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Research Article

Principal eigenvalues of elliptic problems with singular potential and bounded weight function

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ABSTRACT. Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{0,1}$ boundary, and let $d_\Omega : \Omega \to \mathbb{R}$ be the distance function $d_\Omega(x) := dist(x, \partial\Omega)$. Our aim in this paper is to study the existence and properties of principal eigenvalues of self-adjoint elliptic operators with weight function and singular potential, whose model problem is $-\Delta u + bu = \lambda m u$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, where $b : \Omega \to \mathbb{R}$ is a nonnegative function such that $d_\Omega^2 b \in L^\infty(\Omega)$, $m : \Omega \to \mathbb{R}$ is a nonidentically zero function in $L^\infty(\Omega)$ that may change sign, and the solutions are understood in weak sense.

Keywords: Weighted principal eigenvalue problems, second order elliptic operators, singular potentials.

2020 Mathematics Subject Classification: 35J20, 35J25.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary if $n > 1$, let m be a real valued function defined on Ω, let $\lambda \in \mathbb{R}$, and let $\mathcal{L}$ be a second order elliptic linear operator on Ω. We recall that λ is said a principal eigenvalue of the operator $\mathcal{L}$ with weight function m and Dirichlet boundary condition, if there exists a solution u to the problem

$$\begin{cases} \mathcal{L}u = \lambda m u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u \geq 0 \text{ in } \Omega \text{ and } u \not\equiv 0 \text{ in } \Omega. \end{cases} \tag{1.1}$$

These problems have received a lot of attention in the literature, in part because they appear naturally when one studies semilinear bifurcation problems via the implicit function theorem (for details see e.g., [8], Chapter 5, Section 5.3). Let us recall some works related to problem (1.1).

Manes and Micheletti in [15] studied the problem (with the solutions understood in weak sense and belonging to $H_0^1(\Omega) \cap C(\overline{\Omega})$)

$$\begin{cases} -\operatorname{div}(A\nabla u) = \lambda m u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega \end{cases} \tag{1.2}$$

in the case when $m \in L^r(\Omega)$ for some $r > \frac{n}{2}$ and $A = (a_{ij}(x))$ is a symmetric uniformly elliptic $n \times n$ whose coefficients belong to $C^{0,1}(\overline{\Omega})$. They proved, by variational methods, the following facts:

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a) If $m \geq 0$, then problem (1.2) has a principal eigenvalue $\lambda_1(m)$, which is positive and simple, and that it is the first positive eigenvalue of the problem

$$\begin{cases} -\operatorname{div}(A\nabla u) = \lambda m u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

that is, if λ is any other eigenvalue λ of (1.3), then $\lambda > \lambda_1(m)$.

b) If $m \leq 0$, then problem (1.2) has a principal eigenvalue $\lambda_{-1}(m)$, which is negative and simple, and satisfies that $\lambda < \lambda_{-1}(m)$ for any other eigenvalue λ of problem (1.3).

c) If $m^+ \not\equiv 0$ and $m^- \not\equiv 0$, then problem (1.2) has two principal eigenvalues $\lambda_1(m)$ and $\lambda_{-1}(m)$, with $\lambda_1(m) > 0$ and $\lambda_{-1}(m) < 0$; both of them are simple eigenvalues, and $\lambda \notin (\lambda_{-1}(m), \lambda_1(m))$ for any eigenvalue λ of problem (1.3).

They proved also a maximum principle with weight, which reads as: If $h \in L^q(\Omega)$ for some $q > n$ and $0 \leq h \not\equiv 0$, and if either $m^+ \not\equiv 0, m^- \not\equiv 0$ and $\lambda_{-1}(m) < \lambda < \lambda_1(m)$, or $m \geq 0$ and $\lambda < \lambda_1(m)$, or $m \leq 0$ and $\lambda > \lambda_{-1}(m)$, then the problem

$$\begin{cases} -\operatorname{div}(A\nabla u) = \lambda m u + h \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1.4}$$

has a unique solution, and it is positive in Ω.

On the other hand, motivated by problems of genetic population dynamics, Brown and Lin in [4] studied the existence and properties of principal eigenvalues for problem (1.2) in the case of the Laplace operator with homogeneous Neumann boundary condition, Hess and Kato in [13] investigated principal eigenvalue problems with weight for a general uniformly elliptic second order linear operator

$$\mathcal{L}u := -\sum_{1\leq i,j\leq n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{1\leq i\leq n} a_i(x) \frac{\partial u}{\partial x_i} + a_0(x) u.$$

Indeed, they studied the problem

$$\begin{cases} \mathcal{L}u = \lambda m u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \tag{1.5}$$

where the weight m may change sign and belongs to $C^\gamma(\overline{\Omega})$ for some $\gamma \in (0,1)$, and with the solutions understood in classical sense (i.e., $u \in C^2(\Omega) \cap C(\overline{\Omega})$). Under standard regularity assumptions on the coefficients of $\mathcal{L}$ (among them that $a_0 \in C^\gamma(\overline{\Omega})$ for some $\gamma \in (0,1)$), they proved, by using the Krein Rutman theorem, that if $a_0 \geq 0$ in Ω, and $m^+ \not\equiv 0$ (respectively $m^- \not\equiv 0$), then problem (1.5) admits a unique positive (resp. negative) principal eigenvalue $\lambda_1(m)$ (resp $\lambda_{-1}(m)$) which is simple. They also showed that the solutions u of (1.5) belong to $C^1(\overline{\Omega})$ and satisfy, for some positive constants c_1 and c_2,

$$c_1 d_\Omega \leq u \leq c_2 d_\Omega \text{ in } \Omega.$$

They proved also the following maximum principle with weight: If $a_0 \geq 0$ in Ω, and if $m^+ \not\equiv 0$ (respectively $m^- \not\equiv 0$) and if $0 \leq \lambda < \lambda_1(m)$ (resp. $\lambda_{-1}(m) < \lambda \leq 0$) then, for any nonidentically zero h such that $0 \leq h \in C^\gamma(\overline{\Omega})$, the problem

$$\begin{cases} \mathcal{L}u = \lambda m u + h \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1.6}$$

has a unique (classical) solution u and it is positive in Ω.

Hess and Senn in [18] studied problem (1.5) with the Dirichlet replaced by the Neumann boundary condition.

Lopez-Gomez in [14] addressed problem (1.5) in the case when a_0 is not necessarily nonnegative and, by using arguments relying on the maximum principle, they stated sufficient conditions for the existence and the nonexistence of principal eigenvalues.

Hernandez, Mancebo and Vega (see [10], Section 2), studied problem (1.5) in situations where some coefficients of $\mathcal{L}$ and the weight m are allowed to have a certain kind of singularity along $\partial\Omega$. They assumed that:

1) Ω is a bounded domain in $\mathbb{R}^n$ with $C^{3+\gamma}$ boundary for some $\gamma \in (0,1)$,
2) $A(x) = (a_{i,j}(x))$ is a symmetric $n \times n$ matrix, uniformly and strongly elliptic in $\overline{\Omega}$, and for each i, j, $a_{ij} \in C^3(\Omega) \cap C(\overline{\Omega})$,
3) $a_i \in C^2(\Omega)$ and there exists a constant K and $\alpha \in (-1,1)$ such that $\left|\frac{\partial a_{ij}}{\partial x_k}\right| + |a_i| \leq K(1 + d_\Omega^\alpha)$ and $\left|\frac{\partial^2 a_{ij}}{\partial x_i \partial x_j}\right| + \left|\frac{\partial a_i}{\partial x_j}\right| \leq K d_\Omega^{\alpha-1}$ for all $x \in \Omega$ and $1 \leq i,j \leq n$; and their assumptions on the functions a_0 and m were:
4) $a_0 \in C^1(\Omega)$ and, for all $k = 1, 2, ..., n$, $d_\Omega^{2-\alpha}\left|\frac{\partial a_0}{\partial x_k}\right| \in L^\infty(\Omega)$, with α as in 3),
5) m is strictly positive in Ω and satisfies the conditions in 4).

Under the hypothesis 1)-5), they proved (see [10, Theorem 2.6]), that there exists a unique real eigenvalue λ with an associated eigenfunction u in the interior of the positive cone of $C^1(\overline{\Omega})$ (i.e., such that $u > 0$ in Ω and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, where ν denotes the unit outward normal to $\partial\Omega$), and that such a λ is a simple eigenvalue of problem (1.5).

Let us mention also that Berestycki, Varadhan an Nirenberg in [2] studied, in a generalized sense, problem (1.5) in the case where each $a_{ij} \in C(\Omega)$, $a_0 \in L^\infty(\Omega)$, and $a_i \in L^\infty(\Omega)$ for $i = 1, 2, ..., n$. Additional results and more references concerning principal eigenvalues for elliptic problems can be found in [6].

Principal eigenvalue problems for periodic parabolic operators with Dirichlet boundary condition were studied by Beltramo and Hess in [1], and applications to semilinear periodic parabolic problems were given in [11]. A very good exposition of these results, including problems with either Neumann or Robin boundary conditions and its nonlinear applications, as well as additional references, can be found in the book [12].

Problems of the form

$$\begin{cases} -\Delta u + bu = \lambda m u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \tag{1.7}$$

were studied in [9] in the case when m is a nonnegative and nonidentically zero function belonging to $L^\infty(\Omega)$, and b is a singular potential of the form $b = av^{-\alpha-1}$, where:

1′) $0 < \alpha < 3$,
2′) $a \in L^\infty(\Omega)$ and there exists $\delta > 0$ such that $ess\inf_{A_\delta} a > 0,$, with $A_\delta := \{x \in \Omega : d_\Omega(x) \leq \delta\}$,
3′) $v \in D_\alpha := \left\{v \in H_0^1(\Omega) : \vartheta_\alpha^{-1} v \in L^\infty(\Omega) \text{ and } ess\inf_\Omega \vartheta_\alpha^{-1} v > 0\right\}$, where $\vartheta_\alpha := d_\Omega$ if $0 < \alpha < 1$, $\vartheta_1 := d_\Omega\left(\log\left(\frac{\omega}{d_\Omega}\right)\right)^{\frac{1}{2}}$, where ω is an arbitrary constant greater than the diameter of Ω, and $\vartheta_\alpha := d_\Omega^{\frac{2}{1+\alpha}}$ if $1 < \alpha < 3$.

Under these assumptions, Lemmas 4.3 and 4.4 in [9] state the existence of a positive principal eigenvalue for problem (1.7), and a maximum principle with weight.

Our aim in this paper is to study principal eigenvalue problems with singular potential and bounded weight function of the form

$$(1.8)\qquad \begin{cases} -\operatorname{div}(A\nabla u)+bu=\lambda mu \text{ in } \Omega, \\ u=0 \text{ on } \partial\Omega, \\ u>0 \text{ in } \Omega, \end{cases}$$

where the solution u is understood in weak sense (see Definition 1.1 below), and Ω, A, b and m satisfy the following assumptions:

H1) Ω is a bounded domain in $\mathbb{R}^n$, with $C^{1,1}$ boundary if $n>1$.

H2) $A:\Omega\to M_n(\mathbb{R})$, with $A=(a_{ij}(x))$ uniformly elliptic (i.e., there exists a constant $\gamma>0$ such that $\langle A(x)\xi,\xi\rangle \geq \gamma|\xi|^2$ for any $x\in\overline{\Omega}$ and $\xi\in\mathbb{R}^n$) and such that $a_{ij}\in C^{0,1}(\overline{\Omega})$, $a_{ij}=a_{ji}$ for $1\leq i,j\leq n$.

H3) The potential $b:\Omega\to\mathbb{R}$ is nonnegative and $bd_\Omega^2\in L^\infty(\Omega)$, where $d_\Omega:\Omega\to\mathbb{R}$ denotes the distance function given by

$$(1.9)\qquad d_\Omega(x):=dist(x,\partial\Omega).$$

H4) $m\in L^\infty(\Omega)$ and $m\not\equiv 0$ in Ω, i.e., $|\{x\in\Omega: m(x)\neq 0\}|>0$.

Observe that H3) allows b to be singular along $\partial\Omega$ and H4) allows m to change sign in Ω. The notion of weak solution we use is the usual one, given by the following:

Definition 1.1. *Let $f:\Omega\to\mathbb{R}$ be such that $f\varphi\in L^1(\Omega)$ for any $\varphi\in H_0^1(\Omega)$, and let $u:\Omega\to\mathbb{R}$. We say that u is a weak solution of the problem*

$$\begin{cases} -\operatorname{div}(A\nabla u)=f \text{ in } \Omega, \\ u=0 \text{ on } \partial\Omega \end{cases}$$

if $u\in H_0^1(\Omega)$ and $\int_\Omega\langle A\nabla u,\nabla\varphi\rangle=\int_\Omega f\varphi$ for any $\varphi\in H_0^1(\Omega)$.

The paper is organized as follows: In Section 2, we present some general facts need later. In Section 3, following the approach of [13] we study, for each $\lambda\in\mathbb{R}$ and under the assumptions H1)-H4), the principal eigenvalue problem without weight (i.e., with weight $\mathbf{1}$)

$$(1.10)\qquad \begin{cases} -\operatorname{div}(A\nabla u)+bu=\lambda mu+\mu u \text{ in } \Omega, \\ u=0 \text{ on } \partial\Omega, \\ u>0 \text{ in } \Omega. \end{cases}$$

We prove that, for each $\lambda\in\mathbb{R}$, problem (1.10) has a unique principal eigenvalue $\mu=\mu_{m,b}(\lambda)$, which has the variational characterization

$$(1.11)\qquad \mu_{m,b}(\lambda):=\inf_{w\in H_0^1(\Omega)\setminus\{0\}}\frac{\int_\Omega\langle A\nabla w,\nabla w\rangle+\int_\Omega(b-\lambda m)w^2}{\int_\Omega w^2}.$$

We prove also that the eigenspace $V_{\mu_{m,b}(\lambda)}$ corresponding to $\mu_{m,b}(\lambda)$ is one dimensional, and that if $0\not\equiv u\in V_{\mu_{m,b}(\lambda)}$ then $u\in H_0^1(\Omega)\cap C^1(\Omega)$ and either $u\equiv 0$ in Ω, or $u>0$ in Ω, or $u<0$ in Ω. In addition, we show that $\mu_{m,b}$ is a concave function which satisfies $\mu_{m,b}(0)>0$, $\lim_{\lambda\to\infty}\mu_{m,b}(\lambda)=-\infty$ if $m^+\not\equiv 0$, and $\lim_{\lambda\to-\infty}\mu_{m,b}(\lambda)=-\infty$ if $m^-\not\equiv 0$. We show also that if $m\geq 0$ in Ω then $\mu_{m,b}(\lambda)>0$ for any $\lambda\leq 0$, and that if $m\leq 0$ in Ω then $\mu_{m,b}(\lambda)>0$ for any $\lambda\geq 0$. From these facts, it follows that if m changes sign in Ω then the equation $\mu_{m,b}(\lambda)=0$ has exactly two roots, $\lambda=\lambda_{-1}(m,b)<0$ and $\lambda=\lambda_1(m,b)>0$, whereas if $m\geq 0$ (respectively $m\leq 0$) the same equation has a unique solution $\lambda=\lambda_1(m,b)>0$ (resp. $\lambda=\lambda_{-1}(m,b)<0$). From these facts, and since the principal eigenvalues of problem (1.8) are exactly the roots of the equation $\mu_{m,b}(\lambda) = 0$, we state, in Section 4 (see Theorem 4.1) the corresponding results for the principal eigenvalues of (1.8). A maximum principle with weight is given in Theorem 4.2, the variational formula for the principal eigenvalues of problem (1.8) is given in Theorem 4.3. In Theorem 4.4 we prove that the eigenfunctions corresponding to these eigenvalues belong to $H_0^1(\Omega) \cap C^1(\Omega) \cap C(\overline{\Omega})$ and we give lower and upper estimates for them (in terms of powers of d_Ω), and in Theorem 4.5 we study the continuity of the maps $(m,b) \to \lambda_1(m,b)$ and $(m,b) \to \Phi_{m,b}$, where $\Phi_{m,b}$ is the positive eigenfunction associated to $\lambda_1(m,b)$ and normalized by $\|\Phi_{m,b}\|_{L^2(\Omega)} = 1$.

2. Preliminaries

For $1 \leq p \leq \infty$, we will write p' for the Hölder conjugate exponent defined by $\frac{1}{p} + \frac{1}{p'} = 1$ (with the convention that $\frac{1}{\infty} = 0$); and p^* will denote the Sobolev critical exponent defined by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ if $p < n$ and by $p^* := \infty$ otherwise.
For a measurable function $v : \Omega \to \mathbb{R}$ such that $v\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$, we will write S_v to denote the functional $S_v : H_0^1(\Omega) \to \mathbb{R}$ defined by $S_v(\varphi) := \int_\Omega v\varphi$; and we will say $v \in \left(H_0^1(\Omega)\right)'$ to mean that $S_v \in \left(H_0^1(\Omega)\right)'$ and, in this case, if no confusion arises, we will write sometimes v instead of S_v. We will denote by d_Ω the distance to the boundary function $d_\Omega : \Omega \to \mathbb{R}$ defined by

$$d_\Omega(x) = dist(x, \partial\Omega).$$

From now on, $\mathcal{L}_0$ will denote the operator $\mathcal{L}_0 : H_0^1(\Omega) \to \left(H_0^1(\Omega)\right)'$ defined by $\mathcal{L}_0 u := -\operatorname{div}(A\nabla u)$ and, for $\zeta \in \left(H_0^1(\Omega)\right)'$, $\mathcal{L}_0^{-1}(\zeta)$ will denote the unique weak solution $u \in H_0^1(\Omega)$ (given by the Riesz theorem) to the problem $\mathcal{L}_0 u = \zeta$ in Ω, $u = 0$ on $\partial\Omega$.

Remark 2.1. *Let us recall the following well known facts:*

i) (Poincaré's inequality, see e.g., [16], Proposition 1.9.6) If $n > 2$ then there exists a positive constant c such that $\|\varphi\|_{L^{2^*}(\Omega)} \leq c\|\nabla\varphi\|_{L^2(\Omega)}$ for all $\varphi \in H_0^1(\Omega)$ and, if $n = 2$ then for each $q \in [1,\infty)$ there exists a positive constant c_q such that $\|\varphi\|_{L^q(\Omega)} \leq c_q\|\nabla\varphi\|_{L^2(\Omega)}$ for all $\varphi \in H_0^1(\Omega)$.

ii) (Hardy's inequality, see e.g., [3], p. 313) There exists a positive constant c such that $\left\|\frac{\varphi}{d_\Omega}\right\|_{L^2(\Omega)} \leq c\|\nabla\varphi\|_{L^2(\Omega)}$ for all $\varphi \in H_0^1(\Omega)$.

iii) (weak maximum principle, see e.g., [8], Theorem 1.3.7) If $g : \Omega \to \mathbb{R}$ is nonnegative and belongs to $\left(H_0^1(\Omega)\right)'$, then $\mathcal{L}_0^{-1} g \geq 0$.

iv) (weak comparison principle) If $g : \Omega \to \mathbb{R}$ and $h : \Omega \to \mathbb{R}$ belong to $\left(H_0^1(\Omega)\right)'$ and $g \leq h$ in Ω, then $\mathcal{L}_0^{-1} g \leq \mathcal{L}_0^{-1} h$.

Remark 2.2. *Let $v : \Omega \to \mathbb{R}$. From the Poincaré's and Hardy's inequalities of Remark 2.1, it follows immediately that if either $v \in L^{(2^*)'}(\Omega)$ or $d_\Omega v \in L^2(\Omega)$, then:*

i) The functional $S_v : H_0^1(\Omega) \to \mathbb{R}$ is well defined, belongs to $\left(H_0^1(\Omega)\right)'$, and there exists a positive constant c, independent of v, such that: If $v \in L^{(2^*)'}(\Omega)$ then $\|S_v\| \leq c\|v\|_{(2^*)'}$, and if $d_\Omega v \in L^2(\Omega)$ then $\|S_v\| \leq c\|d_\Omega v\|_2$.

ii) The problem $\mathcal{L}_0 z = v$ in Ω, $z = 0$ on $\partial\Omega$, has a unique weak solution $z \in H_0^1(\Omega)$, and it satisfies, for some positive constant c independent of v, $\|z\|_{H_0^1(\Omega)} \leq c\|v\|_{(2^*)'}$ when $v \in L^{(2^*)'}(\Omega)$, and $\|z\|_{H_0^1(\Omega)} \leq c\|d_\Omega v\|_2$ when $d_\Omega v \in L^2(\Omega)$.

Remark 2.3. *If $v : \Omega \to \mathbb{R}$ be a measurable function such that $v\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$ and if $S_v \in \left(H_0^1(\Omega)\right)'$, then, by the Riesz theorem, the problem*

$$\mathcal{L}_0 z = v \text{ in } \Omega, \quad z = 0 \text{ on } \partial\Omega$$

has a unique weak solution $z \in H_0^1(\Omega)$, and it satisfies $\|z\|_{H_0^1(\Omega)} = \|S_v\|_{\left(H_0^1(\Omega)\right)'}$.

If g and h are real functions defined $a.e.$ in Ω, we will write sometimes $f \approx g$ to mean that there exist positive constants c_1 and c_2 such that $c_1 f \le g \le c_2 f$ $a.e.$ in Ω. We will write also $f \lesssim g$ to mean that there exists a positive constant c such that $f \le cg$ $a.e.$ in Ω.
For $\delta > 0$, we set $\Omega_\delta := \{x \in \Omega : d_\Omega(x) > \delta\}$.

Lemma 2.1. *If w and φ belong to $H_0^1(\Omega)$, then $d_\Omega^{-2} w\varphi \in L^1(\Omega)$ and there exists a positive constant, independent of w and φ, such that*

$$\left\|d_\Omega^{-2} w\varphi\right\|_1 \le c\,\|w\|_{H_0^1(\Omega)}\,\|\varphi\|_{H_0^1(\Omega)}. \tag{2.12}$$

Proof. The lemma follows immediately from the Hardy's inequality. □

Lemma 2.2. *Let $b : \Omega \to \mathbb{R}$ be a nonnegative function such that $d_\Omega^2 b \in L^\infty(\Omega)$, and let $h : \Omega \to \mathbb{R}$ be such that $h \in \left(H_0^1(\Omega)\right)'$. Then:*

i) There exists a unique weak solution $u \in H_0^1(\Omega)$ to the problem

$$\begin{cases} \mathcal{L}_0 u + bu = h \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.13}$$

ii) If $h \ge 0$, and if u is the weak solution of (2.13), then $u \ge 0$ $a.e$ in Ω.
iii) If $h \ge 0$ and $h \not\equiv 0$, and if u is the weak solution of (2.13), then, for any $\delta > 0$ such that $\Omega_\delta \ne \varnothing$, there exists a positive constant c such that $u \ge c d_{\Omega_\delta}$ $a.e$ in Ω_δ. In particular, $u > 0$ $a.e.$ in Ω.

Proof. Let $B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be defined by

$$B(\varphi, \psi) := \int_\Omega \left(\langle A\nabla\varphi, \nabla\psi\rangle + b\varphi\psi\right).$$

By Lemma 2.1, B is a continuous bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$ and, since $b \ge 0$, B is also coercive. Then $i)$ follows from the Lax Milgram theorem. Suppose now $h \ge 0$. By taking $-u^-$ as a test function in (2.13), we get

$$\int_\Omega \left(\left\langle A\nabla u^-, \nabla u^-\right\rangle + b\left(u^-\right)^2\right) = \int_\Omega \left(\left\langle A\nabla u, -\nabla u^-\right\rangle + bu\left(-u^-\right)\right) = -\int_\Omega hu^- \le 0,$$

which gives $u^- = 0$ $a.e.$ in Ω. Thus $ii)$ holds.

To prove $iii)$, observe that if $h \ge 0$ $a.e$ in Ω and $h \not\equiv 0$ in Ω, then, for δ positive and small enough, there exist $\varepsilon > 0$ and a measurable set $E \subset \Omega_\delta$ such that $|E| > 0$ and $h \ge \varepsilon\chi_E$ in Ω_δ. For such a δ, let Ω' be a regular domain such that $\Omega_\delta \subset\subset \Omega' \subset\subset \Omega$, and consider the problem

$$\begin{cases} -\mathcal{L}_0 z + bz = \varepsilon\chi_E \text{ in } \Omega', \\ z = 0 \text{ on } \partial\Omega'. \end{cases}$$

Since $0 \le b_{|\Omega'} \in L^\infty(\Omega')$ and $\varepsilon\chi_E \in L^\infty(\Omega')$, by the inner elliptic estimates in ([7], Theorem 9.11), we have $z \in W^{2,q}(\Omega') \cap W_0^{1,q}(\Omega')$ for any $q \in [1, \infty)$ and so $z \in C^1\left(\overline{\Omega'}\right)$. By the maximum principle (as stated e.g., in [7, Theorem 9.1]) we have $z(x) > 0$ for any $x \in \Omega'$, and by the Hopf's boundary lemma (as stated e.g., in [17, Theorem 1.1]), we have also $\frac{\partial z}{\partial \nu} < 0$ on $\partial\Omega'$ and from these two facts it follows that z belongs to the interior of the positive cone of $C^1\left(\overline{\Omega'}\right)$, and so there exists a constant $c > 0$ (which may depend on Ω') such that $z \geq c d_{\Omega'}$ in Ω'. Therefore, since $d_{\Omega'} \geq d_{\Omega_\delta}$ in Ω_δ, we have $z \geq c d_{\Omega_\delta}$ in Ω_δ. Now,

$$\begin{cases} \mathcal{L}_0 (u - z) + b (u - z) = h - \varepsilon \chi_E \geq 0 \text{ in } D' (\Omega'), \\ u - z \geq 0 \text{ on } \partial \Omega', \end{cases}$$

with the inequality on $\partial\Omega'$ understood in the sense of the trace. Thus, by the maximum principle (as stated, e.g., in [7, Theorem 9.1]), $u \geq z$ in Ω' and then $u \geq c d_{\Omega_\delta}$ *a.e.* in Ω_δ. Thus $iii)$ holds for δ positive and small enough, and so $iii)$ holds also for any $\delta > 0$ such that $\Omega_\delta \neq \varnothing$ (because if $0 < \delta_1 < \delta_2$ and $\Omega_{\delta_2} \neq \varnothing$ then $d_{\Omega_1} \leq d_{\Omega_2}$ in Ω_{δ_2}). □

Remark 2.4. *Let $b : \Omega \to \mathbb{R}$ be a nonnegative function such that $d_\Omega^2 b \in L^\infty (\Omega)$, and let $(\mathcal{L}_0 + b)^{-1} : L^2 (\Omega) \to H_0^1 (\Omega)$ be the solution operator of problem (2.13), i.e., the operator defined by $(\mathcal{L}_0 + b)^{-1} h = u$, where u is the weak solution of (2.13). Then $(\mathcal{L}_0 + b)^{-1} : L^2 (\Omega) \to H_0^1 (\Omega)$ is continuous and $(\mathcal{L}_0 + b)^{-1} : L^2 (\Omega) \to L^2 (\Omega)$ is a compact operator. Indeed, for $h \in L^2 (\Omega)$ and $u = (\mathcal{L}_0 + b)^{-1} h$, we have*

$$c \|u\|^2_{H_0^1(\Omega)} \leq \int_\Omega \langle A\nabla u, \nabla u\rangle + \int_\Omega b u^2 = \int_\Omega h u \leq c_P \|h\|_2 \|u\|_{H_0^1(\Omega)},$$

where c is the ellipticity constant of A and c_P is the constant of the Poincaré's inequality, and so, if $u \not\equiv 0$, then $\|u\|_{H_0^1(\Omega)} \leq c^{-1} c_P \|h\|_2$. Since clearly this inequality holds also when $u \equiv 0$, it follows that $(\mathcal{L}_0 + b)^{-1} : L^2 (\Omega) \to H_0^1 (\Omega)$ is continuous. Then, since $H_0^1 (\Omega)$ has compact inclusion in $L^2 (\Omega)$, we conclude that $(\mathcal{L}_0 + b)^{-1} : L^2 (\Omega) \to L^2 (\Omega)$ is a compact operator.

3. A ONE PARAMETER EIGENVALUE PROBLEM WITH SINGULAR POTENTIAL

From now on, b and m will denote, respectively, a nonnegative function $b : \Omega \to \mathbb{R}$ such that $d_\Omega^2 b \in L^\infty (\Omega)$, and a nonidentically zero function $m \in L^\infty (\Omega)$, which (except if otherwise is explicitly stated) may change sign.

Definition 3.2. *For $\lambda \in \mathbb{R}$, let*

$$\mu_{m,b} (\lambda) := \inf_{w \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega \langle A\nabla w, \nabla w\rangle + \int_\Omega (b - \lambda m) w^2}{\int_\Omega w^2}. \tag{3.14}$$

Notice that, by the Hardy's inequality,

$$0 \leq \int_\Omega b w^2 = \int_\Omega d_\Omega^2 b \frac{w^2}{d_\Omega^2} \leq \left\|d_\Omega^2 b\right\|_\infty \|w\|^2_{H_0^1(\Omega)} \leq c \|w\|^2_{H_0^1(\Omega)} \tag{3.15}$$

for any $w \in H_0^1 (\Omega)$, where c is a positive constant independent of w. Also,

$$\begin{aligned} &\frac{\int_\Omega \langle A\nabla w, \nabla w\rangle + \int_\Omega (b - \lambda m) w^2}{\int_\Omega w^2} \\ &\geq \frac{\int_\Omega \langle A\nabla w, \nabla w\rangle + \int_\Omega b w^2}{\int_\Omega w^2} - \|m\|_\infty |\lambda| \geq - \|m\|_\infty |\lambda|, \end{aligned}$$

and then $\mu_{m,b} (\lambda)$ is well defined and finite for any $\lambda \in \mathbb{R}$.

Proposition 3.1. *For any $\lambda \in \mathbb{R}$, we have:*

i) If $\mu \in \mathbb{R}$ and if u is a weak solution of the problem

$$\begin{cases} -\operatorname{div} (A\nabla u) + b u = \lambda m u + \mu u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{3.16}$$

then $u \in C^1(\Omega)$ *and* $\mu_{m,b}(\lambda) \leq \mu$.

ii) The infimum in (3.14) is achieved at some nonnegative and nonidentically zero $u \in H_0^1(\Omega)$.

Proof. To prove $i)$, it is enough to see that: if u is a weak solution of (3.16), and if Ω' is an arbitrary regular domain such that $\Omega' \subset\subset \Omega$, then $u \in C^1(\Omega')$. We consider first the case $n = 2$. For Ω' as above, let U_0 be a regular domain such that $\Omega \supset\supset U_0 \supset\supset \Omega'$. Since $n = 2$, we have $u \in H_0^1(\Omega) \subset L^q(\Omega)$ for any $q \in [1, \infty)$, and so $u \in L^q(U_0)$, $\lambda m u + \mu u \in L^q(U_0)$ for some $q > 2$. Also, $b \in L^\infty(U_0)$. Then, taking into account (3.16), and the inner elliptic estimates in ([7], Theorem 9.11), we get $u \in W^{2,q}(\Omega') \subset C^1(\Omega')$. Suppose now $n > 2$, and let $\{\Omega_j\}_{j \in \mathbb{N} \cup \{0\}}$ and $\{U_j\}_{j \in \mathbb{N} \cup \{0\}}$ be two sequences of regular domains such that $\Omega_0 = \Omega$ and $\Omega_j \supset\supset U_j \supset\supset \Omega_{j+1} \supset\supset \Omega'$ for all $j \in \mathbb{N} \cup \{0\}$. For $j \in \mathbb{N} \cup \{0\}$, let q_j be inductively defined by $q_0 = 2$, and by $q_{j+1} = q_j^*$ (with $q_j^* := \infty$ if $q_j \geq n$). Let $j_0 = \max\left\{j \in \mathbb{N} \cup \{0\} : q_j^* < \infty\right\}$. Thus $q_{j_0} < n$ and $q_{j_0}^* \geq n$. Let us show, inductively, that

$$u \in W^{2,q_j}(\Omega_{j+1}) \text{ for } j = 0, 1, ..., j_0. \tag{3.17}$$

Since $u \in L^2(\Omega)$, we have $u \in L^2(U_0)$, $\lambda m u + \mu u \in L^2(U_0)$. Also, $b \in L^\infty(U_0)$ and thus, by (3.16) and ([7], Theorem 9.11), $u \in W^{2,2}(\Omega_1) = W^{2,q_0}(\Omega_1)$. Then (3.17) holds for $j = 0$. Suppose now that (3.17) holds for some $j \in \{0, 1, ..., j_0 - 1\}$. Then $u \in L^{q_j^*}(U_{j+1})$, $\lambda m u + \mu u \in L^{q_j^*}(U_{j+1})$, and also $b \in L^\infty(U_{j+1})$, and so, again now from (3.16) and ([7], Theorem 9.11), $u \in W^{2,q_j^*}(\Omega_{j*2}) = W^{2,q_{j+1}}(\Omega_{j*2})$, which completes the inductive proof of (3.17). Then $u \in W^{2,q_{j_0}}(\Omega_{j_0+1})$ and so, by using again now the above argument, $u \in W^{2,q_{j_0}^*}(\Omega_{j_0+2})$. If $q_{j_0}^* > n$ then $W^{2,q_{j_0}^*}(\Omega_{j_0+2}) \subset C^1(\Omega_{j_0+2}) \subset C^1(\Omega')$ and we are done. If $q_{j_0}^* = n$ then $W^{2,q_{j_0}^*}(\Omega_{j_0+2}) \subset L^r(\Omega_{j_0+2})$ for any $r \in [1, \infty)$. We take $r > n$ to obtain, proceeding as above, $u \in W^{2,r}(\Omega_{j_0+3}) \subset C^1(\Omega_{j_0+3}) \subset C^1(\Omega')$. Thus the first assertion of $i)$ holds.

On the other hand, from (3.16),

$$\int_\Omega \left(\langle A\nabla u, \nabla u\rangle + (b - \lambda m) u^2\right) = \mu \int_\Omega u^2$$

and so $\mu = \left(\int_\Omega u^2\right)^{-1} \int_\Omega \left(\langle A\nabla u, \nabla u\rangle + (b - \lambda m) u^2\right) \geq \mu_{m,b}(\lambda)$, the last inequality by (3.14), which completes the proof of $i)$. To prove $ii)$ consider a minimizing sequence $\{w_j\}_{j \in \mathbb{N}}$ for (3.14). After normalizing it, and by replacing, if necessary, w_j by $|w_j|$ we can assume that $w_j \geq 0$ and $\|w_j\|_2 = 1$ for each j. From (3.14), we have

$$\mu_{m,b}(\lambda) = \lim_{j \to \infty} \left(\int_\Omega \langle A\nabla w_j, \nabla w_j\rangle + \int_\Omega (b - \lambda m) w_l^2\right) \tag{3.18}$$

$$\geq \liminf_{j \to \infty} \int_\Omega \langle A\nabla w_j, \nabla w_j\rangle - |\lambda| \, \|m\|_\infty \tag{3.19}$$

and so, after pass to a further subsequence if necessary, we can assume that $\{w_j\}_{j \in \mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Thus there exist $u \in H_0^1(\Omega)$ and a subsequence, still denoted by $\{w_j\}_{j \in \mathbb{N}}$, such that $\{\nabla w_j\}_{j \in \mathbb{N}}$ converges weakly in $L^2(\Omega, \mathbb{R}^n)$ to ∇u and $\{w_j\}_{j \in \mathbb{N}}$ converges strongly in $L^2(\Omega)$ to u. Thus $\|u\|_2 = 1$. After pass to a further subsequence if necessary, we can assume also that $\{w_j\}_{j \in \mathbb{N}}$ converges to u $a.e.$in Ω and so, since each w_j is nonnegative, we have $u \geq 0$. Let $k \in \mathbb{R}$

such that $b - \lambda m + k \geq 0$. From the equality in (3.18) and since $\|w_j\|_2 = 1$, we have

$$\begin{aligned}\mu_{m,b}(\lambda) + k &= \lim_{j\to\infty}\left(\int_\Omega \langle A\nabla w_j, \nabla w_j\rangle + \int_\Omega (b - \lambda m + k)\, w_l^2\right)\\ &\geq \liminf_{j\to\infty}\int_\Omega \langle A\nabla w_j, \nabla w_j\rangle + \liminf_{j\to\infty}\int_\Omega (b - \lambda m + k)\, w_l^2\\ &\geq \int_\Omega \langle A\nabla u, \nabla u\rangle + \int_\Omega (b - \lambda m + k)\, u^2\\ &= \int_\Omega \langle A\nabla u, \nabla u\rangle + \int_\Omega (b - \lambda m)\, u^2 + k,\end{aligned}$$

where in the last inequality it was used the Fatou's Lemma and the fact that $\|\langle A\nabla u, \nabla u\rangle\|_2 \leq \liminf_{j\to\infty}\|\langle A\nabla w_j, \nabla w_j\rangle\|_2$. Then $\mu_{m,b}(\lambda) \geq \int_\Omega \langle A\nabla u, \nabla u\rangle + \int_\Omega (b - \lambda m)\, u^2$. On the other hand, from the definition of $\mu_{m,b}(\lambda)$, we get the opposite inequality. Then $\mu_{m,b}(\lambda) = \int_\Omega \langle A\nabla u, \nabla u\rangle + \int_\Omega (b - \lambda m)\, u^2$ and so $ii)$ holds. ☐

Proposition 3.2. *For any $\lambda \in \mathbb{R}$, we have:*

i) If u is a minimizer of (3.14), then u is a weak solution of the problem

$$\begin{cases} -\operatorname{div}(A\nabla u) + bu = \lambda mu + \mu_{m,b}(\lambda)\, u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.20}$$

ii) For $\mu \in \mathbb{R}$, if u is a nonidentically zero weak solution of the problem

$$\begin{cases} -\operatorname{div}(A\nabla u) + bu = \lambda mu + \mu u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{3.21}$$

such that $u \geq 0$ in Ω, then $\mu = \mu_{m,b}(\lambda)$ and u is a minimizer of (3.14).

Proof. To prove $i)$, consider a minimizer w of (3.14). Thus

$$\mu_{m,b}(\lambda) = \frac{\int_\Omega \left(\langle A\nabla w, \nabla w\rangle + (b - \lambda m)\, w^2\right)}{\int_\Omega w^2}. \tag{3.22}$$

Let $\psi \in H_0^1(\Omega)$. Then there exists $\varepsilon_0 > 0$ such that $w + t\psi \in H_0^1(\Omega) \setminus \{0\}$ for any $t \in (-\varepsilon_0, \varepsilon_0)$. Then, for such a t,

$$\mu_{m,b}(\lambda) \leq \frac{\int_\Omega \left(\langle A\nabla(w + t\psi), \nabla(w + t\psi)\rangle + (b - \lambda m)(w + t\psi)^2\right)}{\int_\Omega (w + t\psi)^2}. \tag{3.23}$$

From (3.23), a computation using gives that, for $t \in (0, \varepsilon_0)$,

$$\begin{aligned}&\mu_{m,b}(\lambda)\left(\int_\Omega w\psi + \frac{t}{2}\int_\Omega \psi^2\right)\\ &\leq \int_\Omega \left(\langle A\nabla w, \nabla\psi\rangle + \frac{t}{2}\langle A\nabla\psi, \nabla\psi\rangle + (b - \lambda m)\left(w\psi + \frac{t}{2}w^2\right)\right),\end{aligned}$$

and so, by taking $\lim_{t\to 0^+}$ we get $\mu_{m,b}(\lambda)\int_\Omega w\psi \leq \int_\Omega \left(\langle A\nabla w, \nabla\psi\rangle + (b - \lambda m)\, w\psi\right)$. By replacing ψ by $-\psi$, the reversed inequality is obtained, and thus $i)$ holds.

To prove $ii)$, suppose that $u \in H_0^1(\Omega)$ is a nonidentically zero weak solution of (3.16) such that $u \geq 0$ in Ω. Let $w \in C_c^\infty(\Omega)$ and let $\varepsilon > 0$. Then $\frac{w^2}{u+\varepsilon} \in H_0^1(\Omega)$. We take $\frac{w^2}{u+\varepsilon}$ as a test function in (3.16) to obtain

$$\int_\Omega \left\langle A\nabla u, \frac{(u+\varepsilon)\,2w\nabla w - w^2\nabla u}{(u+\varepsilon)^2}\right\rangle + \int_\Omega bw^2\frac{u}{u+\varepsilon}$$
$$= \lambda\int_\Omega mw^2\frac{u}{u+\varepsilon} + \int_\Omega w^2\frac{\mu u}{u+\varepsilon},$$

that is

$$\int_\Omega \left\langle A\nabla u, \frac{2w\nabla w}{u+\varepsilon}\right\rangle - \int_\Omega \left\langle A\nabla u, \frac{w^2\nabla u}{(u+\varepsilon)^2}\right\rangle + \int_\Omega bw^2\frac{u}{u+\varepsilon}$$
$$= \lambda\int_\Omega mw^2\frac{u}{u+\varepsilon} + \int_\Omega w^2\frac{\mu u}{u+\varepsilon},$$

i.e.,

$$\int_\Omega 2\left\langle Aw\nabla \ln(u+\varepsilon), \nabla w\right\rangle - \int_\Omega \left\langle Aw\nabla \ln(u+\varepsilon), w\nabla \ln(u+\varepsilon)\right\rangle + \int_\Omega bw^2\frac{u}{u+\varepsilon}$$
$$= \lambda\int_\Omega mw^2\frac{u}{u+\varepsilon} + \int_\Omega w^2\frac{\mu u}{u+\varepsilon},$$

that is

$$-\int_\Omega \left\langle A\left(w\nabla \ln(u+\varepsilon) - \nabla w\right), w\nabla \ln(u+\varepsilon) - \nabla w\right\rangle + \int_\Omega \left\langle A\nabla w, \nabla w\right\rangle + \int_\Omega bw^2\frac{u}{u+\varepsilon}$$
$$= \lambda\int_\Omega mw^2\frac{u}{u+\varepsilon} + \mu\int_\Omega w^2\frac{u}{u+\varepsilon},$$

and so

$$\int_\Omega w^2\frac{\mu u}{u+\varepsilon} \le \int_\Omega \left\langle A\nabla w, \nabla w\right\rangle + \int_\Omega bw^2\frac{u}{u+\varepsilon} - \lambda\int_\Omega mw^2\frac{u}{u+\varepsilon}. \tag{3.24}$$

From (3.24), by taking $\lim_{\varepsilon\to 0^+}$ and using the Lebesgue's dominated convergence theorem, we get

$$\mu\int_\Omega w^2 \le \int_\Omega \left\langle A\nabla w, \nabla w\right\rangle + \int_\Omega bw^2 - \lambda\int_\Omega mw^2. \tag{3.25}$$

Since this holds for any $w \in C_c^\infty(\Omega)$, and taking into account Lemma 2.1, a density argument gives that (3.25) holds also for any $w \in H_0^1(\Omega)$. Therefore,

$$\mu \le \frac{\int_\Omega \left\langle A\nabla w, \nabla w\right\rangle + \int_\Omega bw^2 - \lambda\int_\Omega mw^2}{\int_\Omega w^2} \tag{3.26}$$

for any $w \in H_0^1(\Omega)\setminus\{0\}$. On the other hand, by taking $w = u$ as a test function in (3.14), we get $\mu = \left(\int_\Omega u^2\right)\int_\Omega\left(\left\langle A\nabla u, \nabla u\right\rangle + bu^2 - \lambda mu^2\right)$. Thus, from this fact and (3.26), $\mu = \mu_{m,b}(\lambda)$. Then $ii)$ holds. □

Proposition 3.3. *For any $\lambda \in \mathbb{R}$, we have:*

i) If u is a nonidentically zero weak solution of problem (3.20), then either $u > 0$ in Ω or $u < 0$ in Ω.

ii) The space of the weak solutions u of (3.20) is one dimensional.

Proof. To prove $i)$ we follow, partly, [15] (see also [5, Theorem 1.13]). We proceed by the way of contradiction. Suppose that $u \in H_0^1(\Omega) \setminus \{0\}$ is a weak solution of (3.20), and that $u^+ \not\equiv 0$ and $u^- \not\equiv 0$. Let

$$\begin{aligned}
\alpha &:= \int_\Omega \left(\langle A\nabla u, \nabla u \rangle + (b - \lambda m)\, u^2 \right), & \beta &:= \int_\Omega u^2, \\
\alpha_1 &:= \int_\Omega \left(\left\langle A\nabla u^+, \nabla u^+ \right\rangle + (b - \lambda m) \left(u^+\right)^2 \right), & \beta_1 &:= \int_\Omega \left(u^+\right)^2, \\
\alpha_2 &:= \int_\Omega \left(\left\langle A\nabla u^-, \nabla u^- \right\rangle + (b - \lambda m) \left(u^-\right)^2 \right), & \beta_2 &:= \int_\Omega \left(u^-\right)^2.
\end{aligned}$$

Thus $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$. Now,

$$\mu_{m,b}(\lambda) = \frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2},$$

and so, since u^+ and u^- belong to $H_0^1(\Omega) \setminus \{0\}$,

$$\frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2} \le \frac{\alpha_1}{\beta_1} \text{ and } \frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2} \le \frac{\alpha_2}{\beta_2},$$

that is

$$\begin{aligned}
\alpha_1\beta_1 + \alpha_2\beta_1 &\le \beta_1\alpha_1 + \beta_2\alpha_1, \\
\alpha_1\beta_2 + \alpha_2\beta_2 &\le \beta_1\alpha_2 + \beta_2\alpha_2,
\end{aligned} \tag{3.27}$$

i.e., $\frac{\alpha_1}{\beta_1} \ge \frac{\alpha_2}{\beta_2}$ and $\frac{\alpha_1}{\beta_1} \le \frac{\alpha_2}{\beta_2}$. Then $\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$ and so $\frac{\alpha_1+\alpha_2}{\beta_1+\beta_2} = \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$. Thus $\mu_{m,b}(\lambda) = \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$. Therefore u^+ and u^- are nonnegative minimizers of (3.14) and then, by Proposition 3.1 $ii)$, they are nonnegative and nonidentically zero weak solutions of (3.20) and so, for $q \in \mathbb{R}$ such that $b - \lambda m + q \ge 0$ and $\mu_{m,b}(\lambda) + q > 0$ we have, in weak sense,

$$\begin{cases} -\operatorname{div}\left(A\nabla u^+\right) + (b - \lambda m + q)\, u^+ = (\mu_{m,b}(\lambda) + q)\, u^+ \text{ in } \Omega, \\ u^+ = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.28}$$

Thus, from Lemma 2.2 (used with b replaced by $b - \lambda m + q$ and with h replaced by $(\mu_{m,b}(\lambda) + q)\, u$), we get that, for any $\delta > 0$ such that $\Omega_\delta \ne \varnothing$, there exists a positive constant c such that $u^+ \ge c d_{\Omega_\delta}$ in Ω_δ. In particular, $u^+ > 0$ in Ω, and so $u^- \equiv 0$ in Ω, which contradicts our assumptions. Then $i)$ holds.

To prove $ii)$, suppose that v and w are two linearly independent solutions of (3.20) and let $x_0 \in \Omega$. Taking into account $i)$ and Proposition 3.1, we can assume (by replacing, if necessary, v and/or w by $-v$ and/or $-w$ respectively) that $v(x_0) > 0$ and $w(x_0) > 0$. Let $t_0 = (v(x_0))^{-1} w(x_0)$ and let $z := t_0 v - w$. Then $t_0 > 0$ and z is a solution of (3.20) such that $z(x_0) = 0$. Thus, by $i)$, z is identically zero on Ω, which contradicts the assumed linear independence of v and w. □

Proposition 3.4. *Let $b : \Omega \to \mathbb{R}$ be a nonnegative function such that $d_\Omega^2 b \in L^\infty(\Omega)$, let $m \in L^\infty(\Omega)$ be a nonidentically zero function, and, for $\lambda \in \mathbb{R}$, let $\mu_{m,b}(\lambda)$ be defined by (3.14). Then:*

i) The map $\lambda \to \mu_{m,b}(\lambda)$ is concave and $\mu_{m,b}(0) > 0$.

ii) If $m^+ \not\equiv 0$ then $\lim_{\lambda\to\infty} \mu_{m,b}(\lambda) = -\infty$; and there exists a unique $\lambda > 0$ such that $\mu_{m,b}(\lambda) = 0$. If, in addition, $m \ge 0$ in Ω, then $\mu_{m,b}(\lambda) > 0$ for any $\lambda \le 0$.

iii) If $m^- \not\equiv 0$ then $\lim_{\lambda\to-\infty} \mu_{m,b}(\lambda) = -\infty$; and there exists a unique $\lambda < 0$ such that $\mu_{m,b}(\lambda) = 0$. If, in addition, $m \le 0$ in Ω, then $\mu_{m,b}(\lambda) > 0$ for any $\lambda \ge 0$.

Proof. The first assertion of $i)$ follows from the facts that $\mu_{m,b}(\lambda)$ is finite for any $\lambda \in \mathbb{R}$, and that $\lambda \to \left(\int_\Omega w^2\right)^{-1}\left(\int_\Omega \langle A\nabla w, \nabla w\rangle + \int_\Omega (b-\lambda m) w^2\right)$ is an affine function for any $w \in H_0^1(\Omega)\backslash\{0\}$. Observe also that, from Proposition 3.1 ii), Proposition 3.2 $i)$ and Proposition 3.3 $i)$, all of them used with $\lambda = 0$, the problem

$$\left\{\begin{array}{c} -\operatorname{div}(A\nabla u) + bu = \mu_m(0)\, u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega \end{array}\right. \tag{3.29}$$

has a weak solution u. By taking u as a test function in (3.29), we get

$$\int_\Omega \langle A\nabla u, \nabla u\rangle + \int_\Omega bu^2 = \mu_m(0)\int_\Omega u^2$$

which gives $\mu_m(0) > 0$. Thus $i)$ holds.

To see $ii)$, suppose $m^+ \not\equiv 0$ and let $w_0 \in H_0^1(\Omega)\setminus\{0\}$ such that $\int_\Omega m w_0^2 > 0$. By normalizing w_0, if necessary, we can assume that $\int_\Omega m w_0^2 = 1$. Then, for any $\lambda \in \mathbb{R}$, $\mu_{m,b}(\lambda) \le \int_\Omega \langle A\nabla w_0, \nabla w_0\rangle + \int_\Omega b w_0^2 - \lambda \int_\Omega m w_0^2$. From this fact, and since μ_m is concave and $\mu_m(0) > 0$, it follows that $\lim_{\lambda\to\infty} \mu_{m,b}(\lambda) = -\infty$; and that there exists a unique $\lambda > 0$ such that $\mu_{m,b}(\lambda) = 0$. On the other hand, if $m \ge 0$ in Ω and $\lambda \le 0$, and if u is a positive solution of the problem

$$\left\{\begin{array}{c} -\operatorname{div}(A\nabla u) + bu = \lambda m u + \mu_{m,b}(\lambda)\, u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{array}\right.$$

then, by taking u as a test function, we get

$$\int_\Omega \left(\langle A\nabla u, \nabla u\rangle + bu^2\right) = \lambda\int_\Omega m u^2 + \mu_{m,b}(\lambda)\int_\Omega u^2$$

and so $\int_\Omega u^2 > 0$, which implies $\mu_{m,b}(\lambda) > 0$. Thus $ii)$ holds.

Finally, $iii)$ follows from $ii)$ by using that, by (3.14), $\mu_{m,b}(\lambda) = \mu_{-m}(-\lambda)$. ☐

4. Principal eigenvalues problems with singular potential and bounded weight

Definition 4.3. *Let $b : \Omega \to \mathbb{R}$ be a nonnegative function such that $d_\Omega^2 b \in L^\infty(\Omega)$ and let $m \in L^\infty(\Omega)\setminus\{0\}$. We say that $\lambda \in \mathbb{R}$ is a principal eigenvalue of the operator $\mathcal{L}_0 + b$ on Ω, with weight function m and homogeneous Dirichlet boundary condition, if the problem*

$$\left\{\begin{array}{c} -\operatorname{div}(A\nabla\phi) + b\phi = \lambda m\phi \text{ in } \Omega, \\ \phi = 0 \text{ on } \partial\Omega \end{array}\right. \tag{4.30}$$

has a weak solution $\phi \in H_0^1(\Omega)$ such that $\phi \ge 0$ a.e. in Ω and $\phi \not\equiv 0$ in Ω. In such a case, any nonidentically zero solution of (4.30) will be called a principal eigenfunction associated to the principal eigenvalue λ.

Theorem 4.1. *Let $b : \Omega \to \mathbb{R}$ be a nonnegative function such that $d_\Omega^2 b \in L^\infty(\Omega)$ and let $m \in L^\infty(\Omega)$ be such that $m \not\equiv 0$. Then:*

i) $\lambda \in \mathbb{R}$ is a principal eigenvalue for problem (4.30) if, and only if, $\mu_{m,b}(\lambda) = 0$.
ii) If $m^+ \not\equiv 0$ (respectively if $m^- \not\equiv 0$) there exists a unique positive (resp. a unique negative) principal eigenvalue for problem (4.30), which will be denoted by $\lambda_1(m,b)$ (resp. by $\lambda_{-1}(m)$).
iii) If $m \ge 0$ (respectively if $m \le 0$), then $\lambda_1(m,b)$ (resp. $\lambda_{-1}(m)$) is the unique principal eigenvalue for problem (4.30).

iv) If $\lambda \in \mathbb{R}$ is a principal eigenvalue for problem (4.30), and if u is an associated eigengunction, then $u \in H_0^1(\Omega) \cap C^1(\Omega)$. Moreover, if $u \in H_0^1(\Omega)$ nonidentically zero then either $u > 0$ in Ω or $u < 0$ in Ω.
v) The space of solutions of (4.30) is one dimensional.

Proof. The proposition follows directly from Propositions 3.1, 3.2, 3.3, and 3.4. □

The following form of the maximum principle for problems with singular potential and weight function holds:

Theorem 4.2. *Let $b : \Omega \to \mathbb{R}$ be a nonnegative function such that $d_\Omega^2 b \in L^\infty(\Omega)$ and let $m \in L^\infty(\Omega)$ be a nonidentically zero function. For $\lambda \in \mathbb{R}$, let $\mu_{m,b}(\lambda)$ be defined by (3.14) and let $h : \Omega \to \mathbb{R}$ be such that $h \in \left(H_0^1(\Omega)\right)'$. Then:*
i) If $\mu_{m,b}(\lambda) > 0$, the problem

$$\begin{cases} -\operatorname{div}(A\nabla u) + bu = \lambda m u + h \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{4.31}$$

has a unique weak solution.
ii) If $\mu_{m,b}(\lambda) > 0$ and $0 \not\equiv h \geq 0$, then the solution u of (4.31) is positive a.e in Ω.
iii) If $0 \not\equiv h \geq 0$ and if (4.31) has a nonnegative solution, then $\mu_{m,b}(\lambda) > 0$.
iv) If $0 \not\equiv h \geq 0$ and $\mu_{m,b}(\lambda) = 0$, then (4.31) has no weak solutions.

Proof. To prove $i)$, suppose $\mu_{m,b}(\lambda) > 0$ and let $k \in [0,\infty)$ be such that $b - \lambda m + k \geq 0$. Let $T : L^2(\Omega) \to L^2(\Omega)$ be defined by $T := (\mathcal{L}_0 + b - \lambda m + k)^{-1}$. Thus T is a continuous, compact, linear and it is self-adjoint operator on $L^2(\Omega)$. Notice that ρ is an eigenvalue of T if and only if $\rho = \frac{1}{k+\mu}$ with μ an eigenvalue of $\mathcal{L}_0 + b + k - \lambda m$ with (homogeneous Dirichlet boundary condition). By Proposition 3.1 $i)$, we have $\mu \geq \mu_{m,b+k}(\lambda) = \mu_{m,b}(\lambda) + k > 0$, and so $\rho < \frac{1}{k}$. Thus, by the Fredholm alternative theorem, $\frac{1}{k}I - T : L^2(\Omega) \to L^2(\Omega)$ is bijective, and so the problem $\frac{1}{k}u - Tu = \frac{1}{k}Th$ has a unique weak solution $u \in H_0^1(\Omega)$, that is, the problem

$$\begin{cases} \dfrac{1}{k}(\mathcal{L}_0 + b - \lambda m + k)\, u - u = \dfrac{1}{k} h \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

has a unique weak solution u. Then $i)$ holds.

To see $ii)$ observe that if $\mu_{m,b}(\lambda) > 0$ and if $u \in H_0^1(\Omega)$ is a weak solution of

$$\begin{cases} -\operatorname{div}(A\nabla u) + (b - \lambda m)\, u = h \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

then, by taking $-u^-$ as a test function,

$$\mu_{m,b}(\lambda) \int_\Omega (u^-)^2 \leq \int_\Omega \left(\langle A\nabla u^-, \nabla u^- \rangle + (b - \lambda m)(u^-)^2\right) = -\int_\Omega h u^- \leq 0$$

and so $u^- = 0$. Thus $u \geq 0$. In addition, since $-\operatorname{div}(A\nabla u) + (b - \lambda m + k)\, u = h + ku$ and $0 \not\equiv h + ku \geq 0$, Lemma 2.2 gives $u > 0$ in Ω. Thus $ii)$ holds.

To see $iii)$ suppose that $0 \not\equiv h \geq 0$ and that u is a nonnegative solution of (4.31). Take k as in the proof of $i)$, to get

$$\begin{cases} -\operatorname{div}(A\nabla u) + (b - \lambda m + k)\, u = h + ku \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

Then, by Lemma 2.2 iii), $u > 0$ $a.e.$ in Ω. Now we can repeat, line by line, the first part of the proof of Lemma 3.2 ii), replacing there, in each appearance, μu by h, to obtain, instead of (3.24), that for any $w \in C_c^{\infty}(\Omega)$ and $\varepsilon > 0$,

$$\int_{\Omega} w^2 \frac{h}{u+\varepsilon} \leq \int_{\Omega} \langle A\nabla w, \nabla w \rangle + \int_{\Omega} bw^2 \frac{u}{u+\varepsilon} - \lambda \int_{\Omega} mw^2 \frac{u}{u+\varepsilon}$$

and so, by taking $\liminf_{\varepsilon \to 0^+}$

$$0 \leq \int_{\Omega} \langle A\nabla w, \nabla w \rangle + \liminf_{\varepsilon \to 0^+} \left(\int_{\Omega} bw^2 \frac{u}{u+\varepsilon} - \lambda \int_{\Omega} mw^2 \frac{u}{u+\varepsilon} \right). \tag{4.32}$$

Notice that $u > 0$ $a.e.$ in Ω, $\lim_{\varepsilon \to 0^+} bw^2 \frac{u}{u+\varepsilon} = bw^2$ $a.e.$ in Ω, and $\lim_{\varepsilon \to 0^+} mw^2 \frac{u}{u+\varepsilon} = mw^2$ $a.e.$ in Ω. Also, $bw^2 \frac{u}{u+\varepsilon} \leq bw^2$ and $mw^2 \frac{u}{u+\varepsilon} \leq mw^2$. Observe also that, by Lemma 2.1 and that, from our assumption on b, $bw^2 \in L^1(\Omega)$. Also, clearly $mw^2 \in L^1(\Omega)$. Thus, from (4.32) and the Lebesgue's dominated convergence theorem,

$$0 \leq \int_{\Omega} \langle A\nabla w, \nabla w \rangle + \int_{\Omega} bw^2 - \lambda \int_{\Omega} mw^2$$

and so

$$\frac{\int_{\Omega} \left(\langle A\nabla w, \nabla w \rangle + bw^2 - \lambda mw^2 \right)}{\int_{\Omega} w^2} \geq 0$$

and thus, since $w \to \int_{\Omega} bw^2$ and $w \to \int_{\Omega} mw^2$ are continuous on $H_0^1(\Omega)$, the same inequality holds for any $w \in H_0^1(\Omega) \setminus \{0\}$. Thus $\mu_{m,b}(\lambda) \geq 0$. If $\mu_{m,b}(\lambda) = 0$, then there exists $\phi \in H_0^1(\Omega)$ such that

$$\begin{cases} -\operatorname{div}(A\nabla\phi) + b\phi = \lambda m\phi \text{ in } \Omega, \\ \phi = 0 \text{ on } \partial\Omega, \\ \phi > 0 \text{ in } \Omega. \end{cases} \tag{4.33}$$

Then , $\int_{\Omega} (\langle A\nabla\phi, \nabla u \rangle + b\phi u) = \lambda \int_{\Omega} m\phi u$ and also $\int_{\Omega} (\langle A\nabla u, \nabla\phi \rangle + bu\phi) = \lambda \int_{\Omega} m\phi u + \int_{\Omega} h\phi$. Then $\int_{\Omega} h\phi = 0$, which is impossible. □

Remark 4.5. *From Proposition 3.4, it follows immediately that:*

i) If $m \geq 0$ in Ω, then $\{\lambda \in \mathbb{R} : \mu_{m,b}(\lambda) > 0\} = (-\infty, \lambda_1(m,b))$.
ii) If $m \leq 0$ in Ω, then $\{\lambda \in \mathbb{R} : \mu_{m,b}(\lambda) > 0\} = (\lambda_{-1}(m), \infty)$.
iii) $m^+ \not\equiv 0$ and $m^- \not\equiv 0$, then $\{\lambda \in \mathbb{R} : \mu_{m,b}(\lambda) > 0\} = (\lambda_{-1}(m), \lambda_1(m,b))$.

Theorem 4.3. *If $m^+ \not\equiv 0$, then*

$$\lambda_1(m,b) = \inf_{\left\{w \in H_0^1(\Omega) : \int_{\Omega} mw^2 > 0\right\}} \frac{\int_{\Omega} \left(\langle A\nabla w, \nabla w \rangle + bw^2 \right)}{\int_{\Omega} mw^2} \tag{4.34}$$

or, equivalently,

$$\lambda_1(m,b) = \inf_{w \in W_m} \int_{\Omega} \left(\langle A\nabla w, \nabla w \rangle + bw^2 \right), \tag{4.35}$$

where $W_m := \left\{ w \in H_0^1(\Omega) : \int_{\Omega} mw^2 = 1 \right\}$.

Proof. For $\lambda > 0$, from (3.14), we have $\mu_{m,b}(\lambda) = 0$ if and only if

$$\inf_{\left\{w \in H_0^1(\Omega) : \int_{\Omega} mw^2 > 0\right\}} \frac{\int_{\Omega} \left(\langle A\nabla w, \nabla w \rangle + (b - \lambda m) w^2 \right)}{\int_{\Omega} mw^2} = 0,$$

i.e., if and only if (4.34) holds. □

Remark 4.6. *From proposition 4.3, it is clear that the following three facts follow:*

i) Let $b_i : \Omega \to \mathbb{R}$, $i = 1, 2$, be nonnegative functions such that $d_\Omega^2 b_i \in L^\infty(\Omega)$, $i = 1, 2$ and let $m \in L^\infty(\Omega) \setminus \{0\}$ be such that $m^+ \not\equiv 0$. If $b_1 \le b_2$ in Ω, then $\lambda_1(m, b_1) \le \lambda_1(m, b_2)$.

ii) Let $b : \Omega \to \mathbb{R}$, $i = 1, 2$, be a nonnegative function such that $d_\Omega^2 b \in L^\infty(\Omega)$, and let $m_i : \Omega \to \mathbb{R}$, $i = 1, 2$, be functions in $L^\infty(\Omega)$ such that $m_1^+ \not\equiv 0$. If $m_1 \le m_2$ in Ω, then $\lambda_1(m_1, b) \ge \lambda_1(m_2, b)$.

iii) Let Ω_1, Ω_2 be bounded domains in $\mathbb{R}^n$ such that $\Omega_1 \subset \Omega_2$, let $m \in L^\infty(\Omega_2)$ be such that $m^+ \not\equiv 0$ in Ω_1 and let $b : \Omega_2 \to \mathbb{R}$ be a nonnegative function such that $d_{\Omega_2}^2 b \in L^\infty(\Omega_2)$. Let $\lambda_1(m, b, \Omega_i)$, $i = 1, 2$, be the positive principal eigenvalue of the operator $\mathcal{L}_0 + b$ on Ω_i with weight function m. Then $\left\{w \in H_0^1(\Omega_1) : \int_{\Omega_1} mw^2 = 1\right\} \subset \left\{w \in H_0^1(\Omega_2) : \int_{\Omega_2} mw^2 = 1\right\}$ and so $\lambda_1(m, b, \Omega_2) \le \lambda_1(m, b, \Omega_1)$.

For $\delta > 0$, we set $A_\delta := \{x \in \Omega : dist(x, \partial\Omega) < \delta\}$.

Remark 4.7. *Let $b : \Omega \to \mathbb{R}$ be a nonnegative function such that $d_\Omega^2 b \in L^\infty(\Omega)$, and let $\delta > 0$ be such that $\Omega_\delta \ne \varnothing$. If $v \in H^1(\Omega) \cap C(\overline{\Omega})$ and $\mathcal{L}_0 v + bv \ge 0$ in $D'(A_\delta)$, $v \ge 0$ on ∂A_δ then $v \ge 0$ in A_δ. Indeed, we have $v^- \in H^1(A_\delta) \cap C(\overline{\Omega})$ and $v^- = 0$ on ∂A_δ, and so $v^- \in H_0^1(A_\delta)$. Let $\{\varphi_j\}_{j \in \mathbb{N}}$ be a sequence in $C_c^\infty(A_\delta)$ such that $\{\varphi_j\}_{j\in\mathbb{N}}$ converges to v^- in $H_0^1(A_\delta)$. By replacing $\{\varphi_j\}_{j\in\mathbb{N}}$ by $\left\{\sqrt{\varphi_j^2 + \frac{1}{j^2}} - \frac{1}{j}\right\}_{j\in\mathbb{N}}$ if necessary, we can assume that each φ_j is nonnegative. Then*

$$\int_{A_\delta} \left(\langle A\nabla v^-, \nabla v^-\rangle + b\left(v^-\right)^2\right) = \lim_{j\to\infty} \int_{A_\delta} \left(\langle A\nabla v^-, \nabla \varphi_j\rangle + bv^-\varphi_j\right)$$
$$= -\lim_{j\to\infty} \int_\Omega \left(\langle A\nabla v, \nabla \varphi_j\rangle + bv\varphi_j\right) \le 0$$

and so $v^- = 0$ on A_δ.

In the case when $0 \le b \in L^\infty(\Omega)$ (and m such that $m \in L^\infty(\Omega)$ and $m^+ \not\equiv 0$), it is well known that any positive eigenfunction u associated to $\lambda_1(b, m)$ satisfies $u \approx d_\Omega$ in Ω (because $u \in C^1(\overline{\Omega})$ and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, see e.g., [5], Proposition 1.6 and the Remark immediately before it). Let us mention that, if we require only that $b \ge 0$ and $d_\Omega^2 b \in L^\infty(\Omega)$, the assertion that $u \approx d_\Omega$ in Ω may not hold, as the following example shows:

Example 4.1. *Let $\gamma_1 > 1$ and let φ_1 be a principal eigenfunction for the problem without weight $-\Delta\varphi_1 = \lambda_1\varphi_1$ in Ω, $\varphi_1 = 0$ on $\partial\Omega$, $\varphi_1 > 0$ in Ω. A computation shows that $-\Delta(\varphi_1^\gamma) = \gamma\lambda_1\varphi_1^\gamma - \gamma(\gamma-1)\varphi_1^{\gamma-2}|\nabla\varphi_1|^2$, i.e., $-\Delta(\varphi_1^\gamma) + b\varphi_1^\gamma = \gamma\lambda_1\varphi_1^\gamma$ in Ω, where $b := \gamma(\gamma-1)\varphi_1^{-2}|\nabla\varphi_1|^2$, and, since $\varphi_1 \approx d_\Omega$ in Ω and $|\nabla\varphi_1| \in L^\infty(\Omega)$, we have $b \ge 0$ and $d_\Omega^2 b \in L^\infty(\Omega)$. It is easy to see that $\varphi_1^\gamma \in H_0^1(\Omega)$ and that φ_1^γ satisfies, in weak sense, $-\Delta(\varphi_1^\gamma) + b\varphi_1^\gamma = \gamma\lambda_1\varphi_1^\gamma$ in Ω, $\varphi_1^\gamma = 0$ on $\partial\Omega$, and so φ_1^γ is a principal eigenfunction corresponding to the potential b and the weight $m = 1$, and clearly $\varphi_1^\gamma \not\approx d_\Omega$ in Ω.*

In order to prove the next theorem, we need the following elementary lemma:

Lemma 4.3. *For $\delta > 0$ such that $\Omega_\delta \ne \varnothing$, we have*

$$\{x \in \Omega : dist(x, \partial\Omega) = \delta\} \subset \overline{\Omega_{\frac{\delta}{2}}}. \tag{4.36}$$

Proof. If $x \in \Omega$ and $dist(x, \partial\Omega) = \delta$, then $dist\left(z, \partial\Omega_{\frac{\delta}{2}}\right) = \frac{\delta}{2}$ for any $z \in \partial\Omega_{\frac{\delta}{2}}$, and so there exists $p_z \in \partial\Omega$ such that $|z - p_z| = \frac{\delta}{2}$. Now,

$$|x - z| = |x - p_z - (z - p_z)| \ge |x - p_z| - |z - p_z| = |x - p_z| - \frac{\delta}{2} \ge \delta - \frac{\delta}{2} = \frac{\delta}{2}$$

then, since $z \in \partial\Omega_{\frac{\delta}{2}}$ was arbitrary, we conclude that $dist\left(x, \partial\Omega_{\frac{\delta}{2}}\right) \geq \frac{\delta}{2}$. Thus (4.36) holds. □

Theorem 4.4. *Let $b : \Omega \to \mathbb{R}$ be a nonnegative function such that $d_\Omega^2 b \in L^\infty(\Omega)$, let $m \in L^\infty(\Omega)$ such that $m \not\equiv 0$ in Ω, and let $\lambda \in \mathbb{R}$. If $u \in H_0^1(\Omega)$ is a weak solution of the problem*

$$\begin{cases} -\operatorname{div}(A\nabla u) + bu = \lambda m u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \tag{4.37}$$

then:

i) There exists a positive constant c_1 such that $u \leq c_1 d_\Omega$ in Ω.
ii) $u \in C\left(\overline{\Omega}\right)$.
iii) If, in addition, $d_\Omega^\beta b \in L^\infty(\Omega)$ for some $\beta < 2$, then for any $\gamma > 1$ there exists a positive constant c_2 such that $u \geq c_2 d_\Omega^\gamma$ in Ω.

Proof. Since $\lambda m = -\lambda(-m)$ it is enough to consider the case when $\lambda > 0$. Notice that, for $k > 0$, the equation $\mathcal{L}_0 u + bu = \lambda m u$ can be written as $\mathcal{L}_0 u + (b + \lambda k) u = \lambda (m + k) u$ and that $b + \lambda k$ satisfies the condition on b assumed in the statements of the lemma. Therefore, by taking k positive and large enough, we can assume that $m \geq 1$.

We first prove $i)$ and $ii)$. For $\delta > 0$ such that $\Omega_\delta \neq \emptyset$ let $b_\delta := b\chi_{\Omega_\delta}$. Then $0 \leq b_\delta \in L^\infty(\Omega)$ and, in weak sense, $\mathcal{L}_0 u + b_\delta u \leq \mathcal{L}_0 u + bu = \lambda m u$ in Ω. Thus

$$0 < u \leq (\mathcal{L}_0 + b_\delta)^{-1}(\lambda m u) \text{ in } \Omega. \tag{4.38}$$

If $2^* = \infty$ (i.e., if $n = 1, 2$) then $(\mathcal{L}_0 + b_\delta)^{-1}(\lambda m u) \in L^r(\Omega)$ for any $r \in [1, \infty)$ (because $\lambda m u \in L^2(\Omega)$) and thus, by (4.38), $u \in L^r(\Omega)$ for any $r \in [1, \infty)$. In particular, $\lambda m u \in L^r(\Omega)$ for some $r > n$ which implies $(\mathcal{L}_0 + b_\delta)^{-1}(\lambda m u) \in C^1\left(\overline{\Omega}\right)$. Then, by (4.38), u is continuous at $\partial\Omega$ and, since by Proposition 3.1 $i)$, $u \in C(\Omega)$ we conclude that $u \in C\left(\overline{\Omega}\right)$. Also, since $(\mathcal{L}_0 + b_\delta)^{-1}(\lambda m u) \in C^1\left(\overline{\Omega}\right)$ and $(\mathcal{L}_0 + b_\delta)^{-1}(\lambda m u) = 0$ on $\partial\Omega$, there exists a positive constant c such that $(\mathcal{L}_0 + b_\delta)^{-1}(\lambda m u) \leq c d_\Omega$ in Ω, and then, by (4.38), $u \leq c d_\Omega$ in Ω.

In the case when $2^* < \infty$, since $u \in H_0^1(\Omega)$ we have $u \in L^{2^*}(\Omega)$. Thus $\lambda m u \in L^{2^*}(\Omega)$ and then $(\mathcal{L}_0 + b_\delta)^{-1}(\lambda m u) \in L^{2^{**}}(\Omega)$ (when $2^{**} < \infty$) and thus, from (4.38), $u \in L^{2^{**}}(\Omega)$ and so $\lambda m u \in L^{2^{**}}(\Omega)$. By iterating this procedure, we get that $\lambda m u \in L^r(\Omega)$ for some $r > n$. Then $(\mathcal{L}_0 + b_\delta)^{-1}(\lambda m u) \in C^1\left(\overline{\Omega}\right)$ and thus, as above, we get that $u \in C\left(\overline{\Omega}\right)$ and that there exists a positive constant c such that $u \leq c d_\Omega$ in Ω. Thus $i)$ and $ii)$ hold.

To prove $iii)$, assume that $d_\Omega^\beta b \in L^\infty(\Omega)$ for some $\beta < 2$. Notice that if $\gamma > r$ then (since Ω is bounded) there exists a constant $c_{r,s}$ such that $d_\Omega^\gamma \leq c_{s,r} d_\Omega^r$ in Ω. Therefore it is enough to prove $iii)$ when $1 < \gamma < 2$. Consider the solution $\psi \in \cap_{1 \leq q < \infty} W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ of the problem

$$\begin{cases} \mathcal{L}_0 \psi = 1 \text{ in } \Omega, \\ \psi = 0 \text{ on } \partial\Omega. \end{cases}$$

The regularity of ψ and the Hopf's boundary lemma give that there exist $\delta > 0$ and a constant $c_3 > 0$ such that

$$\langle A\nabla\psi, \nabla\psi \rangle \geq c_3^2 \text{ in } A_\delta. \tag{4.39}$$

From this fact, the strong maximum principle and the fact that $\psi \in C^1\left(\overline{\Omega}\right)$, it follows that, for some positive constants c_4 and c_5,

$$c_4 d_\Omega < \psi \leq c_5 d_\Omega \text{ in } \Omega. \tag{4.40}$$

Let $c_6 \in (0,\infty)$ be such that $d_\Omega^\beta b < c_6$ in Ω. A computation shows that

$$\mathcal{L}_0\left(\psi^\gamma\right) + b\psi^\gamma = \gamma\psi^{\gamma-1} - \gamma\left(\gamma-1\right)\psi^{\gamma-2}\left\langle A\nabla\psi, \nabla\psi\right\rangle + b\psi^\gamma \text{ in } \Omega,$$

and so, for δ as above,

$$\begin{aligned}\mathcal{L}_0\left(\psi^\gamma\right) + b\psi^\gamma &\leq \gamma c_5^{\gamma-1} d_\Omega^{\gamma-1} - \gamma\left(\gamma-1\right) c_3^{\gamma-2} c_5^2 d_\Omega^{\gamma-2} + c_6 c_5^\gamma d_\Omega^{-\beta+\gamma} \\ &= d_\Omega^{\gamma-2}\left(-\gamma\left(\gamma-1\right) c_3^{\gamma-2} c_5^2 + \gamma c_3^{\gamma-1} d_\Omega + c_6 c_5^\gamma d_\Omega^{-\beta+2}\right)\end{aligned}$$

and thus, by diminishing δ if necessary,

$$\mathcal{L}_0\left(\psi^\gamma\right) + b\psi^\gamma \leq 0 \text{ in } A_\delta.$$

Then, for any $\varepsilon > 0$,

$$\{\mathcal{L}_0\left(u - \varepsilon\psi^\gamma\right) + b\left(u - \varepsilon\psi^\gamma\right) \geq 0 \text{ in } D'\left(A_\delta\right).$$

Let us show that, for ε small enough, $u - \varepsilon\psi^\gamma \geq 0$ on ∂A_δ. Indeed, clearly $u - \varepsilon\psi^\gamma = 0$ on $\partial\Omega$. Also, by Lemma 2.2 iii), there exists a positive constant c_7 such that

(4.41) $$u \geq c_7 d_{\Omega_{\frac{\delta}{2}}} \text{ in } \Omega_{\frac{\delta}{2}}.$$

Thus, since $u \in C\left(\overline{\Omega}\right)$ we have

(4.42) $$u \geq c_7\frac{\delta}{2} \text{ in } \overline{\Omega_{\frac{\delta}{2}}}.$$

Then, by (4.42), (4.36) and (4.40), for ε small enough (perhaps depending on δ) we have

$$\begin{aligned}u - \varepsilon\psi^\gamma &\geq c_6\frac{\delta}{2} - \varepsilon c_5^\gamma d_\Omega^\gamma \geq c_6\frac{\delta}{2} - \varepsilon c_5^\gamma \delta^\gamma \\ &= \delta\left(\frac{c_6}{2} - \varepsilon c_5^\gamma \delta^{\gamma-1}\right) > 0 \text{ in } \left\{x \in \Omega : dist\left(x, \partial\Omega\right) = \delta\right\}.\end{aligned}$$

Then, by Remark 4.7,

$$u - \varepsilon\psi^\gamma \geq 0 \text{ in } A_\delta.$$

On the other hand, since $\psi \leq M := c_5 diam\left(\Omega\right)$ in Ω, by diminishing ε if necessary we have $u - \varepsilon\psi^\gamma \geq c_6\frac{\delta}{2} - \varepsilon M^\gamma > 0$ in $\Omega_{\frac{\delta}{2}}$ and so $u - \varepsilon\psi^\gamma > 0$ in $\overline{\Omega_\delta}$). Then $u - \varepsilon\psi^\gamma \geq 0$ in Ω and the Proposition follows from (4.40). □

Let us to introduce some convenient notation. We set

$$\mathcal{B} := \left\{b : \Omega \to \mathbb{R} : d_\Omega^2 b \in L^\infty\left(\Omega\right)\right\}$$

and for $b \in \mathcal{B}$, we set $\|b\|_{\mathcal{B}} := \left\|d_\Omega^2 b\right\|_\infty$ and $\mathcal{B}^+ := \{b \in \mathcal{B} : b \geq 0\}$. Thus $(\mathcal{B}, \|b\|_{\mathcal{B}})$ is a Banach space and $\mathcal{B}^+$ is its positive cone. We set also $\mathcal{P} := \{m \in L^\infty\left(\Omega\right) : m^+ \not\equiv 0\}$.

For $m \in \mathcal{P}$ and $b \in \mathcal{B}^+$, we will write $\lambda_1\left(m, b\right)$ for the (unique) positive principal eigenvalue of problem (4.33), and we will denote by $\phi_{m,b}$ the (unique) associated positive principal eigenfunction, normalized by $\|\phi_{m,b}\|_2 = 1$.

Lemma 4.4. *Let $(m,b) \in \mathcal{P}\times\mathcal{B}^+$ and let $\{(m_j, b_j)\}_{j\in\mathbb{N}}$ be a sequence in $\mathcal{P}\times\mathcal{B}^+$ such that $\{(m_j, b_j)\}_{j\in\mathbb{N}}$ converges to (m,b) in $\mathcal{P}\times\mathcal{B}$ (with $\mathcal{P}$ endowed with the topology of the norm of $L^\infty\left(\Omega\right)$ and $\mathcal{B}^+$ endowed with the topology of the norm $\|.\|_{\mathcal{B}}$). Then:*

i) $\{\lambda_1\left(m_j, b_j\right)\}_{j\in\mathbb{N}}$ *is bounded.*

ii) $\left\{\phi_{m_j,b_j}\right\}_{j\in\mathbb{N}}$ *is bounded in* $H_0^1\left(\Omega\right)$.

Proof. To see $i)$, consider an arbitrarily chosen function $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $z > 0$ $a.e.$ in Ω. Since $\{b_j\}_{j\in\mathbb{N}}$ converges to b in $\mathcal{B}$, there exists a positive constant c such that $b_j \leq c d_\Omega^{-2}$ $a.e.$ in Ω for any $j \in \mathbb{N}$ and, by Lemma 2.1, $\int_\Omega d_\Omega^{-2} z^2 < \infty$. Then, for $j \in \mathbb{N}$,

$$\int_\Omega b_j z^2 \leq c'' \tag{4.43}$$

with c'' a positive constant independent of j. Also, taking into account that $\{m_j\}_{j\in\mathbb{N}}$ converges to m in $L^\infty(\Omega)$ and that $z^2 \in L^1(\Omega)$, the Lebesgue's dominated convergence gives $\lim_{j\to\infty} \int_\Omega m_j z^2 = \int_\Omega m z^2 > 0$. Then there exists a positive constant c''' such that, for any $j \in \mathbb{N}$,

$$\int_\Omega m_j z^2 \geq c''' \tag{4.44}$$

then $i)$ follows from (4.43), (4.44) and from the fact that

$$\lambda_1(m_j, b_j) \leq \frac{\int_\Omega \left[|\nabla z|^2 + b_j z^2\right]}{\int_\Omega m_j z^2}.$$

To prove $ii)$, observe that

$$\int_\Omega \left|\nabla \phi_{m_j,b_j}\right|^2 = \lambda_1(m_j, b_j) \int_\Omega m_j \phi_{m_j,b_j}^2 - \int_\Omega b_j \phi_{m_j,b_j}^2 \leq \lambda_1(m_j, b_j) \int_\Omega m_j \phi_{m_j,b_j}^2,$$

and so, since $\{m_j\}_{j\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, $ii)$ follows from $i)$. $\square$

Theorem 4.5. *i) The map $(m, b) \to \lambda_1(m, b)$ is continuous from $\mathcal{P} \times \mathcal{B}_+$ into $\mathbb{R}$.*
ii) The map $(m, b) \to \phi_{m,b}$ is continuous from $\mathcal{P} \times \mathcal{B}_+$ into $H_0^1(\Omega)$.

Proof. To prove the lemma, it is enough to see that if $(m, b) \in \mathcal{P} \times \mathcal{B}_+$ and if $\{(m_j, b_j)\}_{j\in\mathbb{N}}$ is a sequence in $\mathcal{P} \times \mathcal{B}_+$ which converges to (m, b) in $\mathcal{P} \times \mathcal{B}$, then there exists a subsequence $\{(m_{j_k}, b_{j_k})\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \lambda_1(m_{j_k}, b_{j_k}) = \lambda_1(m, b)$ and $\lim_{k\to\infty} \left\|\phi_{m_{j_k},b_{j_k}} - \phi_{m,b}\right\|_{H_0^1(\Omega)} = 0$. To do it, consider a pair $(m, b) \in \mathcal{P} \times \mathcal{B}_+$ and a sequence $\{(m_j, b_j)\}_{j\in\mathbb{N}} \subset \mathcal{P} \times \mathcal{B}_+$ such that $\lim_{j\to\infty} (m_j,_j) = (m, b)$ with convergence in $\mathcal{P} \times \mathcal{B}$. From Lemma 4.4 $i)$ and $ii)$, after pass to a subsequence if necessary (still denoted by $\{(m_j, b_j)\}_{j\in\mathbb{N}}$, we can assume that $\{\lambda_1(m_j, b_j)\}_{j\in\mathbb{N}}$ converges to some $\mu \in [0, \infty)$, and that there exists $\phi \in H_0^1(\Omega)$ such that $\{\phi_{m_j,b_j}\}_{j\in\mathbb{N}}$ converges to ϕ strongly in $L^2(\Omega)$ and $a.e.$ in Ω, and $\{\nabla\phi_{m_j,b_j}\}_{j\in\mathbb{N}}$ converges weakly to $\nabla\phi$ in $L^2(\Omega, \mathbb{R}^n)$. In particular, this implies $\|\phi\|_2 = 1$, and then ϕ is nonnegative (because each ϕ_{m_j,u_j} is positive) and nonidentically zero in Ω.

Let us show that $\{\phi_{m_j,b_j}\}_{j\in\mathbb{N}}$ converges to ϕ strongly in $H_0^1(\Omega)$. For $j, k \in \mathbb{N}$ we have, in weak sense,

$$\begin{aligned} \mathcal{L}_0\left(\phi_{m_j,b_j} - \phi_{m_k,b_k}\right) &= -\left(b_j \phi_{m_j,b_j} - b_k \phi_{m_k,b_k}\right) \\ &\quad + \lambda_1(m_j, b_j) m_j \phi_{m_j,b_j} - \lambda_1(m_k, b_k) m_k \phi_{m_k,b_k} \text{ in } \Omega, \\ \phi_{m_j,b_j} - \phi_{m_k,b_k} &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{4.45}$$

and so, by taking $\phi_{m_j,b_j} - \phi_{m_k,b_k}$ as a test function in (4.45), we get

$$\int_\Omega \left\langle A\nabla\left(\phi_{m_j,b_j} - \phi_{m_k,b_k}\right), \left(\phi_{m_j,b_j} - \phi_{m_k,b_k}\right)\right\rangle = I_{j,k} + II_{j,k},$$

where

$$I_{j,k} := -\int_\Omega \left(b_j \phi_{m_j,b_j} - b_k \phi_{m_k,b_k}\right)\left(\phi_{m_j,b_j} - \phi_{m_k,b_k}\right),$$
$$II_{j,k} := \int_\Omega \left(\lambda_1(m_j,b_j)\, m_j \phi_{m_j,b_j} - \lambda_1(m_k,b_k)\, m_k \phi_{m_k,b_k}\right)\left(\phi_{m_j,b_j} - \phi_{m_k,b_k}\right).$$

Now, $b_j = \beta_j d_{\Omega j}^{-2}$ in Ω, with $\beta_j \in L^\infty(\Omega)$ such that, for some positive constant c and for all $j \in \mathbb{N}$, $\|\beta_j\|_\infty \le c$. Thus

$$\begin{aligned} I_{j,k} &= -\int_\Omega (b_j - b_k)\,\phi_{m_j,b_j}\left(\phi_{m_j,b_j} - \phi_{m_k,b_k}\right) - \int_\Omega b_k \left(\phi_{m_j,b_j} - \phi_{m_k,b_k}\right)^2 \\ &\le \int_\Omega \phi_{m_j,b_j}\, |b_j - b_k|\, \left|\phi_{m_j,b_j} - \phi_{m_k,b_k}\right| \\ &= \int_\Omega \frac{\phi_{m_j,b_j}}{d_\Omega} d_\Omega^2\, |b_j - b_k| \left|\frac{\phi_{m_j,b_j} - \phi_{m_k,b_k}}{d_\Omega}\right| \\ &= \int_\Omega \frac{\phi_{m_j,b_j}}{d_\Omega}\, |\beta_j - \beta_k| \left|\frac{\phi_{m_j,u_j} - \phi_{m_k,u_k}}{d_\Omega}\right|. \end{aligned} \tag{4.46}$$

Then, by the Hardy's inequality,

$$\begin{aligned} I_{j,k} &\le c\,\|\beta_j - \beta_k\|_\infty \left\|\frac{\phi_{m_j,b_j} - \phi_{m_k,b_k}}{d_\Omega}\right\|_2 \left\|\frac{\phi_{m_j,b_j}}{d_\Omega}\right\|_2 \\ &\le c'\,\|\beta_j - \beta_k\|_\infty \left\|\phi_{m_j,b_j} - \phi_{m_k,b_k}\right\|_{H_0^1(\Omega)} \left\|\phi_{m_j,b_j}\right\|_{H_0^1(\Omega)} \\ &\le c''\varepsilon(j,k) \left\|\phi_{m_j,b_j} - \phi_{m_k,b_k}\right\|_{H_0^1(\Omega)}, \end{aligned}$$

where $\varepsilon(j,k) := \|\beta_j - \beta_k\|_\infty$ and where c, c' and c'' are positive constants independent of j and k. Therefore

$$I_{j,k} \le c''\varepsilon(j,k) \left\|\phi_{m_j,b_j} - \phi_{m_k,b_k}\right\|_{H_0^1(\Omega)}. \tag{4.47}$$

On the other hand,

$$\begin{aligned} II_{j,k} &\le \int_\Omega \left|\left(\lambda_1(m_j,b_j) - \lambda_1(m_k,b_k)\right) m_j \phi_{m_j,b_j}\left(\phi_{m_j,b_j} - \phi_{m_k,b_k}\right)\right| \\ &\quad + \int_\Omega \left|\lambda_1(m_k,b_k)(m_j - m_k)\,\phi_{m_j,b_j}\left(\phi_{m_j,b_j} - \phi_{m_k,b_k}\right)\right| \\ &\quad + \int_\Omega \lambda_1(m_k,b_k)\, m_k \left(\phi_{m_j,b_j} - \phi_{m_k,b_k}\right)\left(\phi_{m_j,b_j} - \phi_{m_k,b_k}\right) \\ &\le c'\delta(j,k) \left\|\phi_{m_j,b_j} - \phi_{m_k,b_k}\right\|_{H_0^1(\Omega)}, \end{aligned} \tag{4.48}$$

where c' is a positive constant independent of j and k and

$$\begin{aligned} \delta(j,k) &:= \left\|\left(\lambda_1(m_j,b_j) - \lambda_1(m_k,b_k)\right) m_j \phi_{m_j,b_j}\right\|_2 \\ &\quad + \left\|\lambda_1(m_k,b_k)(m_j - m_k)\,\phi_{m_j,b_j}\right\|_2 + \left\|\lambda_1(m_k,b_k)\, m_k \left(\phi_{m_j,b_j} - \phi_{m_k,b_k}\right)\right\|_2. \end{aligned}$$

Now, $\lim_{j,k\to\infty}\left(\lambda_1(m_j,b_j) - \lambda_1(m_k,b_k)\right) = 0$, $\{m_j\}_{j\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, and $\left\{\phi_{m_j,b_j}\right\}_{j\in\mathbb{N}}$ converges to ϕ in $L^2(\Omega)$. Then

$$\lim_{j,k\to\infty} \left\|\left(\lambda_1(m_j,b_j) - \lambda_1(m_k,b_k)\right) m_j \phi_{m_j,u_j}\right\|_2 = 0.$$

Also, $\{\lambda_1(m_k,b_k)\}_{k\in\mathbb{N}}$ is bounded, $\lim_{j\to\infty} m_j = m$ with convergence in $L^\infty(\Omega)$, and $\{\phi_{m_j,u_j}\}_{j\in\mathbb{N}}$ is bounded in $L^2(\Omega)$. Thus

$$\lim_{j,k\to\infty} \left\|\lambda_1(m_k,b_k)(m_j-m_k)\phi_{m_j,b_j}\right\|_2 = 0,$$

and, since $\{\lambda_1(m_k,b_k)\}_{k\in\mathbb{N}}$ and $\{m_k\}_{k\in\mathbb{N}}$ are bounded in $\mathbb{R}$ and $L^\infty(\Omega)$ respectively, and $\{\phi_{m_j,b_j}\}_{j\in\mathbb{N}}$ converges to ϕ in $L^2(\Omega)$, we have

$$\lim_{j,k\to\infty} \left\|\lambda_1(m_k,b_k)m_k\left(\phi_{m_j,b_j}-\phi_{m_k,b_k}\right)\right\|_2 = 0.$$

Then $\lim_{j,k\to\infty}\delta(j,k)=0$ and, since $\{b_j\}_{j\in\mathbb{N}}$ converges to b in $\mathcal{B}$, we have also that $\lim_{j,k\to\infty}\varepsilon(j,k) = 0$. Now,

$$\begin{aligned}
&\left\|\phi_{m_j,b_j}-\phi_{m_k,b_k}\right\|^2_{H_0^1(\Omega)} \\
&= I_{j,k} + II_{j,k} \\
&\le c\varepsilon_{j,k}\left\|\phi_{m_j,b_j}-\phi_{m_k,b_k}\right\|_{H_0^1(\Omega)} + c'\delta_{j,k}\left\|\phi_{m_j,b_j}-\phi_{m_k,b_k}\right\|_{H_0^1(\Omega)}
\end{aligned}$$

and so

$$\lim_{j,k\to\infty}\left\|\phi_{m_j,b_j}-\phi_{m_k,b_k}\right\|_{H_0^1(\Omega)} = 0.$$

Thus $\{\phi_{m_j,b_j}\}_{j\in\mathbb{N}}$ converges in $H_0^1(\Omega)$ to some $\widetilde{\phi}$. Since ϕ_{m_j,b_j} converges $a.e.$ in Ω to ϕ, we conclude that $\widetilde{\phi}=\phi$. Therefore,

$$\{\phi_{m_j,b_j}\}_{j\in\mathbb{N}} \text{ converges to } \phi \text{ in } H_0^1(\Omega). \tag{4.49}$$

To complete the proof of the lemma, it only remains to see that $\mu=\lambda_1(m,b)$ and $\phi=\phi_{m,b}$. For $\varphi\in H_0^1(\Omega)$ and $j\in\mathbb{N}$, we have

$$\int_\Omega\left(\left\langle A\nabla\phi_{m_j,b_j},\nabla\varphi\right\rangle + b_j\phi_{m_j,b_j}\varphi\right) = \lambda_1(m_j,b_j)\int_\Omega m_j\phi_{m_j,b_j}\varphi, \tag{4.50}$$

and, by (4.49), $\lim_{j\to\infty}\int_\Omega\left\langle\nabla\phi_{m_j,b_j},\nabla\varphi\right\rangle = \int_\Omega\langle\nabla\phi,\nabla\varphi\rangle$. Also, $b_j\phi_{m_j,b_j}\varphi$ converges to $b\phi\varphi$ $a.e.$ in Ω and, by Lemma 4.4 i), we have

$$|b_j\phi\varphi| \le c d_\Omega^{-2}\phi|\varphi|$$

with c a positive constant independent of j and, by Lemma 2.1, $d_\Omega^{-2}\phi|\varphi|\in L^1(\Omega)$. Thus, by the Lebesgue's dominated convergence theorem,

$$\lim_{j\to\infty}\int_\Omega b_j\phi_{m_j,b_j}\varphi = \int_\Omega b\phi\varphi.$$

Also, since $\lim_{j\to\infty}\lambda_1(m_j,b_j)=\mu$, $\lim_{j\to\infty}m_j=m$ with convergence in $L^\infty(\Omega)$, and $\lim_{j\to\infty}\phi_{m_j,b_j} = \phi$ with convergence in $H_0^1(\Omega)$, we have

$$\lim_{j\to\infty}\lambda_1(m_j,b_j)\int_\Omega m_j\phi_{m_j,b_j}\varphi = \mu\int_\Omega m\phi\varphi.$$

Then, from (4.50),

$$\int_\Omega\left(\langle A\nabla\phi,\nabla\varphi\rangle + b\phi\varphi\right) = \mu\int_\Omega m\phi\varphi$$

and so $\mu=\lambda_1(m,b)$ and $\phi=\phi_{m,b}$. $\square$

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