and Dag (SAKA and DAG (2008)). Numerical solutions of modified Burgers' equation was obtained by using sextic B-spline collocation method by Irk (IRK (2009)). Grienwank and El-Danaf (GRIENWANK and EL-DANAF (2009)) obtained the numerical solutions of the modified Burgers' equation by using a non-polynomial spline based method. Bratsos and Petrakis (BRATSOS and PETRAKIS (2011)) used an explicit numerical scheme for solving the modified Burgers' equation. The equation has been numerically solved by Roshan and Bhamra (ROSHAN and BHAMRA (2011)) by the Petrov-Galerkin Method.

The explict exponential finite difference method have been improved by Bhattacharya (BHATTACHARYA (1985)) for the solution of heat equation. Bhattacharya (BHATTACHARYA (1990)) and Handschuh-Keith (HANDSCHUH and KEITH (1992)) used the exponential finite difference technique to obtain numerical solutions of Burgers' equation. Bahadır (BAHADIR (2005)) obtained the numerical solutions of KdV equation for small times by applying the exponential finite difference method. Implicit exponential finite difference method and fully implicit exponential finite difference method were used to solve the Burgers' equation by Inan and Bahadır (INAN and BAHADIR (2013)) The Burgers' equation has been numerically solved by Inan and Bahadır (INAN and BAHADIR (2013)) by applying the Hopf-Cole transformation to equation and then employed the explicit exponential finite difference method to approximate the resulting heat equation.

Model Problem and Numerical Method

Model Problem

We consider the initial boundary value problem for modified Burgers' equation

$$u_t + u^2 u_x = v u_{xx}, \quad 0 \le x \le 1, \quad t \ge 1 \quad (1)$$

with initial condition

$$u(x,1) = \frac{x}{1 + (1/c_0)\exp(x^2/4v)} \quad (2)$$

and boundary conditions

$$u(0,t) = 0, \quad u(1,t) = \frac{1/t}{1 + (\sqrt{t}/c_0)\exp(1/4vt)}. \quad (3)$$

Following (HARRIS (1996)) modified Burgers' equation (1) has the analytic solution

$$u(x,t) = \frac{x/t}{1 + (\sqrt{t}/c_0)\exp(x^2/4vt)}$$

where c_0 is a constant, $0 < c_0 < 1$.

Numerical Method

To obtain numerical solutions, the solution region of the problem $0 \le x \le 1$ is divided by N equal subintervals of length h. We indicate the finite difference approximation of $u(x,t)$ at the mesh point (x_i, t_n) by u_i^n in which $x_i = ih \ (i = 0,1,\ldots,N)$, $\quad t_n = t_0 + nk$ $(n = 0,1,2,\ldots)$, $h = \dfrac{1-0}{N}$ is the mesh size in x direction and k is the time step.

Now let us express the finite difference schemes of the initial condition (2) and the boundary condition (3), respectively,

- $u_i^0 = \dfrac{x_i/1}{1 + (1/c_0)\exp(x_i^2/4v)}$ and

- $u_0^n = 0$,

- $u_N^n = \dfrac{1/t_n}{1 + (\sqrt{t_n}/c_0)\exp(1/4vt_n)}$.

We follow the procedure of ref. (BAHADIR (2005)) to examine the numerical method. If we suppose that $F(u)$ is any continuous differential function and multiplying equation (1) by $\dfrac{\partial F}{\partial u}$ the following equation is obtained:

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} = F'(u)\left(-u^2\frac{\partial u}{\partial x} + v\frac{\partial^2 u}{\partial x^2}\right)$$

and

$$\frac{\partial F}{\partial t} = F'(u)\left(-u^2\frac{\partial u}{\partial x} + v\frac{\partial^2 u}{\partial x^2}\right). \quad (4)$$

Using the forward difference approximation for $\dfrac{\partial F}{\partial t}$ the finite difference representation of equation (4) is obtained as:

$$F(u_i^{n+1}) = F(u_i^n) + kF'(u_i^n)\left[-\left(u^2\frac{\partial u}{\partial x}\right)_i^n + v\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n\right]$$

in which k is the time step. Let $F(u) = \ln u$ then the explicit exponential finite difference scheme for equation (1) is obtained as:

$$u_i^{n+1} = u_i^n \exp\left\{\frac{k}{u_i^n}\left[-\left(u^2\frac{\partial u}{\partial x}\right)_i^n + v\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n\right]\right\} \quad (5)$$

If we use central difference approximation in place of $\frac{\partial^2 u}{\partial x^2}$ in equation (5)

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n \cong \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2}, \quad 1 \leq i \leq N\text{-}1$$

and then apply the following linearization techniques in place of the non-linear term $u^2\frac{\partial u}{\partial x}$

FD-I. $\left(u^2\frac{\partial u}{\partial x}\right)_i^n \cong (u_i^n)^2\left(\frac{u_{i+1}^n - u_{i-1}^n}{2h}\right), \quad 1 \leq i \leq N\text{-}1$

FD-II. $\left(u^2\frac{\partial u}{\partial x}\right)_i^n \cong \left(\frac{u_i^n + u_{i+1}^n}{2}\right)^2\left(\frac{u_{i+1}^n - u_{i-1}^n}{2h}\right), \quad 1 \leq i \leq N\text{-}1$

FD-III. $\left(u^2\frac{\partial u}{\partial x}\right)_i^n \cong \left(\frac{u_{i-1}^n + u_i^n}{2}\right)^2\left(\frac{u_{i+1}^n - u_{i-1}^n}{2h}\right), \quad 1 \leq i \leq N\text{-}1$

FD-IV. $\left(u^2\frac{\partial u}{\partial x}\right)_i^n \cong \left(\frac{u_{i+1}^n + u_i^n + u_{i-1}^n}{3}\right)^2\left(\frac{u_{i+1}^n - u_{i-1}^n}{2h}\right), \quad 1 \leq i \leq N\text{-}1$

respectively, we obtained the following explicit exponential finite difference schemes

E-EFDM-I. $u_i^{n+1} = u_i^n \exp\left\{-\frac{k}{2h}u_i^n\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{rv}{u_i^n}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)\right\}$

E-EFDM-II.

$$u_i^{n+1} = u_i^n \exp\left\{-\frac{k}{8h}\frac{\left(u_i^n + u_{i+1}^n\right)^2}{u_i^n}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{rv}{u_i^n}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)\right\},$$

E-EFDM-III. $u_i^{n+1} = u_i^n \exp\left\{-\frac{k}{8h}\frac{\left(u_{i-1}^n + u_i^n\right)^2}{u_i^n}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{rv}{u_i^n}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)\right\},$

E-EFDM-IV.

$$u_i^{n+1} = u_i^n \exp\left\{-\frac{k}{18h}\frac{\left(u_{i+1}^n + u_i^n + u_{i-1}^n\right)^2}{u_i^n}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{rv}{u_i^n}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)\right\},$$

respectively, where $r = \frac{k}{h^2}$, $1 \leq i \leq N\text{-}1$.

Numerical Results and Discussion

Numerical solutions of model problem are obtained by explicit exponential finite difference method. To show the accuracy of the results, L_2 and L_∞ error norms:

$$L_2 = \|u - U_N\|_2 = \sqrt{h\sum_{j=0}^{N}\left|u_j - (U_N)_j\right|^2},$$

$$L_\infty = \|u - U_N\|_\infty = \max_j\left|u_j - (U_N)_j\right|$$

are used, in which u and U_N represent analytic and computed numerical solutions respectively. Numerical solutions are obtained at different times for different values of v, h, Δt and $c_0 = 0.5$. The obtained results are displayed in Table 1-6 and Figure 1-2. Table 1 and Table 2 present L_2 and L_∞ error norms with $v = 0.001$, $h = 0.0125$, $\Delta t = 0.01$ at different times. L_2 and L_∞ error norms of explicit exponential finite difference schemes at $t_f = 2$ for $v = 0.01$ and $\Delta t = 0.001$ for different values of h are given in Table 3 and Table 4, respectively. It is seem that the values of L_2 and L_∞ decrease with decrease of h in Table 3 and Table 4, respectively. The obtained error norms L_2 and L_∞ of present study are compared with other methods (RAMADAN and EL-DANAF (2005), IRK (2009)) for $v = 0.005$, $h = 0.005$ and $\Delta t = 0.001$ at times $t = 2, 6, 10$ in Table 5. The obtained error norms L_2 and L_∞ of the present study are compared with other methods (RAMADAN et. al. (2005), SAKA and DAG (2008), IRK (2009)) for $v = 0.01$, $h = 0.02$ and $\Delta t = 0.01$ at times $t = 2, 6, 10$ in Table 6. As seen from the Tables 5-6, it is observed that the obtained results using the explicit exponential finite difference schemes are in good agreement with those available in the literature. Figures 1 and 2 show behavior of the numerical solutions for $v = 0.01$ and $v = 0.005$ with $h = 0.05$, $\Delta t = 0.01$ at times $t = 1, 2, 4, 6, 8, 10$ for E-EFDM-I. The top and bottom curves are at $t = 1$ and $t = 10$, respectively. It can be seen from the figures that the curve of the numerical solution decays as the time increases. Note that as the viscosity parameter v gets small the decay gets fast.

Table 1. $L_2 \times 10^3$ error norms with $v=0.001$, $h = 0.0125$ and $\Delta t = 0.01$.

t	E-EFDM-I	E-EFDM-II	E-EFDM-III	E-EFDM-IV
2	0.070907	0.070925	0.070888	0.070916
3	0.061455	0.061509	0.061394	0.061455
4	0.054713	0.054778	0.054640	0.054710
5	0.050197	0.050263	0.050122	0.050193
6	0.046844	0.046908	0.046772	0.046840
7	0.044160	0.044220	0.044092	0.044156
8	0.041909	0.041966	0.041845	0.041905
9	0.039966	0.040020	0.039906	0.039963
10	0.038257	0.038307	0.038200	0.038254

Table 2. $L_\infty \times 10^3$ error norm with $v = 0.001$, $h = 0.0125$ and $\Delta t = 0.01$.

t	E-EFDM-I	E-EFDM-II	E-EFDM-III	E-EFDM-IV
2	0.257336	0.257714	0.256925	0.257335
3	0.223175	0.223464	0.222852	0.223164
4	0.186933	0.187179	0.186657	0.186920
5	0.160985	0.161194	0.160751	0.160973
6	0.141988	0.142166	0.141787	0.141977
7	0.127328	0.127483	0.127154	0.127319
8	0.115506	0.115642	0.115353	0.115497
9	0.105649	0.105769	0.105513	0.105641
10	0.097656	0.097771	0.097526	0.097648

Table 3. $L_2 \times 10^3$ error norms with $v = 0.01$, $\Delta t = 0.001$ and $t_f = 2$.

h	E-EFDM-I	E-EFDM-II	E-EFDM-III	E-EFDM-IV
$h = 0.05$	0.43128	0.43130	0.43074	0.43128
$h = 0.025$	0.39014	0.39040	0.38975	0.39014
$h = 0.0125$	0.38071	0.38088	0.38051	0.38071
$h = 0.01$	0.37961	0.37975	0.37944	0.37961

Table 4. $L_\infty \times 10^3$ error norms with $v = 0.01$, $\Delta t = 0.001$ and $t_f = 2$.

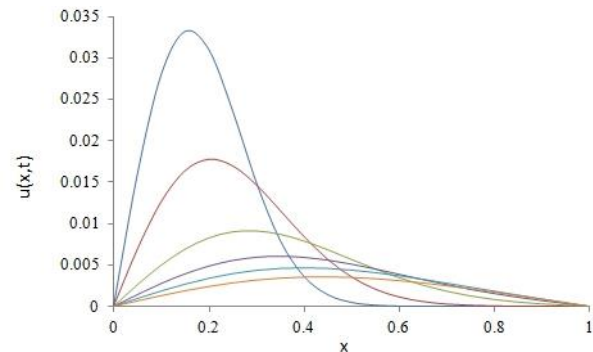| h	E-EFDM-I	E-EFDM-II	E-EFDM-III	E-EFDM-IV
$h = 0.05$	0.87308	0.87702	0.86863	0.87308
$h = 0.025$	0.82601	0.82843	0.82347	0.82601
$h = 0.0125$	0.81577	0.81701	0.81449	0.81577
$h = 0.01$	0.81580	0.81678	0.81480	0.81580

Figure 1. Numerical solutions for $v = 0.01$ with $h = 0.05$, $\Delta t = 0.01$ at times $t = 1, 2, 4, 6, 8, 10$ for E-EFDM-I.
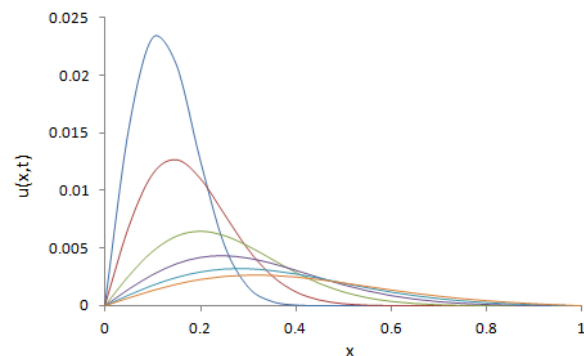

Figure 2. Numerical solutions for $v = 0.005$ with

Conclusions

In this study, explicit exponential finite difference schemes based on four different linearization techniques have been proposed for the numerical solutions of the Modified Burgers' equation. A model problem is used to verify the efficiency and accuracy of the schemes that we proposed. Also comparisons were made with the relevant ones in the literature. L_2 and L_∞ error norms have been calculated and given. The obtained error norms are sufficiently small in all computer runs. The results show that this method is a successful numerical scheme for solving the modified Burgers' equation. $h = 0.05$, $\Delta t = 0.01$ at times $t = 1, 2, 4, 6, 8, 10$ for E-EFDM-I.

Table 5. Comparison of the error norms L_2 and L_∞ with those in other studies in the literature at $t = 2, 6, 10$ for $h = 0.005$, $\Delta t = 0.001$ and $v = 0.005$.

	$t = 2$		$t = 6$		$t = 10$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
E-EFDM-I	0.22610	0.57843	0.16368	0.32834	0.13882	0.22770
E-EFDM-II	0.22615	0.57877	0.16377	0.32851	0.13890	0.22782
E-EFDM-III	0.22605	0.57808	0.16358	0.32816	0.13875	0.22759
E-EFDM-IV	0.22610	0.57843	0.16368	0.32833	0.13882	0.22770
(RAMADAN AND EL-DANAF (2005))	0.25786	0.72264	0.22569	0.43082	0.18735	0.30006
(IRK (2009)) (SBCM1)	0.22890	0.58623	-	-	0.14042	0.23019
(IRK (2009)) (SBCM2)	0.23397	0.58424	-	-	0.13747	0.22626

Table 6. Comparison of the error norms L_2 and L_∞ with those in other studies in the literature at $t = 2, 6, 10$ for $h = 0.02$, $\Delta t = 0.01$ and $v = 0.01$.

	$t = 2$		$t = 6$		$t = 10$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
E-EFDM-I	0.37027	0.78740	0.31581	0.52579	0.55159	1.28125
E-EFDM-II	0.37053	0.78941	0.31636	0.52579	0.55187	1.28125
E-EFDM-III	0.37000	0.78531	0.31524	0.52579	0.55129	1.28125
E-EFDM-IV	0.37031	0.78740	0.31580	0.52579	0.55158	1.28125
(RAMADAN ET. AL. (2005))	0.79043	1.70309	0.51672	0.76105	0.80026	1.80239
(SAKA AND DAG (2008)) (QBCA1)	0.37911	0.81254	0.32941	0.52579	0.55848	1.28125
(SAKA AND DAG (2008)) (QBCA2)	0.39473	0.88383	0.31588	0.53910	0.52425	1.28125
(IRK (2009)) (SBCM1)	0.38474	0.82611	-	-	0.55985	1.28127
(IRK (2009)) (SBCM2)	0.41321	0.81502	-	-	0.55095	1.28127

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