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Coordinates of the Midpoint of a Segment in an Extended Hyperbolic Space

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

In this article, we find an analytical characteristic of the type of a line and derive the formulae for calculating the coordinates of the midpoints and quasi-midpoints of elliptic, hyperbolic, and parabolic segments in an extended hyperbolic space H^3 in the frame of the first type. The space H^3 we consider in the Cayley – Klein projective model as a projective three-dimensional space with an oval quadric γ fixed in it.

Keywords: *Extended hyperbolic space; hyperbilic space; hyperbolic space of positive curvature; segment; midpoint; quasi-midpoint.*
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1. Introduction

In the Cayley – Klein projective model, an *extended hyperbolic space* H^3 consists of two connected components, a hyperbolic space $\widehat{H}^3$ of positive curvature and a Lobachevskii space Λ^3, which is also called a hyperbolic space of negative curvature. The space $\widehat{H}^3$ (Λ^3) can be realized on the exterior (interior) domain with respect to an oval quadric [2, p. 392], [3, p.150] γ in the projective space P^3. This quadric is called the *absolute* of spaces H^3, $\widehat{H}^3$, and Λ^3. The projective automorphisms of the quadric γ form the fundamental group of each of these spaces [3, 11, 12, 13], we denote this group by G.

A plane of the space H^3, depending on its location with respect to the absolute quadric γ, can be elliptic, extended hyperbolic or co-Euclidean. An elliptic (extended hyperbolic) plane intersects the absolute γ along a zero (oval) curve. A co-Euclidean plane has two imaginary conjugate lines from γ [2, 6, 11, 12, 13].

A line of the space H^3, depending on its location with respect to the absolute, can be elliptic, hyperbolic or parabolic. An elliptic (hyperbolic) line has two imaginary conjugate (real) points from the absolute γ. A parabolic line touches the absolute. Objects and ratios on lines of all types are described in [7].

Investigating analogs of known fractal objects of Euclidean geometry in the space H^3 (see, for instance, [10]), we faced the problem of derivation formulae for division a segment in a given ratio. Both these formulae themselves and their derivation in hyperbolic geometry are more cumbersome than in Euclidean geometry. To simplify the task, we first find formulae for calculating the coordinates of the midpoint and the quasi-midpoint of a segment of each type. We solve this task for non-parabolic segments of the extended hyperbolic plane H^2 in the canonical frame of the first type in [5], and for segments in the co-Euclidean plane in [6]. In this paper, we derive the required formulae for segments of all types in the space H^3 in a canonical frame of the first type. In Section 2, we present the main definitions used in this study. In Section 3, we obtain an analytical characteristic of the line type in the space H^3 in Plücker coordinates. Sections 4 and 5 are devoted to the derivation of the desired formulae for non-parabolic and parabolic segments, respectively. In Section 6, we generalize the obtained formulae and formulate the main results of the study in Theorem 6.1.

We note that deriving the desired formulae in the space H^3, we have obtained similar formulae for a segment in an elliptic space, in particular, in an elliptic plane (see Remark 4.3 and Section 6). Not being sure that these formulae are new, we do not state the result in a theorem.

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In future publications, we plan to present the first applications of Theorem 6.1. In particular, we will study the homothety of the space H^3 with coefficients 2 and 0.5 and construct analogues of some fractal objects from Euclidean geometry.

2. Main definitions

In the presented study, the following definitions will be used.

Definition 2.1. Let p be a parabolic line of the space H^3 with a point K from the absolute quadric γ. Any finite points A, B of the line p define a segment $[AB]$ on this line. On the line p there is a single point, say S, that harmonically separated with the point K the pair of points A, B. For the point S the cross-ratio $(ABSK)$ of the quadruple of points on the line p satisfies the condition $(ABSK) = -1$. The point S we call the *midpoint* of the segment $[AB]$ (see, for instance, [6, p. 18], [7, p. 92]).

The construction of the midpoint of a parabolic segment in an extended hyperbolic plane is based on the following lemma.

Lemma 2.1. *[9, Lemma 1] On the plane H^2, the midpoint of a segment of a parabolic line l lies on a hyperbolic line parallel to the distinct from l parabolic lines passing through the ends of this segment.*

Using the costruction algorithm proposed in [8], [9] and choosing some extended hyperbolic plane of the space H^3 containing the given parabolic segment, it is possible to construct the midpoint of this segment. In Figure 1a, we shown the construction of the midpoint of the parabolic segment $[AB]$ in the extended hyperbolic plane H^2 with the absolute conic γ_0.

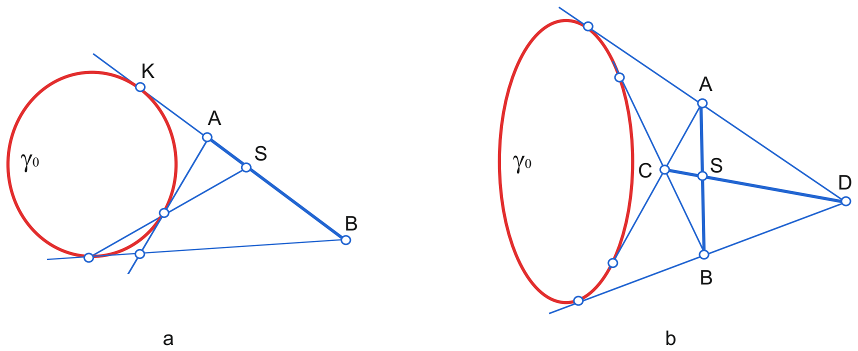

Figure 1. The construction of the midpoint S of the parabolic segment $[AB]$ (a); of the elliptic (hyperbolic) segment $[AB]$ ($[CD]$) (b).

Definition 2.2. Assume that a non-parabolic line p of the space H^3 possesses points K_1 and K_2 from the absolute quadric γ. Points A and B on p we call *orthogonal*, if they are harmonically separated by the points K_1, K_2, that is, if the equality $(ABK_1K_2) = -1$ holds (see, for instance, [7, p. 95, 97]).

Orthogonal points divide the elliptic (hyperbolic) line containing them into two congruent segments (quasisegments) (see [7, p. 95, 101]).

Definition 2.3. Let p be an elliptic line with imaginary conjugate points K_1 and K_2 from the absolute quadric γ. Distinct real points A, B of the line p define two segments on this line. There is a single pair of orthogonal real points, say S and S^*, which harmonically separated the points A and B on the line p. The points S and S^* belong to different segments between the points A and B. The point S or S^* lying on the segment $[AB]$, we call the *midpoint* of this segment, the other point we call the *quasi-midpoint* (see, for instance, [6, p. 23], [7, p. 95]).

The elliptic line p from definition 2.3 can belong to both the space H^3 and the elliptic space. In the case of elliptic space, the quadric γ is a zero quadric or, in other terms, an elliptic quadric (see [2, p. 392] or [3, p. 150], respectively).

Definition 2.4. Let the point A^* (B^*) be orthogonal to the point A (B) on the elliptic line p. The points A^* and B^* belong to the same segment between the points A, B. This segment we called *long*, another segment between these points we called *short*. The length of a short (long) elliptic segment is less (more) than half the length of an elliptic line, that is, less (more) than $\pi\rho/2$, where the number ρ is the curvature radius of the space H^3 (see [6, p.97, 118]).

Definition 2.5. Let p be a hyperbolic line of the space H^3 with the real points K_1 and K_2 from the absolute quadric γ. Assume that real points A and B belong to one branch of the line p between the points K_1 and K_2, that is, $(ABK_1K_2) > 0$. There is a single pair of orthogonal real points, say S and S^*, which harmonically separated the points A and B on the line p. One of the points S, S^* belongs to the segment $[AB]$, we call it the *midpoint* of this segment. The other point lies on a branch of the line p, that does not contain the points A, B, we call it the *quasi-midpoint* of the segment $[AB]$ (see, for instance, [6, p. 127], [7, p. 98]).

In Figure 1b, we shown the construction of the midpoint S of an elliptic segment $[AB]$ (hyperbolic segment $[CD]$) using parabolic lines containing the ends of this segment (see [8]).

Definition 2.6. A *canonical frame of the first type* in the space H^3 is a projective frame $R^* = \{A_1, A_2, A_3, A_4, E\}$, whose the fourth vertex A_4 lies in the interior domain with respect to the absolute quadric γ, and each vertex is a pole with respect γ of the plane containing three other vertices. The unit point E of the canonical frame R^* is chosen so that each of the planes A_1A_2E, A_2A_3E, and A_1A_3E is co-Euclidean.

In any canonical frame R^* of the first type the absolute quadric γ is given by the equation

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0. \tag{2.1}$$

The quadratic form $\phi = x_1^2 + x_2^2 + x_3^2 - x_4^2$ is the *metric form* of the spaces H^3, $\widehat{H}^3$, and Λ^3. The number $\phi(a_i) = a_1^2 + a_2^2 + a_3^2 - a_4^2$ we call the *characteristic* of the real coordinates (a_i), $i = 1, 2, 3, 4$, of a point A in the frame R^*. For the real coordinares (a_i) of any proper point A of the space $\widehat{H}^3$ (Λ^3) the inequality holds

$$a_1^2 + a_2^2 + a_3^2 - a_4^2 > 0 \quad \left(a_1^2 + a_2^2 + a_3^2 - a_4^2 < 0\right).$$

The conjugacy of points coordinates with respect to the bilinear form ψ, which is polar to the form ϕ, is an analytical characteristic of orthogonality of the points in canonical frames of the first type. For orthogonal points $A(a_i)$ and $B(b_i)$ we have $\psi(a_i, b_i) = 0$, that is,

$$a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4 = 0. \tag{2.2}$$

3. Characteristic of the line type in Plücker coordinates

Let A and B be distinct real points on a line p in the space H^3. Assume that these points are given by coordinates (a_i), (b_i), $i = 1, 2, 3, 4$, respectively, in a canonical frame R^* of the first type. Let us agree that for all i the mumbers a_i, b_i are real. These conditions will help us to uniquely determine the type of the line p by the sign of the characteristic of its Plücker coordinates.

We denote the Plücker coordinates (see, for instance, [4, 14]) of the line p in R^* by $(p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34})$, where real numbers p_{jk}, $j, k = 1, 2, 3, 4$, $j < k$, are determined by the equalities

$$p_{jk} = \begin{vmatrix} a_j & a_k \\ b_j & b_k \end{vmatrix}$$

and satisfy the quadratic Plúcker relation

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0. \tag{3.1}$$

The bilinear form, polar to the quadratic form from the left side of Eq. (3.1), defines the condition of the complanarity of the lines p and q with the Plücker coordinates (p_{jk}) and (q_{jk}), respectively. This condition has form

$$p_{12}q_{34} - p_{13}q_{24} + p_{14}q_{23} + p_{34}q_{12} - p_{24}q_{13} + p_{23}q_{14} = 0. \tag{3.2}$$

The coordinate plane $A_1A_2A_3$ of the canonical frame R^* of the first type in the space H^3 is elliptic. Therefore, every line lying in this plane is also elliptic. When the line p lies in the plane $A_1A_2A_3$, it is complanar with each of coordinate lines A_1A_2, A_1A_3, and A_2A_3. Using equality (3.2) and the Plücker coordinates of the lines

$$A_1A_2\,(1:0:0:0:0:0), \quad A_1A_3\,(0:1:0:0:0:0), \quad A_2A_3\,(0:0:0:1:0:0),$$

we find the analytical conditions of belonging of the line p to the plane $A_1A_2A_3$

$$p_{14} = p_{24} = p_{34} = 0. \tag{3.3}$$

So, under conditions (3.3) the line p lies in the plane $A_1A_2A_3$ and is elliptic.

Now assume that the line p does not lie in the plane $A_1A_2A_3$. In this case, at least one of the numbers p_{14}, p_{24}, p_{34} is non-zero. Without loss of generality, we agree that $p_{34} \neq 0$. This condition means that the lines p and A_1A_2 are not complanar.

Let us find the common points K_1, K_2 of the line p and the absolute quadric γ using the condition

$$\{K_1, K_2\} = ABA_1 \cap ABA_2 \cap \gamma.$$

The planes ABA_1 and ABA_2 are given in the frame R^* by the following equations

$$ABA_1 : \; x_2p_{34} - x_3p_{24} + x_4p_{23} = 0, \quad ABA_2 : \; x_1p_{34} - x_3p_{14} + x_4p_{13} = 0. \tag{3.4}$$

From Eqs. (3.4) we obtain

$$x_1 = \frac{x_3p_{14} - x_4p_{13}}{p_{34}}, \quad x_2 = \frac{x_3p_{24} - x_4p_{23}}{p_{34}}. \tag{3.5}$$

The consequence of the system of Eqs. (2.1), (3.5) is the equation

$$x_3^2\left(p_{14}^2 + p_{24}^2 + p_{34}^2\right) - 2x_3x_4\left(p_{13}p_{14} + p_{23}p_{24}\right) + x_4^2\left(p_{13}^2 + p_{23}^2 - p_{34}^2\right) = 0. \tag{3.6}$$

Since $p_{34} \neq 0$, from Eqs. (3.5), (3.6) we obtain the coordinates of the points K_1 and K_2 in the frame R^*

$$K_{1,2}\left(-p_{12}p_{24} - p_{13}p_{34} \pm p_{14}\sqrt{\mu} : p_{12}p_{14} - p_{23}p_{34} \pm p_{24}\sqrt{\mu} : p_{13}p_{14} + p_{23}p_{24} \pm p_{34}\sqrt{\mu} : p_{14}^2 + p_{24}^2 + p_{34}^2\right), \tag{3.7}$$

where

$$\mu = -p_{12}^2 - p_{13}^2 - p_{23}^2 + p_{14}^2 + p_{24}^2 + p_{34}^2. \tag{3.8}$$

The number μ from (3.8) is the value of the quadratic form $\Phi = -x_{12}^2 - x_{13}^2 - x_{23}^2 + x_{14}^2 + x_{24}^2 + x_{34}^2$ on the Plücker coordinates p_{jk} of the line p in the frame R^*: $\mu = \Phi(p_{jk})$.

The sign of the number $\Phi(p_{jk})$ uniquely determines the nature of the points K_1, K_2. These points are distinct imaginary conjugate (real) if and only if $\Phi(p_{jk}) < 0$ ($\Phi(p_{jk}) > 0$), and they coincide when $\Phi(p_{jk}) = 0$. Thus the sign of the number $\Phi(p_{jk})$ uniquely determines the type of the line p. For the elliptic, hyperbolic or parabolic line p we have, respectively,

$$\Phi(p_{jk}) < 0, \quad \Phi(p_{jk}) > 0 \quad \text{or} \quad \Phi(p_{jk}) = 0. \tag{3.9}$$

Note that conditions (3.9) are also valid when the line p lies in the plane $A_1A_2A_3$ and is elliptic, that is, when conditions (3.3) hold.

Each transformation of the fundamental group G of the space H^3 keeps the type of a line, hense the sign of the number $\Phi(p_{jk})$ is an invariant of the group G. By direct calculations, we get the expression the number $\Phi(p_{jk})$ in terms of the metric forms of the space H^3

$$\Phi(p_{jk}) = \psi^2(a_i, b_i) - \phi(a_i)\phi(b_i). \tag{3.10}$$

We call the number $\Phi(p_{jk})$ the *characteristic* of the real Plücker coordinates p_{jk} of the line p of the space H^3 in the canonical frame of the first type. Since the projective coordinates of points, and hence the Plücker coordinates p_{jk}, are given up to a non-zero factor, the number itself $\Phi(p_{jk})$ has no geometric meaning. We are only interested in the sign of this number.

4. Coordinates of the midpoint and the quasi-midpoint of a non-parabolic segment

4.1. Derivation of main formulae

Assume that distinct real points A and B of a non-parabolic line p in the space H^3 are given in a canonical frame R^* of the first type by the coordinates (a_i) and (b_i), $i = 1, 2, 3, 4$, respectively. Let us find the coordinates (s_v) and (s_v^*), $v = 1, 2, 3, 4$, of the midpoint S and the quasi-midpoint S^* of the segment $[AB]$ using Definitions 2.2, 2.3, and 2.5.

I. At the first stage, assume that the line p does not belong to the coordinate plane $A_1A_2A_3$. In this case, at least one of the Plücker coordinates p_{14}, p_{24}, p_{34} of the line p is not equal to zero. Let us agree that $p_{34} \neq 0$.

Then the line p can be considered as a common line of planes ABA_1 and ABA_2. Sinse $S \in p$ and $S^* \in p$, the coordinates of the points S and S^* satisfy Eqs. (3.5). Consequently,

$$s_1 = \frac{s_3 p_{14} - s_4 p_{13}}{p_{34}}, \quad s_2 = \frac{s_3 p_{24} - s_4 p_{23}}{p_{34}}, \quad s_1^* = \frac{s_3^* p_{14} - s_4^* p_{13}}{p_{34}}, \quad s_2^* = \frac{s_3^* p_{24} - s_4^* p_{23}}{p_{34}}. \tag{4.1}$$

According to condition (2.2), for coordinates (s_v), (s_v^*) of the orthogonal points S, S^* the equality holds

$$s_1 s_1^* + s_2 s_2^* + s_3 s_3^* - s_4 s_4^* = 0. \tag{4.2}$$

From the system of conditions (4.1), (4.2) we obtain

$$s_3 s_3^* \left(p_{14}^2 + p_{24}^2 + p_{34}^2\right) - \left(s_3 s_4^* + s_4 s_3^*\right)\left(p_{13}p_{14} + p_{23}p_{24}\right) + s_4 s_4^* \left(p_{13}^2 + p_{23}^2 - p_{34}^2\right) = 0. \tag{4.3}$$

The requirement of harmonic separation of pairs of the points A, B and S, S^* from definitions 2.3, 2.5 is equivalent to the equality $(ABSS^*) = -1$. Rewriting this equality in coordinates, we have

$$\frac{\begin{vmatrix} a_3 & a_4 \\ s_3 & s_4 \end{vmatrix} \begin{vmatrix} b_3 & b_4 \\ s_3^* & s_4^* \end{vmatrix}}{\begin{vmatrix} a_3 & a_4 \\ s_3^* & s_4^* \end{vmatrix} \begin{vmatrix} b_3 & b_4 \\ s_3 & s_4 \end{vmatrix}} = -1. \tag{4.4}$$

Equality (4.4) leads to the condition

$$2 s_3 s_3^* a_4 b_4 - \left(s_3 s_4^* + s_4 s_3^*\right)\left(a_3 b_4 + a_4 b_3\right) + 2 s_4 s_4^* a_3 b_3 = 0. \tag{4.5}$$

Excluding the expression $s_3 s_4^* + s_4 s_3^*$ from equalities (4.3), (4.5), we get

$$s_3 s_4^* + s_4 s_3^* = \frac{2 s_3 s_3^* a_4 b_4 + 2 s_4 s_4^* a_3 b_3}{\Delta}, \quad \Delta = a_3 b_4 + a_4 b_3, \tag{4.6}$$

$$\begin{aligned} & s_3 s_3^* \left[\Delta\left(p_{14}^2 + p_{24}^2 + p_{34}^2\right) - 2 a_4 b_4 \left(p_{13}p_{14} + p_{23}p_{24}\right)\right] + \\ & + s_4 s_4^* \left[\Delta\left(p_{13}^2 + p_{23}^2 - p_{34}^2\right) - 2 a_3 b_3 \left(p_{13}p_{14} + p_{23}p_{24}\right)\right] = 0. \end{aligned} \tag{4.7}$$

Dividing both parts of equalities (4.6), (4.7) by $s_4 s_4^*$, we obtain

$$\frac{s_3}{s_4} + \frac{s_3^*}{s_4^*} = \frac{2 \dfrac{s_3}{s_4} \dfrac{s_3^*}{s_4^*} a_4 b_4 + 2 a_3 b_3}{\Delta}, \quad \frac{s_3}{s_4} \frac{s_3^*}{s_4^*} = -\frac{\Delta\left(p_{13}^2 + p_{23}^2 - p_{34}^2\right) - 2 a_3 b_3 \left(p_{13}p_{14} + p_{23}p_{24}\right)}{\Delta\left(p_{14}^2 + p_{24}^2 + p_{34}^2\right) - 2 a_4 b_4 \left(p_{13}p_{14} + p_{23}p_{24}\right)}. \tag{4.8}$$

Consider expressions from (4.8) as a system of equations with respect to variables s_3/s_4 and s_3^*/s_4^*. Solving this system, we find

$$\frac{s_3}{s_4} = \Omega + \varepsilon\sqrt{\Omega^2 - \Theta}, \quad \frac{s_3^*}{s_4^*} = \Omega - \varepsilon\sqrt{\Omega^2 - \Theta}, \quad \varepsilon = \pm 1, \tag{4.9}$$

where

$$\Theta = -\frac{\Delta\left(p_{13}^2 + p_{23}^2 - p_{34}^2\right) - 2 a_3 b_3 \left(p_{13}p_{14} + p_{23}p_{24}\right)}{\Delta\left(p_{14}^2 + p_{24}^2 + p_{34}^2\right) - 2 a_4 b_4 \left(p_{13}p_{14} + p_{23}p_{24}\right)}, \quad \Omega = \frac{a_3 b_3 + \Theta\, a_4 b_4}{\Delta}, \quad \Delta = a_3 b_4 + a_4 b_3. \tag{4.10}$$

Using expressions (4.1), (4.9), and (4.10), we write down the coordinates of the points S and S^* in the following form

$$\left(p_{14}\Omega + \varepsilon p_{14}\sqrt{\Omega^2 - \Theta} - p_{13} \,:\, p_{24}\Omega + \varepsilon p_{24}\sqrt{\Omega^2 - \Theta} - p_{23} \,:\, p_{34}\Omega + \varepsilon p_{34}\sqrt{\Omega^2 - \Theta} \,:\, p_{34}\right), \ \varepsilon = \pm 1. \tag{4.11}$$

In each specific task, the choice of the number ε depends on the location of the points S, S^* with respect to the given segment $[AB]$. We discuss this problem in more detail in §4.2.

If the line p does not lie in the plane $A_1A_2A_3$, but is complanar with the line A_1A_2, that is, $p_{34} = 0$, then at least one of the conditions $p_{14} \neq 0$, $p_{24} \neq 0$ is true. By making the appropriate indexes substitution, we can find the coordinates of the points S, S^* in each of the possible cases.

In some problems, it is convenient to use pre-calculated values of the quadratic form ϕ. To facilitate solutions of such problems, we find the corresponding block of coordinates for the points S, S^*. To this end, we first express the numbers Θ, Ω, and $\sqrt{\Omega^2 - \Theta}$ from (4.10) in terms of the characteristics of real coordinates (a_i), (b_i) of the points A and B by means direct symbolic calculations. In this way, we get

$$\Theta = \frac{a_3^2\phi(b_i) - b_3^2\phi(a_i)}{a_4^2\phi(b_i) - b_4^2\phi(a_i)}, \quad \Omega = \frac{a_3a_4\phi(b_i) - b_3b_4\phi(a_i)}{a_4^2\phi(b_i) - b_4^2\phi(a_i)}, \quad \sqrt{\Omega^2 - \Theta} = p_{34}\frac{\sqrt{\phi(a_i)\phi(b_i)}}{a_4^2\phi(b_i) - b_4^2\phi(a_i)}. \tag{4.12}$$

Using expressions (4.12), we write down the coordinates of the points S, S^* from (4.11) in the following form

$$\Big(a_1a_4\phi(b_i) - b_1b_4\phi(a_i) + \varepsilon p_{14}\sqrt{\phi(a_i)\phi(b_i)} \,:\, a_2a_4\phi(b_i) - b_2b_4\phi(a_i) + \varepsilon p_{24}\sqrt{\phi(a_i)\phi(b_i)} \,:$$
$$a_3a_4\phi(b_i) - b_3b_4\phi(a_i) + \varepsilon p_{34}\sqrt{\phi(a_i)\phi(b_i)} \,:\, a_4^2\phi(b_i) - b_4^2\phi(a_i)\Big), \ \varepsilon = \pm 1. \tag{4.13}$$

Considering that $p_{jj} = 0$, the coordinates (s_v) from (4.13) can be written as follows

$$s_v = a_va_4\phi(b_i) - b_vb_4\phi(a_i) + \varepsilon p_{v4}\sqrt{\phi(a_i)\phi(b_i)}, \ v = 1, 2, 3, 4, \ \varepsilon = \pm 1. \tag{4.14}$$

Remark 4.1. Let us pay attention to the fact that moving from formulae (4.11) to formulae (4.14), we divided each coordinate of the points S, S^* by p_{34}. As a result, formulae (4.14) allow us to calculate the coordinates of these points and in the case when $p_{34} = 0$. Thus formulae (4.14) are true for any location of the line p outside the plane $A_1A_2A_3$.

Remark 4.2. Assume that the points A and B lie in the hyperbolic coordinate plane $A_1A_2A_4$, that is, $a_3 = b_3 = 0$. Then the points S and S^* lie in the plane $A_1A_2A_4$ too. In this case, coordinates (4.14) can be rewritten in the form

$$\Big(a_1a_4\phi(b_i) - b_1b_4\phi(a_i) + \varepsilon p_{14}\sqrt{\phi(a_i)\phi(b_i)} \,:\, a_2a_4\phi(b_i) - b_2b_4\phi(a_i) + \varepsilon p_{24}\sqrt{\phi(a_i)\phi(b_i)} \,:\, 0 \,:\, a_4^2\phi(b_i) - b_4^2\phi(a_i)\Big),$$

where

$$\phi(a_j) = a_1^2 + a_2^2 - a_4^2, \quad \phi(b_j) = b_1^2 + b_2^2 - b_4^2.$$

Replacing the index 4 in these expressions with the index 3, we get formulae (9) from [5]. Consequently, formulae (4.14) are a generalization of formulae (9) from [5].

II. Now assume that the line p belongs to the coordinate plane $A_1A_2A_3$, that is, equalities (3.3) hold. In this case, we can consider the line p as the intersection of planes $A_1A_2A_3$, ABA_4 and give it in the frame R^* by the system of equations

$$x_4 = 0, \quad x_1p_{23} - x_2p_{13} + x_3p_{12} = 0. \tag{4.15}$$

Since the line p is non-parabolic, we have $\Phi(p_{jk}) \neq 0$. This means that under conditions (3.3) at least one of the numbers p_{12}, p_{13}, p_{23} is different from zero. Let $p_{12} \neq 0$. Then for coordinates of the points A, B, S, and S^* from Eqs. (4.15) we obtain the following conditions

$$s_3 = \frac{s_2p_{13} - s_1p_{23}}{p_{12}}, \quad s_3^* = \frac{s_2^*p_{13} - s_1^*p_{23}}{p_{12}}, \quad a_4 = b_4 = s_4 = s_4^* = 0. \tag{4.16}$$

In the case under consideration, condition (2.2) for the orthogonal points S and S^* has form

$$s_1s_1^* + s_2s_2^* + s_3s_3^* = 0. \tag{4.17}$$

Excluding coordinates s_3, s_3^* from expressions in (4.16), (4.17), we get to the equality

$$s_1s_1^*\left(p_{12}^2 + p_{23}^2\right) - p_{13}p_{23}\left(s_1s_2^* + s_2s_1^*\right) + s_2s_2^*\left(p_{12}^2 + p_{13}^2\right) = 0. \tag{4.18}$$

The harmonic separation of pairs of the points A, B and S, S^* implies the equality $(ABSS^*) = -1$. Rewriting this equality in coordinates, we get to the condition

$$2s_1s_1^*a_2b_2 - \left(s_1s_2^* + s_2s_1^*\right)\left(a_1b_2 + a_2b_1\right) + 2s_2s_2^*a_1b_1 = 0. \tag{4.19}$$

Excluding the expression $s_1s_2^* + s_2s_1^*$ from equalities (4.18), (4.19), we obtain

$$s_1s_2^* + s_2s_1^* = \frac{2s_1s_1^*a_2b_2 + 2s_2s_2^*a_1b_1}{\Delta}, \quad \Delta = a_1b_2 + a_2b_1, \tag{4.20}$$

$$s_1 s_1^* \left[\Delta\left(p_{12}^2+p_{23}^2\right)-2a_2b_2p_{13}p_{23}\right]+s_2 s_2^* \left[\Delta\left(p_{12}^2+p_{13}^2\right)-2a_1b_1p_{13}p_{23}\right]=0. \tag{4.21}$$

Dividing both parts of equalities (4.20), (4.21) by $s_2 s_2^*$, we obtain

$$\frac{s_1}{s_2}+\frac{s_1^*}{s_2^*}=\frac{2\,\dfrac{s_1}{s_2}\,\dfrac{s_1^*}{s_2^*}\,a_2b_2+2a_1b_1}{\Delta}, \qquad \frac{s_1}{s_2}\,\frac{s_1^*}{s_2^*}=-\frac{\Delta\left(p_{12}^2+p_{13}^2\right)-2a_1b_1p_{13}p_{23}}{\Delta\left(p_{12}^2+p_{23}^2\right)-2a_2b_2p_{13}p_{23}}. \tag{4.22}$$

From expressions (4.22) we find

$$\frac{s_1}{s_2}=\Omega+\varepsilon\sqrt{\Omega^2-\Theta}, \qquad \frac{s_1^*}{s_2^*}=\Omega-\varepsilon\sqrt{\Omega^2-\Theta}, \qquad \varepsilon=\pm1, \tag{4.23}$$

where

$$\Theta=-\frac{\Delta\left(p_{12}^2+p_{13}^2\right)-2a_1b_1p_{13}p_{23}}{\Delta\left(p_{12}^2+p_{23}^2\right)-2a_2b_2p_{13}p_{23}}, \qquad \Omega=\frac{a_1b_1+\Theta\,a_2b_2}{\Delta}, \qquad \Delta=a_1b_2+a_2b_1. \tag{4.24}$$

Using expressions (4.16), (4.23), and (4.24), we give the coordinates of the points S and S^* in the following form

$$\left(p_{12}\Omega+\varepsilon p_{12}\sqrt{\Omega^2-\Theta}\ :\ p_{12}\ :\ p_{13}-p_{23}\left(\Omega+\varepsilon\sqrt{\Omega^2-\Theta}\right)\ :\ 0\right). \tag{4.25}$$

By rewriting expressions from (4.24) in terms of the characteristics of the real coordinates of points, we get

$$\Theta=\frac{a_1^2\phi(b_i)-b_1^2\phi(a_i)}{a_2^2\phi(b_i)-b_2^2\phi(a_i)}, \qquad \Omega=\frac{a_1a_2\phi(b_i)-b_1b_2\phi(a_i)}{a_2^2\phi(b_i)-b_2^2\phi(a_i)}, \qquad \sqrt{\Omega^2-\Theta}=p_{12}\frac{\sqrt{\phi(a_i)\phi(b_i)}}{a_2^2\phi(b_i)-b_2^2\phi(a_i)}.$$

Hence, according to expressions (4.25), the midpoint S and the quasi-midpoint S^* of the segment $[AB]$ in the plane $A_1A_2A_3$ can be set in the frame R^* by coordinates

$$\Big(a_1a_2\phi(b_i)-b_1b_2\phi(a_i)+\varepsilon p_{12}\sqrt{\phi(a_i)\phi(b_i)}\ :\ a_2^2\phi(b_i)-b_2^2\phi(a_i)\ :$$

$$:\ a_2a_3\phi(b_i)-b_2b_3\phi(a_i)-\varepsilon p_{23}\sqrt{\phi(a_i)\phi(b_i)}\ :\ 0\Big),\ \varepsilon=\pm1. \tag{4.26}$$

The number ε in coordinates (4.26) can be determined only in a specific task (see Example 4.2 in §4.2).

Remark 4.3. Formulae (4.26) can be considered as formulae of elliptic geometry. Since $p_{32}=-p_{23}$, in an elliptic plane with the metric form

$$\phi=x_1^2+x_2^2+x_3^2,$$

the midpoint S and the quasi-midpoint S^* of a segment between the points $A(a_i)$, $B(b_i)$, $i=1,2,3$, can be given by the coordinates (s_v), where

$$s_v=a_va_2\phi(b_i)-b_vb_2\phi(a_i)+\varepsilon p_{v2}\sqrt{\phi(a_i)\phi(b_i)},\ v=1,2,3,\ \varepsilon=\pm1. \tag{4.27}$$

4.2. *On choosing the number ε*

In obtained formulae (4.14) and (4.26) for calculating coordinates of the midpoint and the quasi-midpoint of a non-parabolic segment, the number ε remains undefined. Naturally, the question arises as to whether the formulae can be refined in the general case. We give a negative answer to this question in the following simple examples.

Example 4.1. Let us consider a hyperbolic segment $[AB]$ in the space $\widehat{H}^3$. Assume that the points A and B are given by the coordinates $(0:0:1:0)$ and $(0:0:b:1)$, respectively, in the canonical frame R^* of the first type. Since the point B lies in $\widehat{H}^3$, the conditions $b^2-1>0$, $b\in\mathbb{R}$ hold. The Plücker coordinates of the line AB satisfy the conditions

$$p_{12}=0,\ p_{13}=0,\ p_{14}=0,\ p_{23}=0,\ p_{24}=0,\ p_{34}=1.$$

Since $\Phi(p_{jk})>0$, the line AB is hyperbolic. By formulae (4.14) we find the coordinates of the points S, S^*

$$\left(0:0:b+\varepsilon\sqrt{b^2+1}:1\right),\ \varepsilon=\pm1.$$

The midpoint S of the segment $[AB]$ lies in the space $\widehat{H}^3$. Consequently, the characteristic of the real coordinates of this point satisfies the condition

$$\left(b+\varepsilon\sqrt{b^2+1}\right)^2-1>0. \tag{4.28}$$

The truth of inequality (4.28) depends on the sign of the number b. If $b>0$, then the inequality (4.28) is true under the condition $\varepsilon=1$. If $b<0$, then this inequality is true under $\varepsilon=-1$.

Example 4.2. Let now $[AB]$ be an elliptic segment in the space $\widehat{H}^3$. Assume that the points A and B are given in the frame R^* by the coordinates $(1:0:0:0)$ and $(b:1:0:0)$, respectively. The Plücker coordinates of the line AB are as follows

$$p_{12}=1,\ p_{13}=0,\ p_{14}=0,\ p_{23}=0,\ p_{24}=0,\ p_{34}=0.$$

Since $\Phi(p_{jk})<0$, the line AB is elliptic. By formulae (4.26) we obtain the coordinates of the points S, S^*

$$\left(b+\varepsilon\sqrt{b^2+1}:1:0:0\right),\ \varepsilon=\pm1.$$

Denote the point orthogonal to the point A (B) on the line AB by A^* (B^*). The points A^* and B^* belong to the long segment (see Definition 2.4) between the points A, B, and $A^*=A_2(0:1:0:0)$.

To clarify the goal, suppose that we want to find the coordinates of the midpoint S of the short segment $[AB]$. In this case, the point S, together with each of the points A^*, B^*, separates the pair A, B. Consequently, the inequalities $(ABA^*S)<0$ and $(ABB^*S)<0$ hold. Rewriting, for example, the first of these inequalities in coordinates, we get

$$-\frac{\varepsilon\sqrt{b^2+1}}{b}<0. \tag{4.29}$$

The truth of inequality (4.29) again depends on the sign of the number b. If $b>0$, then the inequality (4.29) is true under the condition $\varepsilon=1$. If $b<0$, then this inequality is true under $\varepsilon=-1$.

The considered examples show us that with the same value of ε we can get coordinates of both the midpoint and the quasi-midpoint of the segment. To refine the formulae (4.14), (4.26), we need to have additional information about coordinates of ends of the given segment. Consequently, in the general case, it is impossible to refine these formulae.

Note that the problem of choosing the sign of an expression arises in almost all tasks of the analytic geometry of the extended hyperbolic plane (see, for instance, [7]). For each specific task, we find the most appropriate way of the solution. When we solve a theoretical problem, for example, prove theorems and derive formulae, we are guided by the following assertions.

1. The midpoint of a hyperbolic segment lies in the same component of the space H^3 as the ends of this segment. Consequently, the characteristic of the real coordinates of the segment midpoint has the same sign as the characteristics of the real coordinates of its ends. When choosing the number ε, can be used any of the inequalities
$$\phi(a_j)\phi(s_j)>0,\quad \phi(b_j)\phi(s_j)>0.$$
2. The midpoint of a hyperbolic segment paired with any absolute point of the line containing this segment, separates the ends of the segment. When choosing the number ε, can be used any of the inequalities
$$(ABSK_1)<0,\quad (ABSK_2)<0.$$
3. The midpoint of each segment paired with any of its points does not separate the ends of this segment.
4. Let the point A^* (B^*) be orthogonal to the point A (B) on the hyperbolic line AB. Then the midpoint of the segment $[AB]$ paired with any of the points A^*, B^* separates the ends of this segment. When choosing ε, we may use any of the inequalities
$$(ABSA^*)<0,\quad (ABSB^*)<0.$$
5. Let the point A^* (B^*) be orthogonal to the point A (B) on the elliptic line AB. Then the midpoint of the short (long) segment $[AB]$ (see Definition 2.4) paired with any of the points A^*, B^* separates (does not separate) the ends of this segment. In the case of the short (long) segment $[AB]$, when choosing ε, we may use any of the inequalities
$$(ABSA^*)<0,\quad (ABSB^*)<0,\qquad ((ABSA^*)>0,\quad (ABSB^*)>0)\,.$$

In software modeling of the studied objects, we preliminarily give the location domain of the sought points. The choice of the segment midpoint from the pair S, S^*, that is, the choice of ε, depends on whether the desired point belongs to the given domain. As an example, let us consider the results of the Chaos game on trihedrals of the hyperbolic plane $\widehat{H}$ of positive curvature. The analogues of the Sierpinski triangle presented here were obtained using the software package piv (see [10], [15]).

In Figure 2a, we show the result of the Chaos game in the plane $\widehat{H}$ on a trihedral of the type $epp(I)$ (see [7, §5.5.4]) with vertices at points $A_1(1:0:0)$, $A_2(0:1:0)$, $B(1:1:1)$. The domain of this trihedral is given by the conditions

$$x > 1, \quad y > 1.$$

In the Euclidean plane, this trihedral is represented by a triangle with an edge on an ideal line. In the figure, we see only its fragment.

In Figure 2b, we show the absolute conic γ_0 of the plane $\widehat{H}$ and the result of the Chaos game on a trihedral of the type $eee(I)$ (see [7, §5.4.1]) with vertices at points $A(2:0:1)$, $B(4:3:1)$, $C(4:-3:1)$. Its domain is given by the conditions

$$3x + 2y - 6 > 0, \quad 3x - 2y - 6 > 0, \quad x < 4.$$

In each case, considering all possible segments, we choose the midpoint or the quasi-midpoint from the given domain.

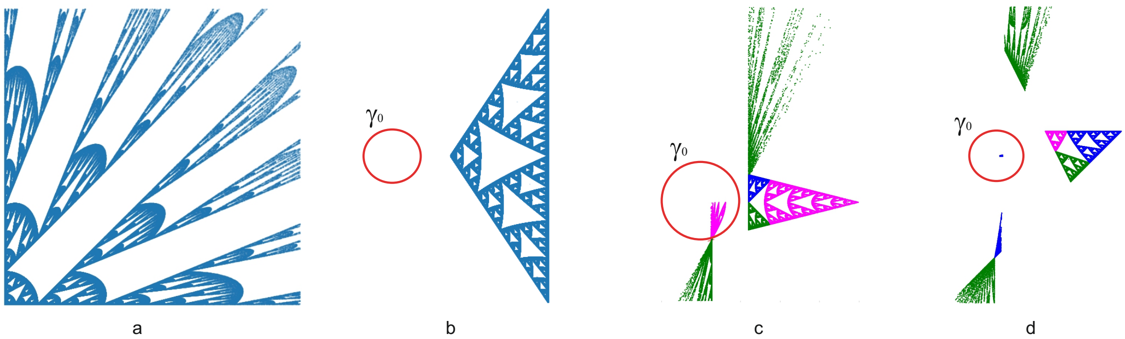

Figure 2. Analogues of the Sierpinski triangle in the plane H^2 on trihedrals of types $epp(I)$ (a), $eee(I)$ (b), $epp(I)$ (c), and $eep(I)$ (d).

In Figures 2c and 2d, we show the result of the Chaos game on trihedrals of types $epp(I)$ and $eep(I)$ (see [7, §5.5.1]), respectively. Here we constructed the midpoint and the quasi-midpoint of each of the considered segments. The points of jump to the same vertex are marked with the same color.

Notice that we depict objects on the Euclidean plane using the transition from projective to Euclidean coordinates by the formulae

$$x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3}.$$

5. Coordinates of the midpoint of a parabolic segment

Let $[AB]$ be a segment on a parabolic line p with the point K from the absolute quadric γ. Assume that points A and B are given in a canonical frame R^* of the first type by coordinates (a_i) and (b_i), $i = 1, 2, 3, 4$, respectively. For the Plücker coordinates p_{jk} of the parabolic line p the condition $\Phi(p_{jk}) = 0$ holds, and coordinates (k_i) of the point K can be find from (3.7) under $\mu = 0$. Here we use only the third and fourth coordinates of the point K, we write them down

$$k_3 = p_{13}p_{14} + p_{23}p_{24}, \quad k_4 = p_{14}^2 + p_{24}^2 + p_{34}^2. \tag{5.1}$$

Since the line p has the point K from γ, it does not lie in the elliptic plane $A_1A_2A_3$. Consequently, at least one of the numbers p_{14}, p_{24}, p_{34} is not equal to zero. Let, for example, $p_{34} \neq 0$. In this case, lines p and A_1A_2 are not complanar, and the line p can be set in R^* by the system of Eqs. (3.5).

Let us find the coordinates (s_v), $v = 1, 2, 3, 4$, of the midpoint S of the segment $[AB]$.

By difinition 2.1, we have $(ABKS) = -1$. Rewriting this condition in coordinates, we get

$$\frac{\begin{vmatrix} a_3 & a_4 \\ k_3 & k_4 \end{vmatrix}\begin{vmatrix} b_3 & b_4 \\ s_3 & s_4 \end{vmatrix}}{\begin{vmatrix} a_3 & a_4 \\ s_3 & s_4 \end{vmatrix}\begin{vmatrix} b_3 & b_4 \\ k_3 & k_4 \end{vmatrix}} = -1. \tag{5.2}$$

Equality (5.2) implies

$$\frac{s_3}{s_4} = -\frac{2a_3b_3k_4 - \Delta k_3}{2a_4b_4k_3 - \Delta k_4}, \quad \Delta = a_3b_4 + a_4b_3. \tag{5.3}$$

From Eqs. (3.5) and expressions (5.1), (5.3), we find the projective coordinates (s_v) of the midpoint S of the parabolic segment $[AB]$

$$s_v = a_v a_4 \left(b_1^2 + b_2^2 + b_3^3\right) - b_v b_4 \left(a_1^2 + a_2^2 + a_3^3\right) - p_{v4}(a_1b_1 + a_2b_2 + a_3b_3). \tag{5.4}$$

For all v, where $v = 1, 2, 3, 4$, the equality holds

$$a_v a_4 b_4^2 - b_v b_4 a_4^2 - p_{v4} a_4 b_4 = 0.$$

Using this equality and the notation $\phi(a_i)$, $\phi(b_i)$, and $\psi(a_i, b_i)$, we rewrite coordinates (s_v) of the point S from (5.4) in the following form

$$s_v = a_v a_4 \phi(b_i) - b_v b_4 \phi(a_i) - p_{v4}\psi(a_i, b_i), \; v = 1, 2, 3, 4. \tag{5.5}$$

Remark 5.1. Owing to expression (3.10) and the condition $\Phi(p_{jk}) = 0$, the equality $\sqrt{\phi(a_i)\phi(b_i)} = \psi(a_i, b_i)$ holds. Therefore, in the case on consideration, formulae (5.5) are equivalent to formulae (4.14) with $\varepsilon = -1$.

6. Generalization of results

Based on Remark 5.1, formulae (4.14) with $\varepsilon = -1$ define the coordinates of the midpoint of a parabolic segment with ends at the points $A(a_i)$, $B(b_i)$ in a canonical frame of the first type. Consequently, formulae (4.14) generalize formulae (5.5).

Let us now turn to the case of a non-parabolic segment.

Replacing the plane $A_1A_2A_3$ in the arguments in §4.1, by any other coordinate plane, we obtain alternative forms of setting the midpoint and the quasi-midpoint of a non-parabolic segment. We can generalize alternative representations of formulae (4.14) as follows.

Since not all projective coordinates of a point are equal to zero, and the points A and B are distinct, among the numbers 1, 2, 3, and 4 there is a number, say w, that satisfies the condition

$$p_{1w}^2 + p_{2w}^2 + p_{3w}^2 + p_{4w}^2 \neq 0.$$

Geometrically, this condition means that the line p does not belong to a coordinate plane that does not contain the vertex A_w of the frame R^*. Replacing the index 4 in formulae (4.14) with the index w, we obtain a generalization of formulae (4.14) and (4.26).

So, concluding the study, we formulate its main result in the following theorem.

Theorem 6.1. *Let A and B be distinct points in the extended hyperbolic space H^3 with the metric form*

$$\phi = x_1^2 + x_2^2 + x_3^2 - x_4^2.$$

Assume that points A, B are given in a canonical frame R^* of the first type by coordinates (a_i), (b_i), $i = 1, 2, 3, 4$, so that the Plücker coordinates of the line AB satisfy the condition

$$p_{1w}^2 + p_{2w}^2 + p_{3w}^2 + p_{4w}^2 \neq 0. \tag{6.1}$$

Denote

$$s_v = a_v a_w \phi(b_i) - b_v b_w \phi(a_i) + \varepsilon p_{vw}\sqrt{\phi(a_i)\phi(b_i)}, \; v = 1, 2, 3, 4, \; \varepsilon = \pm 1. \tag{6.2}$$

Then the following assertions are true.

1. *If the segment $[AB]$ is non-parabolic, then expressions (s_v) can be taken as the projective coordinates of the midpoint and the quasi-midpoint of the segment $[AB]$.*
2. *If the segment $[AB]$ is parabolic, then expressions (s_v) with $\varepsilon = -1$ can be taken as the projective coordinates of the midpoint of this segment.*

Note that the reasoning in §4.1, conducted for a segment of an elliptic space with metric form

$$\phi = x_1^2 + x_2^2 + x_3^2 + x_4^2, \tag{6.3}$$

leads to a generalization of formulae (4.27) to the three-dimensional case. Consequently, under condition (6.1) expressions (s_v) from (6.2) with values $\phi(a_i)$, $\phi(b_i)$ of the quadratic form ϕ (6.3) can be taken as the projective coordinates of the midpoint and the quasi-midpoint of the segment with ends at the points $A(a_i)$ and $B(b_i)$ in the elliptic 3-space.

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