Some Spectrum Properties in C*- Algebras

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Abstract

In this study, we show that a * - homomorphism \( \varphi : A \rightarrow B \) between unital commutative C*- algebras A and B with \( A^{-1} = \varphi^{-1}(B^{-1}) \) satisfies the property to preserve spectrum and adjoint mapping \( \varphi^* : \Delta(B) \rightarrow \Delta(A) \) is surjective, that is, \( \varphi^* \) maps maximal ideal space of B to maximal ideal space of A.

Keywords: C*- algebra, Gelfand transform, maximal ideal, spectrum, complex homomorphism

C* - Cebirlerinde Bazı Spektrum Özellikleri

Özet

Bu çalışmada, birimli değişmişi A ve B C*- cebirleri arasında tanımlı \( A^{-1} = \varphi^{-1}(B^{-1}) \) şartını sağlayan bir \( \varphi \) * - homomorfizminin spektrumu koruma özelliğini sağladığı ve \( \varphi^* : \Delta(B) \rightarrow \Delta(A) \) adjoint dönüşümünün örten olduğu yani B cebirinin maksimal idealler uzayına A cebirinin maksimal idealler uzayına dönüştürdüğü gösterildi.

Anahtar Kelimeler: C*- cebiri, Gelfand dönüşümü, maksimal ideal, spektrum, kompleks homomorfizm

1. Introduction

There are many studies on invertible elements of C*- algebras and the property to preserve spectrum of a homomorphism between C*- algebras. The related studies can be found in references as [1-3]. In this paper, the relation between the property preserve spectrum of a homomorphism from one C*- algebra to another, invertible elements of these C*- algebras and the mapping of their maximal ideals is examined.

In this section, basic definitions and properties related to C*- algebras will be given.

Let A be a complex algebra. An involution on A is a mapping \( * : x \mapsto x^* \) from A into A satisfying the following conditions.

i. \( (x + y)^* = x^* + y^* \),
ii. \( (\lambda x)^* = \bar{\lambda} x^* \),
iii. \( (xy)^* = y^* x^* \),
iv. \( (x^*)^* = x \)

for all \( x, y \in A \) and \( \lambda \in \mathbb{C} \). Then A is called a *- algebra or an algebra with involution.

If *- algebra A is a Banach algebra and involution on it is isometric; that is, \( \|x\| = \|x^*\| \) for all \( x \in A \), then A is called a Banach *- algebra.

If *- algebra A is a Banach algebra and its norm satisfies the equation \( \|x^*x\| = \|x\|^2 \) for all \( x \in A \), then A is said to be a C*- algebra. [4]

Let A and B be C*- algebras, \( \varphi : A \rightarrow B \) be a mapping. If \( \varphi \) satisfies the following conditions for all \( x, y \in A \) and \( \lambda \in \mathbb{C} \), then this mapping is called a *- homomorphism.

\begin{align*}
\text{i. } & \varphi(x + y) = \varphi(x) + \varphi(y), \\
\text{ii. } & \varphi(\lambda x) = \lambda \varphi(x), \\
\text{iii. } & \varphi(xy) = \varphi(x)\varphi(y), \\
\text{iv. } & \varphi(x^*) = \varphi(x)^*.
\end{align*}

It is said to be a * - isomorphism if \( \varphi \) is a bijection. [5]

If A is a unital Banach algebra, then the set \( \{ \lambda \in \mathbb{C} : (x - \lambda 1_A) \notin A^{-1} \} \) is called spectrum of x in A, denoted by \( \sigma_A(x) \), where \( A^{-1} \) denotes the set of invertible elements of A. \( \sigma_A(x) \) is a nonempty compact subset of \( \mathbb{C} \) for every x in A. The resolvent set of x is defined by \( \rho_A(x) = \mathbb{C} \setminus \sigma_A(x) \).
The spectral radius of $x$ is characterized by $r_A(x) = \sup \{ |\lambda| : \lambda \in \sigma_A(x) \}$.

If $A$ is a unital commutative Banach algebra, then for every $x \in A$, the limit

$$r_A(x) = \lim_{n \to \infty} \|x^n\|^{1/n}$$

exists and $r_A(x) = \|x\|$. Also for every $x, y \in A$, $r_A(x + y) \leq r_A(x) + r_A(y)$ and $r_A(xy) \leq r_A(x)r_A(y)$.

When $A$ is a commutative complex algebra with unit, every proper ideal of $A$ is contained in a maximal ideal and every maximal ideal of $A$ is closed. The set of all maximal ideals in $A$ is denoted by $\mathcal{M}(A)$.

Let $A$ be a complex algebra and $\phi$ is a linear functional on $A$. If $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$, then $\phi$ is called a complex homomorphism on $A$. The set of nonzero complex homomorphisms on $A$ is denoted by $\Delta(A)$.

For $x \in A$, $\hat{x} : \Delta(A) \to \mathbb{C}$, Gelfand transform of $x$, is defined by $\hat{x}(h) = h(x)$ for every $h$ in $\Delta(A)$. The set $\hat{A} = \{ \hat{a} : a \in A \}$ is called the set of Gelfand transforms on $A$. [6]

The $\varepsilon$-open neighbourhood $U_{\varepsilon}(h_0, a_1, ..., a_n)$ at any $h_0 \in \Delta(A)$ with respect to the Gelfand topology is given

$$\{ h \in \Delta(A) : |\hat{a}_i(h_0) - \hat{a}_i(h)| < \varepsilon \}$$

where $\varepsilon > 0$, $n \in \mathbb{N}$ and $a_1, ..., a_n$ are arbitrary elements of $A$. [7]

The following is true when $A$ is a unital commutative Banach algebra.

i. Every maximal ideal of $A$ is the kernel of some $h \in \Delta(A)$.

ii. If $h \in \Delta(A)$, then the kernel of $h$ is a maximal ideal of $A$.

iii. An element $x \in A$ is invertible in $A$ if and only if $h(x) \neq 0$ for every $h \in \Delta(A)$.

iv. $\lambda \in \sigma(x)$ if and only if $h(x) = \lambda$ for some $h \in \Delta(A)$. [6]

2. Spectrum Properties in $C^*$- Algebras

In this section, it will be obtained that under what conditions equality $\sigma_A(x) = \sigma_B(\varphi(x))$ for any $x \in A$, the property to preserve spectrum of $\varphi : A \to B$, will be satisfied when $A$ and $B$ are unital commutative $C^*$- algebras and $\varphi$ is a $*-$ homomorphism from $A$ to $B$.

**Proposition 2.1.** Let $A$ and $B$ be unital commutative $C^*$- algebras, $\varphi$ be a $*-$ homomorphism from $A$ to $B$ and $\varphi(1_A) = 1_B$. Then for every $x \in A$, $\sigma_B(\varphi(x)) \subset \sigma_A(x)$. [5]

**Theorem 2.2.** Let $A$ and $B$ be unital commutative $C^*$- algebras, $\varphi : A \to B$ be a $*-$ homomorphism with $\varphi(1_A) = 1_B$. Then $\varphi(x) \in B^{-1}$ for any $x \in A^{-1}$.

**Proof.** $0 \notin \sigma_A(x)$ for an arbitrary $x \in A^{-1}$. $0 \notin \sigma_B(\varphi(x))$ follows from Proposition 2.1 and this proves $\varphi(x) \in B^{-1}$.

**Corollary 2.3.** $A^{-1} \subset \varphi^{-1}(B^{-1})$.

**Theorem 2.4.** Let $A$ and $B$ be unital commutative $C^*$- algebras, $\varphi : A \to B$ be a $*-$ homomorphism with $\varphi(1_A) = 1_B$. In this case, $A^{-1} = \varphi^{-1}(B^{-1})$ if and only if $0 \notin \sigma_A(x)$ whenever $0 \notin \sigma_B(\varphi(x))$ for any $x \in A$.

**Proof.** Let $A^{-1} = \varphi^{-1}(B^{-1})$. Suppose that $0 \notin \sigma_B(\varphi(x))$ for an arbitrary $x \in A$. In this case, $\varphi(x) \in B^{-1}$ and hence $x \in A^{-1}$ so that $0 \notin \sigma_A(x)$ for every $x \in A$.

Conversely, assume that $0 \notin \sigma_A(x)$ whenever $0 \notin \sigma_B(\varphi(x))$ for any $x \in A$. Since $0 \notin \sigma_B(\varphi(a))$ for any $a \in \varphi^{-1}(B^{-1})$, $a \in A^{-1}$ by hypothesis and hence $\varphi^{-1}(B^{-1}) \subset A^{-1}$. According to Corollary 2.3, $A^{-1} = \varphi^{-1}(B^{-1})$.

**Corollary 2.5.** Let $A$ and $B$ be unital commutative $C^*$- algebras, $\varphi : A \to B$ be a $*-$ homomorphism with $\varphi(1_A) = 1_B$. Then $\sigma_A(x) = \sigma_B(\varphi(x))$ for every $x \in A$ if and only if $A^{-1} = \varphi^{-1}(B^{-1})$.

**Proof.** First, suppose that $\sigma_A(x) = \sigma_B(\varphi(x))$ for every $x \in A$. In that case, one says $0 \notin \sigma_B(\varphi(a))$ for any $a \in \varphi^{-1}(B^{-1})$. Hence $\sigma_A(a) = \sigma_B(\varphi(a))$ implies $0 \notin \sigma_A(a)$, that is, $a \in A^{-1}$. Then $\varphi^{-1}(B^{-1}) \subset A^{-1}$. Again, using Corollary 2.3, it follows that $A^{-1} = \varphi^{-1}(B^{-1})$.

Conversely, let $A^{-1} = \varphi^{-1}(B^{-1})$. Given any $\lambda \in \mathbb{C} - \sigma_B(\varphi(x))$, $\varphi(x - \lambda 1_A) \in B^{-1}$ for
any \( x \in A \), that is, \( x - \lambda 1_A \in \varphi^{-1}(B^{-1}) \) for any \( x \in A \) and hence it is clear that \( \lambda \notin \sigma_A(x) \), since \( x - \lambda 1_A \in A^1 \) by hypothesis. Thus, we have seen that \( \sigma_A(x) \subset \sigma_B(\varphi(x)) \) for every \( x \in A \) and we obtain \( \sigma_A(x) = \sigma_B(\varphi(x)) \) for every \( x \in A \) by Proposition 2.1.

**Corollary 2.6.** If \( A^{-1} = \varphi^{-1}(B^{-1}) \), then \( r_A(x) = r_B(\varphi(x)) \) for every \( x \in A \).

3. Mapping of Maximal Ideals in \( C^* \)- Algebras

Let \( \varphi \) be a \( * \)-homomorphism between unital commutative \( C^* \)-algebras \( A \) and \( B \) and also \( A^* \) and \( B^* \) be algebraic duals of \( A \) and \( B \), respectively. Surjectivity of \( \varphi^* : \Delta(B) \to \Delta(A) \) which is obtained from \( \varphi^* : B^* \to A^* \) means that \( \varphi^* \) maps \( M(B) \) to \( M(A) \). In this section, it will be obtained that under what conditions this property will be satisfied.

**Theorem 3.1.** Let \( A \) and \( B \) be unital commutative \( C^* \)-algebras, \( \varphi : A \to B \) be a \( * \)-homomorphism. Then \( \varphi^* f \) is also a \( * \)-homomorphism for every \( f \in \Delta(B) \).

**Proof.** For every \( f \in \Delta(B) \) and \( x, y \in A \),

\[
(\varphi^* f)(xy) = f(\varphi(x)y)
\]

\[
= f(\varphi(x))f(\varphi(y))
\]

\[
= (\varphi^* f)(x)(\varphi^* f)(y)
\]

and hence \( \varphi^* f \in \Delta(A) \). Also, since

\[
(\varphi^* f)(x^*) = f(\varphi(x)^*)
\]

\[
= f(\varphi(x))^*
\]

\[
= f(\varphi(x))
\]

\[
= (\varphi^* f)(x)
\]

for every \( f \in \Delta(B) \) and \( x \in A \), it is clear that \( \varphi^* f \) is a \( * \)-homomorphism.

**Corollary 3.2.** Let \( A \) and \( B \) be unital commutative \( C^* \)-algebras, \( \varphi : A \to B \) be a \( * \)-homomorphism. Then \( \varphi^* \Delta(B) \subset \Delta(A) \).

**Theorem 3.3.** Let \( A \) and \( B \) be unital commutative \( C^* \)-algebras, \( \varphi : A \to B \) be a \( * \)-homomorphism with \( \varphi(1_A) = 1_B \).

In that case, \( A^{-1} = \varphi^{-1}(B^{-1}) \) if and only if \( \varphi^* \Delta(B) = \Delta(A) \).

**Proof.** Let \( A^{-1} = \varphi^{-1}(B^{-1}) \). Then for every \( g \in \Delta(A) \), there exists \( l \in M(A) \) such that \( Kerg = I \). If we denote by \( J_0 \) the smallest ideal of \( B \) containing \( \varphi(I) \), then \( J_0 = B \) or \( J_0 \neq B \). If \( J_0 \neq B \), then there exists \( f \in M(B) \) such that \( J_0 \subset f \) and also \( f \in \Delta(B) \) such that \( Kerg = I \). Since \( l \in M(A) \) and \( A/l \equiv \mathbb{C} \), there exists \( \lambda \in \mathbb{C} \) and \( t \in I \) such that \( a = \lambda + t \) for any \( a \in A \). Therefore,

\[
(\varphi^* f)(a) = (\varphi^* f)(\lambda + t) = \lambda + f(\varphi(t)).
\]

Again for \( t \in I \), \( \varphi(t) \in Kerg \) and hence \( (\varphi^* f)(a) = \lambda \). Thus, we can write \( a = (\varphi^* f)(a) = 1 + t \).

Using the fact that \( t \in I = Kerg \), \( g(a) = (\varphi f)(a) \). Then it is easily seen that \( g = \varphi^* f \in \varphi^* \Delta(B) \) and obtained that \( \Delta(A) \subset \varphi^* \Delta(B) \).

If \( J_0 \) were all of \( B \), then there would be \( b_1, b_2, ..., b_n \in B \) and \( a_1, a_2, ..., a_n \in I \) such that

\[
\sum_{i=1}^{n} b_i \varphi(a_i) = 1.
\]

Since \( b_i \| a_i \| \in B \) and \( \frac{a_i}{\| a_i \|} \in I \), we can assume that \( \| a_i \| = 1 \) for each \( i = 1, 2, ..., n \). Let

\[
\max_{i \leq n} \| b_i \| = M
\]

and a neighbourhood \( U \) at \( g \in \Delta(A) \) with respect to the Gelfand topology for \( 0 < \varepsilon < 1 \) be

\[
\{ h \in \Delta(A) : |\hat{a}_i(h) - \hat{a}_i(g)| < \frac{\varepsilon}{M \cdot n}, 1 \leq i \leq n \}.
\]

Then, since \( a_i \in I = Kerg \) for each \( i = 1, 2, ..., n \),

\[
U = \{ h \in \Delta(A) : |\hat{a}_i(h)| < \frac{\varepsilon}{M \cdot n}, 1 \leq i \leq n \}
\]

As \( A \) is regular, there is a \( m \in A \) such that

\[
\hat{m}(h) = \begin{cases} 
1 & h = g \\
0 & h \in \Delta(A) - U \\
\leq 1 & \text{otherwise}
\end{cases}
\]

Thus for any \( k \in \Delta(A) \),

\[
|\{a_i, m\}^*(k) = |\hat{a}_i(k), \hat{m}(k)| = |\hat{a}_i(k)|, \hat{m}(k)|
\]

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and hence 
\[ \sup\{ |(a_i, m)^\sim(k)| : k \in \Delta(A) \} < \frac{\varepsilon}{M \cdot n}. \]

Also 
\[
\begin{align*}
  r(a_i m) &= \sup\{ |k(a_i, m)| : k \in \Delta(A) \} \\
  &= \sup\{ |(a_i, m)^\sim(k)| : k \in \Delta(A) \}
\end{align*}
\]

implies 
\[ r(a_i m) < \frac{\varepsilon}{M \cdot n}. \]

On the other hand, if we remember 
\[ \sum_{i=1}^{n} b_i, \varphi(a_i) = 1, \]
then it is clear that 
\[
\begin{align*}
  \varphi(m) &= \varphi(m) \sum_{i=1}^{n} b_i, \varphi(a_i) \\
  &= \sum_{i=1}^{n} b_i, \varphi(a_i) \cdot \varphi(m) \\
  &= \sum_{i=1}^{n} b_i, \varphi(a_i m).
\end{align*}
\]

Then, 
\[
\begin{align*}
  r(\varphi(m)) &= r\left( \sum_{i=1}^{n} b_i, \varphi(a_i m) \right) \\
  &\leq \sum_{i=1}^{n} r(b_i) r(\varphi(a_i m)) \\
  &\leq \sum_{i=1}^{n} \|b_i\| r(\varphi(a_i m)) \\
  &\leq M \cdot \sum_{i=1}^{n} \frac{\varepsilon}{M \cdot n} \\
  &= \varepsilon
\end{align*}
\]

and hence 
\[ r(\varphi(m)) < \varepsilon < 1. \] Moreover, since 
\[
\begin{align*}
  r(m) &= \sup\{ |k(m)| : k \in \Delta(A) \} \\
  &= \sup\{ |\tilde{m}(k)| : k \in \Delta(A) \} \\
  &= 1,
\end{align*}
\]

\[ r(\varphi(m)) = 1 \] by hypothesis, which is a contradiction. This contradiction shows that 
\[ J_0 \neq B. \] Consequently, \( \varphi^* \Delta(B) = \Delta(A) \) by Corollary 3.2.

Conversely, let \( \varphi^* \Delta(B) = \Delta(A) \). For any point \( x \in \varphi^{-1}(B^{-1}) \) and \( h \in \Delta(A) \), \( \varphi(x) \in B^{-1} \) and there exists \( u \in \Delta(B) \) such that \( \varphi^* u = h \). Thus \( u(\varphi(x)) \neq 0 \), so that \( \varphi^* u(x) = h(x) \neq 0 \) and hence \( x \in A^{-1} \). Thus it is obtained that \( \varphi^{-1}(B^{-1}) \subset A^{-1} \) and \( \varphi^{-1}(B^{-1}) = A^{-1} \) by Corollary 2.3.

**Corollary 3.4.** \( A^{-1} = \varphi^{-1}(B^{-1}) \) if and only if \( \varphi^* M(B) = M(A) \).

### 4. References


