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Weierstrass Representation of Lightlike Surfaces in Lorentz-Minkowski 4-Space

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

We present a Weierstrass-type representation formula which locally represents every regular two-dimensional lightlike surface in Lorentz-Minkowski 4-Space $\mathbb{M}^4$ by three dual functions (ρ, f, g) and generalizes the representation for regular lightlike surfaces in $\mathbb{M}^3$. We give necessary and sufficient conditions on the functions ρ, f, g for the surface to be minimal, ruled or l-minimal. For ruled lightlike surfaces, we give necessary and sufficient conditions for the representation itself to be ruled. Furthermore, we give a result on totally geodesic half-lightlike surfaces which holds only in $\mathbb{M}^4$.

Keywords: *Weierstrass representation, lightlike surface, minimal surface, conformal parametrization, ruled surface, Lorentz-Minkowski 4-Space.*
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1. Introduction

The first Weierstrass-type representation was developed by K. Weierstrass and A. Enneper. It is a local conformal parametrization of a surface in Euclidean space $\mathbb{E}^3$ which initially represented only minimal surfaces, that is, surfaces of mean curvature $H = 0$. Every such surface is represented by a pair (f, g) of functions, where f is a holomorphic and g a meromorphic complex function. The representation was later generalized to regular two-dimensional surfaces (not necessarily minimal) in various ambient spaces.

The Weierstrass representation of maximal surfaces (spacelike surfaces with $H = 0$) in the Lorentz-Minkowski space $\mathbb{M}^3$ was found by L. McNertney ([14]). As in $\mathbb{E}^3$, they are represented by a holomorphic and a meromorphic complex function f and g of a complex variable $z = u + vi$. Its counterpart for minimal surfaces (timelike surfaces with $H = 0$) in $\mathbb{M}^3$ was found by M. Magid ([13]). Minimal surfaces are represented by four real-valued functions (q, f, r, g), which depend on only one of their two real variables (parameters) u and v. Each of these representations is generalized to any regular surface by using complex functions (with C^∞ real and imaginary part) which are not necessarily holomorphic or meromorphic (spacelike case) or by using any C^∞ real functions of two variables (timelike case).

The representation of spacelike surfaces was generalized to surfaces in $\mathbb{M}^4$ by H. Liu ([12]). It represents every maximal surface in $\mathbb{M}^4$ by three complex functions (ρ, f, g). The surface is maximal if and only if at least one of the functions f and g is holomorphic. The representation of timelike surfaces in $\mathbb{M}^4$, which generalizes the three-dimensional result, was found by one of the authors ([2]). It uses real functions (f, g) and complex functions (q, r) of two real variables to represent every surface.

The representation for lightlike surfaces in $\mathbb{M}^3$ was not known until recently, when the authors found it by using dual functions f and g ([3]). In this paper, this formula will be generalized to lightlike surfaces in $\mathbb{M}^4$ in the analogous way as for spacelike and timelike surfaces.

Finally, let us mention that, B. Konopelchenko and G. Landolfi have found a different form of a Weierstrass-type parametrization for Euclidean surfaces that represents a surface by a pair of functions (φ, ψ) that satisfy

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the so-called Dirac system of linear first-order differential equations. They have generalized this result to two-dimensional surfaces in $\mathbb{E}^n$ and also found its counterpart for spacelike surfaces in $\mathbb{M}^3$ and $\mathbb{M}^4$ ([10, 9]). The counterpart for timelike surfaces was found by S. Lee in $\mathbb{M}^3$ ([11]) and was generalized to surfaces in $\mathbb{M}^4$ by one of the authors ([2]). A counterpart of this representation for lightlike surfaces does not exist ([3]).

2. Preliminaries

2.1. Half-lightlike surfaces

Let us recall the local theory of half-lightlike surfaces from Duggal's book [6] in short. The Lorentz-Minkowski space $\mathbb{M}^n$ is the real vector space $\mathbb{R}^n$ equipped with the pseudoscalar product $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, given by

$$x \cdot y := -x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

A surface S in $\mathbb{M}^n$ is said to be spacelike, timelike or lightlike at the point $p \in S$ if its induced metric in the tangent space T_pS is positive definite, indefinite or degenerate respectively.

Let S be a regular $(n-2)$-dimensional lightlike surface in $\mathbb{M}^n$, $n \geq 4$, and $p \in S$ any point. The vector space

$$Rad\, T_pS := T_pS \cap T_pS^{\perp} \neq \{0\}$$

is called the radical space of the surface S at the point p. We say that the surface S is half-lightlike at p if $\dim Rad\, T_pS = 1$. In $\mathbb{M}^n$, every $(n-2)$-dimensional surface is half-lightlike since $\dim Rad\, T_pS = 2$ can never occur (due to the Minkowski pseudometric). Surfaces with this property exist in pseudo-Riemannian spaces of index greater than 1 and they are called coisotropic surfaces. More on theory of coisotropic surfaces can be read in [6].

Notice that two-dimensional lightlike surfaces in $\mathbb{M}^4$ are half-lightlike. For half-lightlike surfaces we can introduce an induced connection and the second fundamental form. The fundamental result from [6] states that for every point p there exists a spacelike subspace $S(T_pS) \subseteq T_pS$ of dimension $n-3$, which is not unique, such that

$$S(T_pS) \oplus_{orth} Rad\, T_pS = T_pS.$$

The vector bundle $S(TS) = \cup_{p \in S} S(T_pS)$ is called the screen distribution of the surface S. Furthermore, locally there exist vector fields U and ξ on the surface S such that $U \cdot U = 1$, $U \cdot \xi = 0$, $\xi \cdot \xi = 0$ and $\xi \cdot X = 0$ for every vector field X tangential to S. The field ξ induces the radical bundle $Rad\, TS$, and it is unique up to a multiplication by a scalar non-zero function. Let us define $D_p = [U(p)]$ and $D_p^{\perp}$ as its orthogonal complement in the subspace $S(T_pS)^{\perp}$, that is

$$D_p^{\perp} = \{v \in S(T_pS) : v \cdot U(p) = 0\}.$$

The space $D_p^{\perp}$ is non-degenerate with respect to the Minkowski pseudometric and $\xi(p) \in D_p^{\perp}$. Finally, there exists a unique vector field N such that $N(p) \in D_p^{\perp}$ and

$$N \cdot \xi = 1, \quad N \cdot N = N \cdot U = 0,$$

(see [6] for details). The field N induces the lightlike transversal bundle $ltr(TS)$ by $ltr(T_pS) := [N(p)]$. If we choose another $\xi^* = \alpha\xi$, then it will be $N^* = \frac{1}{\alpha}N$, so $ltr(TS)$ depends only on the choice of U. Now we define the transversal bundle $tr(TS)$ by $tr(T_pS) := D_p \oplus_{orth} ltr(T_pS)$. In this way we obtain the following decomposition

$$T_p\mathbb{M}^n = (S(T_pS) \oplus_{orth} Rad\, T_pS) \oplus (D_p \oplus_{orth} ltr(T_pS)) = T_pS \oplus tr(T_pS).$$

The space $tr(T_pS)$ plays the role of a normal space of the surface S at p, while U and N are the spacelike unit and the lightlike normal vector fields of S. Let now $\tilde{\nabla}$ be the Levi-Civita connection of $\mathbb{M}^n$. Then with respect to the above decomposition, every vector field $\tilde{\nabla}_X Y$ has a unique tangential and normal component, so the following Gauss-Weingarten equation holds

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y). \tag{2.1}$$

The mapping ∇ is the induced connection on the surface S and the bilinear map h is the second fundamental form of the surface. The connection $\tilde{\nabla}$ is a metric connection, which means that

$$\tilde{\nabla}_X (Y \cdot Z) = (\tilde{\nabla}_X Y) \cdot Z + Y \cdot (\tilde{\nabla}_X Z) \tag{2.2}$$

for any vector fields X, Y and Z. This is not true in general for the induced connection ∇.

2.2. *Minimal lightlike surfaces*

The mean curvature H cannot be defined for lightlike surfaces in the same way as it is defined for spacelike and timelike surfaces. Therefore, to define minimal lightlike surfaces, a different approach is used ([7]).

Let S and $\tilde{S}$ be any regular surfaces in $\mathbb{M}^n$. A smooth bijective map $\varphi : S \to \tilde{S}$ is called a G-transformation if the planes T_pS and $T_{\varphi(p)}\tilde{S}$ are parallel for every $p \in S$. A trivial example of a G-transformation is a translation. A G-transformation is non-trivial if it is not a translation. For spacelike resp. timelike surfaces, in [14] it was proved that $H = 0$ if and only if there exists a one-parameter family $(S_\theta)_{\theta\in\mathbb{R}}$ of surfaces such that

1. $S_0 = S$,
2. for every $\theta \in \mathbb{R}$, there exists a locally isometric G-transformation $\varphi_\theta : S \to S_\theta$,
3. the G-transformations φ_θ and $\varphi_\beta \circ \varphi_\alpha^{-1}$ are non-trivial for all $\theta, \alpha, \beta \in \mathbb{R}$.

The surfaces S_θ are called the associated surfaces of S, and they are all maximal resp. minimal.

This characterization was used as a definition of a minimal lightlike surface in [7]. It was further proved that locally there are two disjoint classes of lightlike surfaces which satisfy the definition: ruled surfaces with lightlike rulings and the so-called l-minimal surfaces.

A lightlike surface S is a ruled surface if it can be parametrized by a map $\mathbf{x} : I \times \mathbb{R} \to S$ of form

$$\mathbf{x}(u, v) = c(u) + ve(u), \tag{2.3}$$

where c is a regular spacelike curve and $e \neq 0$ a lightlike vector field along c such that $c' \cdot e = 0$.

A lightlike surface S is l-minimal if it can be parametrized by a map $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$, $\mathbf{x}(u, v)$ such that

1. the curves $u = const.$ are lightlike and $\mathbf{x}_u \cdot \mathbf{x}_v = 0$,
2. the fields $\mathbf{x}_v$ and $\mathbf{x}_{vv}$ are linearly independent for all $(u, v) \in U$,
3. the fields $\mathbf{x}_v$ and $\mathbf{x}_{uv}$ are linearly dependent for all $(u, v) \in U$.

In $\mathbb{M}^3$, every regular lightlike surface is ruled (and therefore, minimal). For $n \geq 4$, there exist both classes of surfaces in $\mathbb{M}^n$ and also lightlike surfaces which are not minimal at all.

Let now S be a regular lightlike surface in $\mathbb{M}^n$ and $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ any parametrization. The functions

$$E := \mathbf{x}_u \cdot \mathbf{x}_u, \quad F := \mathbf{x}_u \cdot \mathbf{x}_v, \quad G := \mathbf{x}_v \cdot \mathbf{x}_v$$

are called the metric coefficients of $\mathbf{x}$. We say that the parametrization $\mathbf{x}$ is conformal if

$$E = \lambda > 0, \quad F = G = 0.$$

The function λ is called the conformal factor of $\mathbf{x}$. Every lightlike surface can be locally parametrized by a conformal map. The proof is similar to the proof of existence of an orthogonal parametrization in Euclidean space (satisfying $F = 0$), ([4]). Notice that the parametrizations in the definition of a ruled lightlike surface and an l-minimal surface are also conformal.

Spacelike and timelike surfaces have different conformality conditions. In the spacelike case, if $\mathbf{x}(u, v)$ is a conformal parametrization and we reparametrize it by complex variables $z = u + vi$ and $\overline{z} = u - vi$, the reparametrization has metric coefficients

$$E = G = 0, \quad F = \frac{\lambda}{2}.$$

The same happens for timelike surfaces when we reparametrize by real variables $s = -u + v$ and $t = u + v$. The above metric coefficients must hold for every Weierstrass parametrization. It was shown in [3] that to obtain the above metric coefficients for a lightlike surface, its conformal parametrization must be reparametrized by dual variables.

2.3. *Dual numbers and functions*

A dual number is a number of form $z = x + y\varepsilon$, where $x, y \in \mathbb{R}$ and $\varepsilon \notin \mathbb{R}$ is an imaginary unit such that $\varepsilon^2 = 0$. The set $\mathbb{D}$ of all dual numbers with standard addition and multiplication is a ring, but not a field (as opposed to $\mathbb{C}$). A dual function is any function $f : U \subseteq \mathbb{D} \to \mathbb{D}$. The theory of holomorphic dual functions is given in [15], and here we will recall some of its main parts.

Theorem 2.1. *A dual function $f = f_1 + \varepsilon f_2 : U \to \mathbb{D}$, where $U \subseteq \mathbb{D} \equiv \mathbb{R}^2$ is an open set, is holomorphic if and only if f_1 and f_2 are differentiable real functions which satisfy*

$$\partial_y f_1 = 0, \;\; \partial_y f_2 = \partial_x f_1.$$

If f is holomorphic, then the derivative with respect to the variable $z = x + y\varepsilon$ is given by

$$f' = \partial_x f = \partial_y f_2 + \varepsilon \partial_x f_2. \tag{2.4}$$

We say that f is meromorphic if there exist holomorphic dual functions g and h such that $f = \frac{g}{h}$. For a dual function $h = h_1 + h_2\varepsilon : [a, b] \subseteq \mathbb{R} \to \mathbb{D}$, such that h_1 and h_2 are integrable functions, we define

$$\int_a^b h(t)\, dt := \int_a^b h_1(t)\, dt + \varepsilon \int_a^b h_2(t)\, dt.$$

Now if $f = f_1 + \varepsilon f_2 : U \subseteq \mathbb{D} \to \mathbb{D}$ is a continuous dual function and $\gamma : [a, b] \to U$ a path of class C^1, we define

$$\int_\gamma f(z)\, dz := \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

Then we have the following version of the Newton-Leibniz formula ([15]):

Theorem 2.2. *If f has a primitive function F (that is, such that $F' = f$), then*

$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a)).$$

If S is a regular lightlike surface in $\mathbb{M}^n$ and $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ a conformal parametrization ($E = \lambda$, $F = G = 0$), then for the reparametrization $\mathbf{x}(z, \overline{z})$, where $z = u + v\varepsilon$, the first fundamental form is

$$\lambda\, dz d\overline{z} = \lambda\, (du + \varepsilon\, dv)(du - \varepsilon\, dv) = \lambda\, du^2 - \lambda\varepsilon^2\, dv^2 = E\, du^2 + 2F\, dudv + G\, dv^2 = ds^2.$$

From this we read that the metric coefficients of the reparametrization are

$$E(z, \overline{z}) = G(z, \overline{z}) = 0, \;\; F(z, \overline{z}) = \frac{\lambda}{2}.$$

This leads to the conclusion that the functions representing a lightlike surface should be dual. In [3], the following Weierstrass-type representation of lightlike surfaces in $\mathbb{M}^3$ was derived, in a similar way as it was done for spacelike surfaces ([14]):

Theorem 2.3. *Let S be a regular lightlike surface in $\mathbb{M}^3$. Then S can be locally parametrized by a conformal map $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$, given by*

$$\begin{aligned} x_1(u, v) &= \frac{1}{2} \operatorname{Im} \int_{z_0}^{z} f(1 + g^2)\, dw \\ x_2(u, v) &= -\frac{1}{2} \operatorname{Im} \int_{z_0}^{z} f(1 - g^2)\, dw \\ x_3(u, v) &= \operatorname{Im} \int_{z_0}^{z} fg\, dw, \end{aligned} \tag{2.5}$$

where $z_0 \in U$, $z = u + v\varepsilon$ and $f, g : U \subseteq \mathbb{D} \to \mathbb{D}$ are dual functions such that f is holomorphic, g is meromorphic and fg^2 is holomorphic.

3. Weierstrass representation of lightlike surfaces in $\mathbb{M}^4$

In [12], a Weierstrass representation describing spacelike surface in $\mathbb{M}^4$ by three complex functions ρ, f and g was given. This result generalizes the representation of spacelike surfaces $\mathbb{M}^3$ given in [14] (which represents the surface by two complex functions f and g) in the way that when we interchange f and g and then substitute

$$\rho = \frac{\tilde{f}\tilde{g}}{2}, \;\; f = \frac{1}{\tilde{g}}, \;\; g = \tilde{g},$$

the expressions for x_1, x_2, x_3 reduce to the formula in $\mathbb{M}^3$ with Weierstrass data $(\tilde{f}, \tilde{g})$ and $x_4 = const.$

Since the representation (2.5) for lightlike surfaces in $\mathbb{M}^3$ was derived in a similar way as for spacelike surfaces, we will generalize it in such a way that lightlike surfaces in $\mathbb{M}^4$ will be represented by three dual functions (ρ, f, g).

Let S be a lightlike surface in $\mathbb{M}^4$ and $p \in S$. Then there exists a conformal parametrization $\mathbf{x} = (x_1, x_2, x_3, x_4) : U \subseteq \mathbb{R}^2 \to S$ such that $p \in \mathbf{x}(U)$. Let $z = u + v\varepsilon$ and $f_k : U \subseteq \mathbb{D} \to \mathbb{D}$ be functions given by

$$f_k := \int \partial_v x_k \, du + \varepsilon x_k.$$

It follows from (2.4) that we can use $\partial_z := \partial_u$ as the generalization of the derivative for dual functions which are not holomorphic (but only have a smooth real and imaginary part). Then

$$\partial_z f_k = \partial_v x_k + \varepsilon \partial_u x_k. \tag{3.1}$$

Notice that (3.1) is a generalization of (2.4) for non-holomorphic functions. Now, since the parametrization $\mathbf{x}$ is conformal, it follows that

$$\begin{aligned} -(\partial_z f_1)^2 + (\partial_z f_2)^2 + (\partial_z f_3)^2 + (\partial_z f_4)^2 &= [-(\partial_v x_1)^2 + (\partial_v x_2)^2 + (\partial_v x_3)^2 + (\partial_v x_4)^2] \\ &+ \varepsilon^2[-(\partial_u x_1)^2 + (\partial_u x_2)^2 + (\partial_u x_3)^2 + (\partial_u x_4)^2] \\ &+ 2\varepsilon[-(\partial_u x_1)(\partial_v x_1) + (\partial_u x_2)(\partial_v x_2) + (\partial_u x_3)(\partial_v x_3) + (\partial_u x_4)(\partial_v x_4)] \\ &= G + \varepsilon^2 \cdot E + 2\varepsilon \cdot F = 0 + 0 \cdot E + 2\varepsilon \cdot 0 = 0 \end{aligned}$$

From this it follows that

$$(\partial_z f_3)^2 + (\partial_z f_4)^2 = (\partial_z f_1 - \partial_z f_2)(\partial_z f_1 + \partial_z f_2). \tag{3.2}$$

Let us now define functions $\rho, f, g : U \subseteq \mathbb{D} \to \mathbb{D}$ by

$$\rho := \frac{\partial_z f_1 - \partial_z f_2}{2}, \quad f := \frac{\partial_z f_3 + \partial_z f_4}{2\sqrt{2}\rho}, \quad g := \frac{\partial_z f_3 - \partial_z f_4}{2\sqrt{2}\rho}. \tag{3.3}$$

In the case when ρ is a purely imaginary function, instead of previous f, g, we define

$$f := \frac{\operatorname{Im}(\partial_z f_3 + \partial_z f_4)}{2\sqrt{2}\operatorname{Im}\rho}, \quad g := \frac{\operatorname{Im}(\partial_z f_3 - \partial_z f_4)}{2\sqrt{2}\operatorname{Im}\rho},$$

because imaginary numbers are not invertible in $\mathbb{D}$. Now we have

$$\begin{aligned} \partial_z f_1 - \partial_z f_2 &= 2\rho \\ \partial_z f_3 + \partial_z f_4 &= 2\sqrt{2}\rho f \\ \partial_z f_3 - \partial_z f_4 &= 2\sqrt{2}\rho g. \end{aligned}$$

From the last two equations we obtain

$$\begin{aligned} \partial_z f_3 &= \sqrt{2}\rho(f + g) \\ \partial_z f_4 &= \sqrt{2}\rho(f - g). \end{aligned} \tag{3.4}$$

Therefore,

$$(\partial_z f_3)^2 + (\partial_z f_4)^2 = 2\rho^2(f + g)^2 + 2\rho^2(f - g)^2 = 4\rho^2(f^2 + g^2).$$

Now from (3.2) it follows

$$\partial_z f_1 + \partial_z f_2 = \frac{4\rho^2(f^2 + g^2)}{2\rho} = 2\rho(f^2 + g^2).$$

And finally we obtain

$$\begin{aligned} \partial_z f_1 &= \rho(1 + f^2 + g^2) \\ \partial_z f_2 &= -\rho(1 - f^2 - g^2). \end{aligned} \tag{3.5}$$

By integrating both sides in (3.4) and (3.5) and applying Theorem 2.3, we obtain the following representation for lightlike surfaces, which is the central theorem of this paper.

Theorem 3.1. *Let S be a regular lightlike surface in $\mathbb{M}^4$. Then S can be locally parametrized by a conformal map $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$, given by*

$$\begin{aligned} x_1(u,v) &= \operatorname{Im} \int \rho(1+f^2+g^2)\,dz \\ x_2(u,v) &= -\operatorname{Im} \int \rho(1-f^2-g^2)\,dz \\ x_3(u,v) &= \sqrt{2}\operatorname{Im} \int \rho(f+g)\,dz \\ x_4(u,v) &= \sqrt{2}\operatorname{Im} \int \rho(f-g)\,dz, \end{aligned} \tag{3.6}$$

where $z = u + v\varepsilon$ and $\rho, f, g : U \subseteq \mathbb{D} \to \mathbb{D}$ are dual functions such that their real and imaginary part is smooth.

Remark 3.1. The parametrization (3.6) is similar to the representation of spacelike surfaces given in [12], it only uses polynomials $1 \pm (f^2 + g^2)$ instead of $1 \pm fg$. This is because the expression $(\partial_z f_3)^2 + (\partial_z f_4)^2$ cannot be factorized into linear factors over $\mathbb{D}$ (while over $\mathbb{C}$ it can).

Remark 3.2. If we substitute

$$\rho = \frac{\tilde{f}}{2}, \quad f = g = \frac{\tilde{g}}{\sqrt{2}}$$

into (3.6), we obtain (2.5) with Weierstrass data $(\tilde{f}, \tilde{g})$ and $x_4 = const$. This shows that the representation (3.6) generalizes the formula (2.5) for surfaces in $\mathbb{M}^4$, in the same way as the analogous formula does for spacelike or timelike surfaces.

Remark 3.3. The real variables u and v cannot be expressed as linear combinations of z and $\overline{z}$ because $v\varepsilon = \dfrac{z-\overline{z}}{2}$ and the imaginary unit ε is not invertible in $\mathbb{D}$ (as opposed to $\mathbb{C}$). However, if $\mathbf{x}$ is a ruled parametrization of form (2.3), the dual variables can be substituted explicitly in the following way:

$$\mathbf{x}(u,v) = \operatorname{Im}\left(\varepsilon c(u) + \varepsilon v e(u)\right) = \operatorname{Im}\left(\varepsilon c\left(\frac{z+\overline{z}}{2}\right) + \varepsilon \frac{z-\overline{z}}{2} e\left(\frac{z+\overline{z}}{2}\right)\right).$$

Remark 3.4. In [2], complex functions of two real variables are used to represent a timelike surface because the expression $(\partial_z x_3)^3 + (\partial_z x_4)^2$ cannot be factorized over $\mathbb{R}$ into linear factors. On the other hand, identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the domain would not provide the analogy with the spacelike case because in the timelike case, a complex reparametrization does not result in metric coefficients $E = G = 0$ and $F = \dfrac{\lambda}{2}$. However, in the lightlike case, we cannot do analogously, factorize over $\mathbb{C}$ and use functions $U \subseteq \mathbb{D} \to \mathbb{C}$, because the product $i \cdot \varepsilon$ is not defined, not even in the set $\{a + bi + c\varepsilon : a, b, c \in \mathbb{R}\}$ (in other words, this set is not a ring). On the other hand, the representation for lightlike surfaces in $\mathbb{M}^3$ ([3]) was constructed in a similar way as for spacelike surfaces ([14]), which suggests that the analogy with the spacelike case should continue in $\mathbb{M}^4$. By this analogy, the domain and the codomain of the functions representing the surface should be the same ring.

4. Ruled lightlike surfaces and l-minimal surface

4.1. Minimal lightlike surfaces

It is easy to see for the classical representation formula given in [14], that complex functions f and g that are not holomorphic, but have a smooth real and imaginary part, represent all regular spacelike surfaces. Furthermore, the surface will be maximal if and only if f is holomorphic, g meromorphic and fg^2 holomorphic. The same result holds for (minimal) lightlike surfaces in $\mathbb{M}^3$, ([3]).

The parametrization given in [12] represents all spacelike surfaces in $\mathbb{M}^4$ by three complex functions (ρ, f, g). The surface is maximal if and only if at least one of the functions f and g is holomorphic. In the timelike case, both in $\mathbb{M}^3$ and $\mathbb{M}^4$, any timelike surface can represented by four functions (q, f, r, g) and it is minimal if and only if $f_v = g_u = q_v = r_u = 0$.

Since in $\mathbb{M}^4$ there exit lightlike surfaces which are not minimal, our aim is to find necessary and sufficient conditions on the functions (ρ, f, g) such that the lightlike surface given by (3.6) is minimal. We require the following lemma, which was proved in [7].

Lemma 4.1. *Let S be a regular lightlike surface in $\mathbb{M}^n$ and $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ a conformal map. Then S does not have an associate family of surfaces if and only if the vector fields $\mathbf{x}_v$, $\mathbf{x}_{vv}$ and $\mathbf{x}_{uv}$ are linearly independent.*

This gives us two possibilities for a minimal lightlike surface. The first is that the vector fields $\mathbf{x}_v$ and $\mathbf{x}_{vv}$ are collinear, which describes the class of ruled lightlike surfaces. The second possibility is that $\mathbf{x}_v$ and $\mathbf{x}_{vv}$ are linearly independent and $\mathbf{x}_{uv}$ is their linear combination, which gives exactly the class of l-minimal surfaces ([7]).

From (3.1) it follows that $\partial_v x_k = \operatorname{Re}\partial_z f_k$ for $k \in \{1,2,3,4\}$. Therefore,

$$\mathbf{x}_v = (\operatorname{Re}(1+f^2+g^2), \operatorname{Re}(-1+f^2+g^2), \sqrt{2}\operatorname{Re}\rho(f+g), \sqrt{2}\operatorname{Re}\rho(f-g)).$$

Let now $\rho := \rho_1 + \varepsilon\rho_2$, $f := f_1 + \varepsilon f_2$ and $g := g_1 + \varepsilon g_2$ (the functions f_1 and f_2 are not the same as above, which will not be used further). Since dual numbers satisfy $\operatorname{Re}(z_1 z_2) = (\operatorname{Re} z_1)(\operatorname{Re} z_2)$, we have

$$\mathbf{x}_v = \rho_1(1+f_1^2+g_1^2, -1+f_1^2+g_1^2, \sqrt{2}(f_1+g_1), \sqrt{2}(f_1-g_1)). \tag{4.1}$$

If we differentiate both sides with respect to v, we get

$$\begin{aligned}\mathbf{x}_{vv} &= (\partial_v\rho_1)(1+f_1^2+g_1^2, -1+f_1^2+g_1^2, \sqrt{2}(f_1+g_1), \sqrt{2}(f_1-g_1))\\ &\quad + \rho_1(2(f_1\partial_v f_1 + g_1\partial_v g_1), 2(f_1\partial_v f_1 + g_1\partial_v g_1), \sqrt{2}(\partial_v f_1 + \partial_v g_1), \sqrt{2}(\partial_v f_1 - \partial_v g_1)).\end{aligned}$$

Since S is regular, it must be $\mathbf{x}_v \neq 0$. Furthermore, S is a ruled surface if and only if $\mathbf{x}_{vv} = \alpha\mathbf{x}_v$ for some scalar function α. Comparing the above expressions for $\mathbf{x}_v$ and $\mathbf{x}_{vv}$, we see that this will hold if and only if

$$\rho_1(\partial_v f_1)(2f_1, 2f_1, \sqrt{2}, \sqrt{2}) + \rho_1(\partial_v g_1)(2g_1, 2g_1, \sqrt{2}, -\sqrt{2}) = 0.$$

It must be $\rho_1 \neq 0$ because S is regular. Now since the above two vectors are linearly independent, it follows that S is ruled if and only if

$$\partial_v f_1 = \partial_v g_1 = 0.$$

Suppose now that $\partial_v f_1 \neq 0$ or $\partial_v g_1 \neq 0$. We want to find necessary and sufficient conditions such that $\mathbf{x}_{uv} = \alpha\mathbf{x}_v + \beta\mathbf{x}_{vv}$ for some scalar functions α and β. The vector fields $\mathbf{x}_v$ and $\mathbf{x}_{vv}$ can be further decomposed as

$$\begin{aligned}\mathbf{x}_v &= \rho_1(1,-1,0,0) + \rho_1 f_1(f_1, f_1, \sqrt{2}, \sqrt{2}) + \rho_1 g_1(g_1, g_1, \sqrt{2}, -\sqrt{2})\\ \mathbf{x}_{vv} &= (\partial_v\rho_1)(1,-1,0,0) + (\partial_v\rho_1)f_1(f_1, f_1, \sqrt{2}, \sqrt{2}) + (\partial_v\rho_1)g_1(g_1, g_1, \sqrt{2}, -\sqrt{2})\\ &\quad + \rho_1(\partial_v f_1)(2f_1, 2f_1, \sqrt{2}, \sqrt{2}) + \rho_1(\partial_v g_1)(2g_1, 2g_1, \sqrt{2}, -\sqrt{2})\\ &= (\partial_v\rho_1)(1,-1,0,0) + (\partial_v(\rho_1 f_1))(f_1, f_1, \sqrt{2}, \sqrt{2}) + (\partial_v(\rho_1 g_1))(g_1, g_1, \sqrt{2}, -\sqrt{2})\\ &\quad + \frac{\rho_1}{2}(\partial_v(f_1^2+g_1^2))(1,1,0,0).\end{aligned} \tag{4.2}$$

Notice that the four vectors which appear in the above decomposition of $\mathbf{x}_{vv}$ are linearly independent, that is, they make a basis for $\mathbb{M}^4$. Now if α and β are any scalar functions, we have that

$$\begin{aligned}\alpha\mathbf{x}_v + \beta\mathbf{x}_{vv} &= (\alpha\rho_1 + \beta\partial_v\rho_1)(1,-1,0,0) + (\alpha\rho_1 f_1 + \beta\partial_v(\rho_1 f_1))(f_1, f_1, \sqrt{2}, \sqrt{2})\\ &\quad + (\alpha\rho_1 g_1 + \beta\partial_v(\rho_1 g_1))(g_1, g_1, \sqrt{2}, -\sqrt{2}) + \frac{\beta\rho_1}{2}(\partial_v(f_1^2+g_1^2))(1,1,0,0).\end{aligned}$$

In the same way, differentiating both sides of (4.1) with respect to u and further decomposing, we get

$$\begin{aligned}\mathbf{x}_{uv} &= (\partial_u\rho_1)(1,-1,0,0) + (\partial_u(\rho_1 f_1))(f_1, f_1, \sqrt{2}, \sqrt{2})\\ &\quad + (\partial_u(\rho_1 g_1))(g_1, g_1, \sqrt{2}, -\sqrt{2}) + \frac{\rho_1}{2}(\partial_u(f_1^2+g_1^2))(1,1,0,0).\end{aligned}$$

Now, since the four vectors are linearly independent, the equality $\mathbf{x}_{uv} = \alpha\mathbf{x}_v + \beta\mathbf{x}_{vv}$ is equivalent to the following system of equations

$$\begin{aligned}\partial_u\rho_1 &= \alpha\rho_1 + \beta\partial_v\rho_1\\ \partial_u(\rho_1 f_1) &= \alpha\rho_1 f_1 + \beta\partial_v(\rho_1 f_1)\\ \partial_u(\rho_1 g_1) &= \alpha\rho_1 g_1 + \beta\partial_v(\rho_1 g_1)\\ \frac{\rho_1}{2}\partial_u(f_1^2+g_1^2) &= \frac{\beta\rho_1}{2}\partial_v(f_1^2+g_1^2).\end{aligned}$$

We have to check under which conditions this system has a solution (for the unknowns α and β). By expressing α from the first three equations, we get that

$$\frac{\partial_u \rho_1 - \beta \partial_v \rho_1}{\rho_1} = \frac{\partial_u(\rho_1 f_1) - \beta \partial_v(\rho_1 f_1)}{\rho_1 f_1} = \frac{\partial_u(\rho_1 g_1) - \beta \partial_v(\rho_1 g_1)}{\rho_1 g_1}. \tag{4.3}$$

The first of the above two equations is equivalent to

$$f_1 \partial_u \rho_1 - \beta f_1 \partial_v \rho_1 = (\partial_u \rho_1) f_1 + \rho_1 \partial_u f_1 - \beta((\partial_v \rho_1) f_1 + \rho_1 \partial_v f_1)$$
$$\Leftrightarrow \beta \rho_1 \partial_v f_1 = \rho_1 \partial_u f_1 \Leftrightarrow \beta = \frac{\partial_u f_1}{\partial_v f_1}.$$

In the same way, from (4.3) we obtain

$$\beta = \frac{\partial_u g_1}{\partial_v g_1}.$$

Now it turns out that it must be

$$\frac{\partial_u f_1}{\partial_v f_1} = \frac{\partial_u g_1}{\partial_v g_1} \Leftrightarrow (\partial_u f_1)\partial_v g_1 = (\partial_u g_1)\partial_v f_1.$$

This last condition is sufficient for the fourth equation of the system also to be satisfied because

$$\begin{aligned}\frac{\beta \rho_1}{2}\partial_v(f_1^2 + g_1^2) &= \frac{\rho_1}{2}\beta \cdot 2 f_1 \partial_v f_1 + \frac{\rho_1}{2}\beta \cdot 2 g_1 \partial_v g_1 \\ &= \frac{\rho_1}{2} \cdot \frac{\partial_u f_1}{\partial_v f_1} \cdot 2 f_1 \partial_v f_1 + \frac{\rho_1}{2} \cdot \frac{\partial_u g_1}{\partial_v g_1} \cdot 2 g_1 \partial_v g_1 \\ &= \frac{\rho_1}{2}\partial_u(f_1^2 + g_1^2)\end{aligned}$$

Notice that the condition $(\partial_u f_1)\partial_v g_1 = (\partial_u g_1)\partial_v f_1$ is also satisfied for ruled lightlike surfaces because then both sides are equal to 0, which implies that this equality characterizes all minimal lightlike surfaces in $\mathbb{M}^4$.

Theorem 4.1. *A regular lightlike surface S, represented by the Weierstrass data (ρ, f, g), is minimal if and only if*

$$(\partial_u \operatorname{Re} f)\partial_v \operatorname{Re} g = (\partial_u \operatorname{Re} g)\partial_v \operatorname{Re} f. \tag{4.4}$$

In particular, the surface S is ruled with lightlike rulings if and only if

$$\partial_v \operatorname{Re} f = \partial_v \operatorname{Re} g = 0. \tag{4.5}$$

From the previous theorem it follows that a lightlike surface S is l-minimal if and only if (4.4) is satisfied and (4.5) is not satisfied (see examples). Furthermore, if we compare (4.5) to the $\mathbb{M}^3$ case (where all lightlike surfaces are ruled) and to Theorem 2.1, we see that only one of the two Cauchy-Riemann conditions must be satisfied in order for S to be ruled. Also, the minimality condition (4.4) is weaker than in the spacelike case.

Remark 4.1. Since taking the real part of a dual number commutes with ∂_u and ∂_v, equation (4.4) can be written as

$$\operatorname{Re}((\partial_u f)\partial_v g - (\partial_u g)\partial_v f) = 0 \Leftrightarrow \operatorname{Re} \det \nabla F = 0,$$

where $F : U \subseteq \mathbb{R}^2 \to \mathbb{D}^2$ is a function given by $F := (f, g)$ and ∇ stands for the Jacobian matrix. This last equality says that the surface is minimal if and only if the Jacobian determinant of F is purely imaginary.

4.2. Ruled parametrizations

A ruled lightlike surface can have parametrizations that are not of the form (2.3). In $\mathbb{M}^3$, substituting dual functions f and g such that f is holomorphic, g meromorphic and fg^2 holomorphic into (2.5) always results in a ruled parametrization. If we substitute functions which are not holomorphic or meromorphic, we get a conformal parametrization which may not ruled. However, substituting non-holomorphic functions will not result in a greater class of surfaces because every lightlike surface in $\mathbb{M}^3$ is ruled.

On the other hand, in $\mathbb{M}^4$ there exist lightlike surfaces which are not ruled and we know that the surface itself is ruled (that is, it has at least one ruled parametrization) if and only if (4.5) holds. Since a ruled surface can have conformal parametrizations which are not ruled, this condition is not sufficient for the parametrization (3.6) itself to be ruled. We want to find out which conditions are necessary and sufficient for the Weierstrass representation itself to be ruled.

First of all, let us notice that a certain counterpart of the three-dimensional case holds in $\mathbb{M}^4$: if we substitute holomorphic functions ρ, f, g into (3.6), we get a ruled parametrization. This is because then from (3.4) and (3.5) it follows that the functions $\partial_z f_k$ (from Section 3) are holomorphic. But then from (3.1) and the Cauchy-Riemann theorem, it follows that $\partial_v^2 x_k = \partial_v \operatorname{Re} f_k = 0$. Now, by integrating twice with respect to v, we can conclude that $\mathbf{x}$ is of form (2.3). However, the condition that ρ, f and g are holomorphic is not necessary for the parametrization to be ruled.

Notice that $\mathbf{x}$ being a ruled parametrization is equivalent to $\mathbf{x}_{vv} = 0$. From (4.2) it follows that this is further equivalent to

$$(\partial_v \rho_1)(1,-1,0,0) + (\partial_v(\rho_1 f_1))(f_1, f_1, \sqrt{2}, \sqrt{2}) + (\partial_v(\rho_1 g_1))(g_1, g_1, \sqrt{2}, -\sqrt{2}) + \frac{\rho_1}{2}(\partial_v(f_1^2 + g_1^2))(1,1,0,0) = 0,$$

where $\rho_1 = \operatorname{Re}\rho$, $f_1 = \operatorname{Re} f$ and $g_1 = \operatorname{Re} g$. Since the above vectors are linearly independent, this is equivalent to

$$\partial_v \rho_1 = \partial_v(\rho_1 f_1) = \partial_v(\rho_1 g_1) = \frac{\rho_1}{2}\partial_v(f_1^2 + g_1^2) = 0. \tag{4.6}$$

On the other hand, if $\mathbf{x}$ is ruled, then S is a ruled surface. Therefore, by Theorem 4.1., f_1 and g_1 must depend only on u. If now ρ_1 also depends only on u, then (4.6) is obviously satisfied. However, the condition that ρ_1 depends only on u is also necessary because it appears in (4.6) as $\partial_v \rho_1 = 0$. This proves the following result.

Theorem 4.2. *Let S be a regular lightlike surface in $\mathbb{M}^4$ parametrized by the map $\mathbf{x}$ given in (3.6) with Weierstrass data (ρ, f, g). Then $\mathbf{x}$ is a ruled parametrization if and only if*

$$\partial_v \operatorname{Re}\rho = \partial_v \operatorname{Re} f = \partial_v \operatorname{Re} g = 0.$$

Notice that if we substitute $f = g$ into (4.4), which reduces the parametrization to the three-dimensional case, it satisfies the equality (for any function), which is consistent with Gorkaviy's definition by which every lightlike surface in $\mathbb{M}^3$ is minimal. There are other possibilities to generalize minimality condition to lightlike surfaces in $\mathbb{M}^n$, which we are about to consider in the next section.

5. Totally geodesic surfaces and examples

We say that the surface S is totally umbilical if it's second fundamental form is of form

$$h(X,Y) = \alpha(X \cdot Y)$$

for some smooth function $\alpha : S \to \mathbb{R}$. If $h(X,Y) = 0$, the surface S is said to be totally geodesic.

It was proved in [5] that every lightlike surface in $\mathbb{M}^3$ is totally umbilical. On the other hand, each of them is also minimal, thus a lightlike surface in $\mathbb{M}^3$ is minimal if and only if is totally umbilical. If we want to generalize this condition to $\mathbb{M}^n$ in the way that minimal surfaces are those which are totally umbilical, we loose the analogy with spacelike and timelike surfaces stating that the surface is maximal, resp. minimal, if and only if it has an associated family of surfaces ([14]).

Let now ∇ be the induced connection on a half-lightlike surface S in $\mathbb{M}^n$, $n \geq 4$. We define

$$\varepsilon_1(X) := (\nabla_X Y)\cdot\xi = (\tilde{\nabla}_X Y)\cdot\xi, \quad \varepsilon_2(X) := (\nabla_X Y)\cdot U = (\tilde{\nabla}_X U)\cdot U.$$

Then $\nabla_X U = \varepsilon_1(X)N + \varepsilon_2(X)U$. If we substitute $Y = Z = U$ into (2.2), since $U \cdot U = 1$, it follows that $\varepsilon_2 = 0$.

In Duggal's book [6], the following definition of minimality was given: a half-lightlike surface in $\mathbb{M}^n$ is minimal if $\varepsilon_1(\xi) = 0$ and $\operatorname{trace} h \mid_{S(TS)} = 0$. This is a counterpart of the definition given in [1], by which a lightlike hypersurface in $\mathbb{M}^n$ is minimal if $\operatorname{trace} II \mid_{S(TS)} = 0$, where II is the scalar second fundamental form of the hypersurface (ε_1 is not defined for hypersurfaces). This definition can be applied to surfaces in $\mathbb{M}^3$. However, this second definition is satisfied only for totally geodesic surfaces. Even more, the only totally geodesic surfaces in $\mathbb{M}^3$ are lightlike planes ([8]), which is a significantly smaller class than all lightlike surfaces. If we compare this to the spacelike case, this would be an analogy only if spacelike planes were the only surfaces with $H = 0$ (because the only totally geodesic spacelike surfaces are spacelike planes). However, this is not the case.

Every totally geodesic half-lightlike surface in $\mathbb{M}^n$ minimal (in the sense of the above definition). It is interesting that the converse is also true under some additional conditions on the surface. The following result was proved in [6]:

Theorem 5.1. *Let S be a totally umbilical half-lightlike surface in $\mathbb{M}^n$. Then S is minimal if and only if S is totally geodesic.*

We will add new result similar to this one. We show that the claim holds if the assumption that S is totally umbilical is replaced by the assumption that the induced connection ∇ on S is a metric connection. It is an interesting result that holds only in $\mathbb{M}^4$.

First we need some preliminaries. Let

$$D_1(X,Y) := (\tilde{\nabla}_X Y)\cdot\xi, \;\; D_2(X,Y) := (\tilde{\nabla}_X Y)\cdot U.$$

Then $h(X,Y) = D_1(X,Y)N + D_2(X,Y)U$. If S is a hypersurface, then h doesn't have the $D_2(X,Y)U$ component, while D_1 becomes II (see [6] or [3] for details). A non-trivial result from [6] is that the induced connection ∇ on a half-lightlike surface is a metric connection if and only if $D_1 = 0$. Furthermore, since h is a bilinear form, trace $h\mid_{S(TS)} = 0$ is equivalent to

$$\sum_{k=1}^{n-3} D_1(e_k,e_k) = \sum_{k=1}^{n-3} D_2(e_k,e_k) = 0$$

for some orthonormal basis $\{e_1,\dots,e_{n-3}\}$ for $S(TS)$. For $n=4$, the condition simplifies to

$$D_1(e_1,e_1) = D_2(e_1,e_1) = 0,$$

and only in the case $n=4$ we do not have more than one term in a sum. This is the reason why the claim holds only for $n=4$.

Let us now prove the claim. The implication that a totally geodesic surface is minimal holds in general, so we only have to prove the converse. Let S be a minimal half-lightlike surface in $\mathbb{M}^4$. Since the induced connection ∇ is a metric connection, it follows that $D_1 = 0$. Applying now (2.1), we get that

$$\begin{aligned}\varepsilon_1(\xi) &= (\nabla_\xi U)\cdot\xi = (\tilde{\nabla}_\xi U - h(\xi,U))\cdot\xi = (\tilde{\nabla}_\xi U)\cdot\xi - h(\xi,U)\cdot\xi \\ &= D_1(\xi,U) - D_2(\xi,U)(U\cdot\xi) = 0 - D_2(\xi,U)\cdot 0 = 0\end{aligned}$$

It is easy to show that in general we have $\varepsilon_1(X) = -D_2(X,\xi)$, for any tangential vector field X. Substituting $X=\xi$, we get

$$D_2(\xi,\xi) = -\varepsilon_1(\xi) = 0.$$

Alternatively, we could obtain the same conclusion by applying (2.2) on $(\nabla_\xi\xi)\cdot U$ because now ∇ is a metric connection. Let now e_1 be any unit spacelike vector field such that $S(TS) = [e_1]$. Since $\xi\cdot U = 0$, applying (2.2) we obtain that

$$D_2(e_1,\xi) = (\tilde{\nabla}_{e_1}\xi)\cdot U = -(\tilde{\nabla}_{e_1}U)\cdot\xi = -D_1(e_1,U) = 0.$$

Since h is symmetric, it follows that $D_2(\xi,e_1) = 0$. Also, since S is minimal, we have that $D_2(e_1,e_1) = 0$. This shows that D_2 is 0 on a basis $\{e_1,\xi\}$ for TS. But then it must be $D_2 = 0$ because D_2 is bilinear. Now $D_1 = D_2 = 0$ implies $h = 0$, so S is totally geodesic.

Theorem 5.2. *Let S be a half-lightlike surface in $\mathbb{M}^4$ such that the induced connection ∇ on S is a metric connection. Then S is minimal if and only if it is totally geodesic.*

Furthermore, we also need the following result from Duggal's book [6], given only in $\mathbb{M}^4$, on how to check if a given surface is totally umbilical.

Theorem 5.3. *Let S be a half-lightlike surface in $\mathbb{M}^4$ and $S(TS) = [X]$. Then S is totally umbilical if and only if there exist functions $h_1, h_2 : S \to \mathbb{R}$ such that*

$$D_1(X,X) = h_1(X\cdot X), \;\; D_2(X,X) = h_2(X\cdot X), \;\; D_2(X,\xi) = D_2(\xi,X) = D_2(\xi,\xi) = 0.$$

Next we will give examples of lightlike surfaces and find their Weierstrass data. The first example is from Duggal's book [6].

Example 5.1. The surface in $\mathbb{M}^4$ given implicitly by the equations $x_3 = h(x_2)$ and $x_1 = x_4$, where h is some smooth function, is a ruled lightlike surface. It can be parametrized by the conformal map

$$\mathbf{x}(u,v) = (0, u, h(u), 0) + v(1,0,0,1).$$

For this surface it was shown in [6] that $D_1 = 0$, $D_2(X,\xi) = D_2(\xi,\xi) = 0$ and $D_2(X,X) = h_2(X,X)$, where

$$h_2(p) = \frac{h''(p_2)}{(1+h'(p_2)^2)^2}.$$

Now Theorem 5.3 implies that the surface is totally umbilical. Further, from $D_1 = 0$ it follows that its induced connection is a metric connection. If we choose h such that $h'' \neq 0$, then the surface is not totally geodesic, so this is an example of a proper totally umbilical surface.

Equations (3.1) and (3.3) provide the Weierstrass data of this surface as

$$\rho = \frac{1-\varepsilon}{2}, \quad f = \frac{1+\varepsilon(h'(u)+1)}{\sqrt{2}}, \quad g = \frac{-1+\varepsilon(h'(u)-1)}{\sqrt{2}}.$$

We see that all the functions depend only on u, as it should be by Theorem 4.2.

Next example is an example of a surface which is neither minimal nor totally umbilical.

Example 5.2. The surface in $\mathbb{M}^4$ given by

$$\mathbf{x}(u,v) = (\sinh v, \cosh v, v\cos u, v\sin u)$$

is a regular lightlike surface and the map $\mathbf{x}$ is conformal.

It can be checked by direct calculation that the vector fields $\mathbf{x}_v$, $\mathbf{x}_{vv}$ and $\mathbf{x}_{uv}$ are linearly independent, so by Lemma 4.1., the surface is not minimal. We can choose $\xi = \mathbf{x}_v$ and $U = \mathbf{x}_{vv}$. Then

$$D_2(\xi,\xi) = (\tilde{\nabla}_{\mathbf{x}_v}\mathbf{x}_v)\cdot \mathbf{x}_{vv} = \mathbf{x}_{vv}\cdot\mathbf{x}_{vv} = 1 \neq 0,$$

so by Theorem 5.3, the surface is not totally umbilical.

The Weierstrass data of this parametrization are

$$\rho = \frac{\cosh v - \sinh v}{2}, \quad f = \frac{\cos u + \sin u + \varepsilon v(\cos u - \sin u)}{\sqrt{2}(\cosh v - \sinh v)}, \quad g = \frac{\cos u - \sin u - \varepsilon v(\cos u + \sin u)}{\sqrt{2}(\cosh v - \sinh v)}.$$

It is easy to check that (4.4) is not satisfied, therefore, the surface is not minimal.

The last example is an l-minimal surface which is obtained by modifying the previous example.

Example 5.3. The surface given by the parametrization

$$\mathbf{x}(u,v) = (\sinh v, \cosh v, \cos u, \sin u).$$

is l-minimal. This parametrization is based on a very general example given in [7] on how to construct an l-minimal translation surface in $\mathbb{M}^n$, $n \geq 4$. We will only comment here that such a parametrization in $\mathbb{M}^4$ cannot be conformal because the only way to decompose $\mathbb{M}^4$ is $\mathbb{M}^2 \oplus \mathbb{E}^2$. The parametrization is of form

$$\mathbf{x}(u,v) = c_1(u) + c_2(v),$$

where c_1 is a curve in $\mathbb{E}^2$ and c_2 a curve in $\mathbb{M}^2$. In order for $\mathbf{x}$ to be conformal, c_2 must be a lightlike curve. However, the only lightlike curve in $\mathbb{M}^2$ is a line, but then the surface does not have a non-trivial family of G-transformations.

It is interesting that in $\mathbb{M}^5 = \mathbb{M}^3 \oplus \mathbb{E}^2$, an l-minimal translation surface can have a conformal parametrization. For example, for c_2 we can choose the nul-helix

$$\gamma(s) = \left(\frac{s^3}{4} + \frac{s}{3}, \frac{s^2}{2}, \frac{s^3}{4} - \frac{s}{3}\right)$$

from [8], which is a non-planar lightlike curve.

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