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Modules That Have a $\delta-$Supplement in Every Torsion Extension

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Abstract

In this paper, we call a module M δ-*TE-module* if M has a δ-supplement in every torsion extension. We obtain various properties of these modules. We show that every direct summand of a δ-TE-module is a δ-TE-module. We prove that the class of δ-TE-modules is closed under extension under a special condition.

Keywords $\delta-$Supplement, Torsion Extension, $\delta-$Small

Her Torsiyon Genişlemesinde Tümleyene Sahip Modüller

Özet

Bu makalede her torsiyon genişlemesinde bir δ-tümleyene sahip M modülüne δ-*TE-modül* diye isimlendirildi. Bu modüllerin birçok özellikleri elde edildi. δ-TE-modüllerin direk toplamlarının da bir δ-TE-modül olduğu gösterildi. δ-TE-modül sınıfının özel bir koşulla genişlemeler altında kapalı olduğu ispatlandı.

Anahtar Kelimeler $\delta-$tümleyen, Torsiyon Genişlemesi, $\delta-$Küçük

1. Introduction

Throughout this paper, R will be a commutative domain and all modules will be unital left $R-$modules, unless otherwise specified. Let M be an $R-$module. By $N \leq M$ we mean that N is a submodule of M. Recall that a submodule N of M is called *small*, denoted by $N \ll M$, if $N+L \neq M$ for all proper submodules L of M. By $Rad(M)$, we denote the sum of all small submodule of M. Nevertheless a submodule L of M is said to be *essential* in M, denoted by $L \lhd M$, if $L \cap K \neq 0$ for each nonzero submodule K of M. A module M is said to be *singular* if $M \cong \frac{N}{L}$ for some module N and a submodule L of N with $L \lhd N$.

Let M and N be $R-$modules. N is called an *extension* of M in case $M \subseteq N$. A module M is said to be *injective* if it is a direct summand of every extension of itself [5].

As a proper generalization of direct summands of a module, one can define supplement submodules. The module M is called *supplemented*, if every submodule N of M has a *supplement* in M, i.e. a submodule K of M minimal with respect to $M = N + K$. K is a supplement of N in M if and only if $M = N + K$ and $N \cap K \ll K$ [10].

As a generalization of small submodules, in [11] $\delta-$small submodules were introduced by Zhou. According to [11], a submodule L of M is called $\delta-$*small* in M, denoted by $L \ll_\delta M$, if for any submodule N of M with $\frac{M}{N}$ singular, $M = N + L$ implies that $M = N$. The sum of all $\delta-$small submodules of a module M is denoted by $\delta(M)$. It is easy to see that every small submodule of a module M is $\delta-$small in M, so $Rad(M) \subseteq \delta(M)$ and

$Rad(M)=\delta(M)$ if M is singular. Also any non-singular semisimple submodule of M is $\delta-$small in M and $\delta-$small submodules of a singular module are small submodules. For more detailed discussion on $\delta-$small submodules we refer to [11].

Let K,N be submodules of module M. N is called a *$\delta-$supplement* of K in M, if $M=N+K$ and $N\cap K \ll_{\delta} N$. A module M is called *$\delta-$supplemented* if every submodule of M has a $\delta-$supplement in M [3,9]. On the other hand, a submodule N of M is said to have *ample $\delta-$supplements* in M if every submodule L of M with $M=N+L$ contains a $\delta-$supplement of N in M. The module M is called *amply $\delta-$supplemented* if every submodule of M has ample $\delta-$supplements in M [7].

Let M be a module and N,K be any submodules of M with $M=N+K$. If $N\cap K\leq\delta(N)$ then N is called a *generalized $\delta-$supplement* of K in M. Following [6], M is called a *generalized $\delta-$supplemented* module (or briefly *$\delta-GS$ module*) if every submodule N of M has a generalized $\delta-$supplement in M.

Modules that have supplements [ample supplements] in every module in which it is contained as a submodule have been studied in [12]. The structure of these modules have been determined over Dedekind domains. These modules are called modules with *the property (E)[(EE)*] in [12]. Such modules are also called *supplementing* modules in [1, p.255].

Let R be a commutative domain and M be an $R-$module. We denote by $T(M)$, the set of all elements m of M for which there exists a non-zero element r of R such that $rm=0$, i.e. $Ann(m)\neq 0$. Then $T(M)$, which is a submodule of M, called the torsion submodule of M. If $M=T(M)$, then M is said to be a torsion module and M is torsion-free precisely when $T(M)=0$ [5].

For modules $M\subseteq N$ over a commutative domain, we say that N is a *torsion extension* of M if the factor module $\frac{N}{M}$ is torsion. In a recent paper [2], modules that have a supplement in every torsion extension have been studied and these modules are called TE-modules. We call a module M *$\delta-$TE-module* if M has a $\delta-$supplement in every torsion extension. In this paper, we study some basic properties of these modules. We 36how the class of $\delta-$TE-modules is closed under direct summands, extensions and finite direct sums.We also prove that every submodule of a module is a $\delta-$TE-module if and only if it has ample $\delta-$supplements in every torsion extension.

2. Main Results

Proposition 2.1. Every direct summand of a $\delta-$TE-module is a $\delta-$TE-module.

Proof. Let M be a $\delta-$TE-module and N be a direct summand of M. Then we can write $M=N\oplus K$ for some submodule K of M. For a torsion extension L of N, we denote by T the external direct sum $L\oplus K$. Consider the canonical embedding $\varphi:M\to T$. Then $M\cong\varphi(M)$ is a $\delta-$TE-module and we have

$$\frac{T}{\varphi(M)}=\frac{L\oplus K}{\varphi(M)}\cong\frac{L}{N}$$

is torsion. Since $\varphi(M)$ is $\delta-$TE-module, $\varphi(M)$ has a $\delta-$supplement U in T, that is, $T=\varphi(M)+U$ and $\varphi(M)\cap U\ll_{\delta} U$. For the projection $\pi:T\to L$, we have that $L=\pi(U)+N$. Also since $Ker(\pi)\subseteq\varphi(M)$, we get

$$\pi(\varphi(M)\cap U)\subseteq\pi(\varphi(M))\cap\pi(U)$$
$$=N\cap\pi(U)\ll_{\delta}\pi(U)$$

by [9, Lemma 1.3.(2)]. Hence, $\pi(U)$ is a $\delta-$supplement of N in L.

Proposition 2.2. Let M be a module. Then the following statements are equivalent:

(1) Every submodule of M is a $\delta-$TE-module.

(2) M has ample $\delta-$supplements in every torsion extension.

Proof.(1)⇒(2) Suppose that every submodule of M is a $\delta-$TE-module. For a torsion extension N of M, let $N=M+K$ for some submodule K of N. Note that $\frac{N}{M}\cong\frac{K}{M\cap K}$ is torsion. Since $M\cap K$ is a $\delta-$TE-module, there exists a submodule L of K such that $K=(M\cap K)+L$ and $(M\cap K)\cap L=M\cap L\ll_{\delta} L$. Then we have $N=M+L$. Hence, L is a $\delta-$supplement of M in N.

(2)⇒(1) Let T be any submodule of M. For a torsion extension N of T, let $F=\frac{M\oplus N}{H}$, where the submodule H is the set of all elements $(a,-a)$ of F with $a\in T$ and let $\alpha:M\to F$ via $\alpha(m)=(m,0)+H$, $\beta:N\to F$ via $\beta(n)=(0,n)+H$ for all $m\in M$, $n\in N$. It is clear that α and β are monomorphisms. Thus we have the following pushout:

$$\begin{array}{ccc} T & \xrightarrow{i_1} & N \\ \downarrow^{i_2} & & \downarrow^{\beta} \\ M & \xrightarrow{\alpha} & F \end{array}$$

where i_1 and i_2 are inclusion mappings. It is easy to prove that $F=\mathrm{Im}(\alpha)+\mathrm{Im}(\beta)$. Consider the epimorphism $\gamma:F\to\frac{N}{T}$ defined by $\gamma((m,n)+H)=n+T$ for all $(m,n)+H\in F$. Since

$$\mathrm{Ker}(\gamma)=\mathrm{Im}(\alpha)$$

we have

$$\frac{N}{T}\cong\frac{F}{\mathrm{Im}(\alpha)}$$

is torsion. By the hypothesis, $\mathrm{Im}(\alpha)$ has ample $\delta-$supplements in every torsion extension because $\mathrm{Im}(\alpha)$ is a monomorphism. Then, we can write

$$F=\mathrm{Im}(\alpha)+V \text{ and } \mathrm{Im}(\alpha)\cap V\ll_{\delta} V$$

with $V\le\mathrm{Im}(\beta)$. Hence we obtain that $N=\beta^{-1}(\mathrm{Im}(\alpha))+\beta^{-1}(V)=T+\beta^{-1}(V)$.

Suppose that $T\cap\beta^{-1}(V)+X=\beta^{-1}(V)$ for some submodule X of $\beta^{-1}(V)$ with $\frac{\beta^{-1}(V)}{X}$ singular. Then we have

$$\begin{aligned} V=V\cap\mathrm{Im}(\beta)&=\beta(\beta^{-1}(V))=\beta(T\cap\beta^{-1}(V)+X)\\ &=\beta(T\cap\beta^{-1}(V))+\beta(X)\\ &=\mathrm{Im}(\alpha)\cap V+\beta(X). \end{aligned}$$

Now we define $\theta:\frac{\beta^{-1}(V)}{X}\to\frac{V}{\beta(X)}$ by $\theta(a+X)=\beta(a)+\beta(X)$ for all $a+X\in\frac{\beta^{-1}(V)}{X}$. Note that θ is an isomorphism. Hence

$$\frac{\beta^{-1}(V)}{X}\cong\frac{V}{\beta(X)}$$

is singular. Since $\mathrm{Im}(\alpha)\cap V<<_{\delta}V$, it follows that $\beta(X)=V$ and so that $X=\beta^{-1}(V)$. Thus $T\cap\beta^{-1}(V)<<_{\delta}\beta^{-1}(V)$, that is, $\beta^{-1}(V)$ is a $\delta-$supplement of T in N.

Theorem 2.1. Let $0\to K\to M\to L\to 0$ be a short exact sequence. If K and L are $\delta-$TE modules with L torsion, so does M.

Proof. Without loss of generality, we can assume that $K\le M$ and N be a torsion extension of M. For $K\le M\le N$, we have

$$\frac{N}{M}\cong\frac{\frac{N}{K}}{\frac{M}{K}}$$

is torsion and so $\frac{N}{K}$ is a torsion extension of $\frac{M}{K}$. Since $L\cong\frac{M}{K}$ is a $\delta-$TE-module, there exists a submodule $\frac{V}{K}$ of $\frac{N}{K}$ such that $\frac{N}{K}=\frac{M}{K}+\frac{V}{K}$ and $\frac{M\cap V}{K} \ll_{\delta} \frac{V}{K}$ and so $N=M+V$. Since

$$\frac{\frac{V}{K}}{\frac{M\cap V}{K}} \cong \frac{V}{M\cap V} \cong \frac{M+V}{M}=\frac{N}{M},$$

L is torsion, we obtain that $\frac{V}{K}$ is torsion. Then K has a $\delta-$supplement K_1 in V, i.e. $V=K+K_1$ and $K\cap K_1 \ll_{\delta} K_1$ because K is a $\delta-$TE-module. Therefore $N=M+V=M+K_1$. Assume that $N=M+X$ for some submodule X of K_1. Then $\frac{M}{K}+\frac{X+K}{K}=\frac{N}{K}$. Note that

$$\frac{K_1}{X} \cong \frac{K_1+K}{X+K}=\frac{V}{X+K}=\frac{\frac{V}{K}}{\frac{X+K}{K}}$$

is singular. It follows from [4, Lemma 2.1], $\frac{V}{K}=\frac{X+K}{K}$ and so $V=X+K$. Since K_1 is $\delta-$supplement of K in V, by [4, Lemma 2.1], by we have that $X=K_1$.Thus K_1 is a $\delta-$supplement of M in N. Thus K_1 is a $\delta-$supplement of M in N.

Corollary 2.1. Let M_1 and M_2 be $\delta-$TE-modules with M_2 torsion and $M=M_1\oplus M_2$. Then M is a $\delta-$TE-module.

Proof. Let $M=M_1\oplus M_2$. By using the following short exact sequence

$$0\to M_1\to M\to M_2\to 0$$

we obtain that M is a $\delta-$TE-module by Theorem 2.1.

Lemma 2.1. Let M be a $\delta-$TE-module and N be a torsion extension of M such that $\delta(N)=0$. Then M is a direct summand of N.

Proof. By assumption, M has a $\delta-$supplement in N, say K. Since $M\cap K \ll_{\delta} K$, it follows that $M\cap K\subseteq \delta(K)=0$. Hence $N=M\oplus K$.

In [8], a ring R is called a *left* $\delta-V-$*ring*, if for any left $R-$module M, $\delta(M)=0$.

Corollary 2.2. Let M be a $\delta-$TE-module over a $\delta-V-$ring. Then M is a direct summand of any module N with $\frac{N}{M}$ torsion.

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