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Research Article

Dimodules

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Abstract: This paper introduces a new algebraic structure called dimodule. This structure is similar to a module. A dimodule occurs on a semigroup and a dimonoid in place of an additive abelian group and a ring, respectively. This paper presents some algebraic properties of the dimodules and supplies some of their examples. We suggest a definition of a distributive dimonoid. This paper includes examples of this notion that a distributive dimonoid does not have to be a commutative and idempotent dimonoid. We also have examples of dimonoids and dimonoid homomorphisms.

Keywords: Dimonoid, semigroup, dimodule.

1. Introduction

Jean-Louis Loday introduces the concept of dimonoid [4] as a tool to investigate Leibniz algebras. Dimonoids are nonempty sets with two associative operations providing some axioms. The dimonoid becomes a semigroup if the operations are the same.

Anatolii V. Zhuchok has made many contributions to the topics related to dimonoids. Some of these are to give some properties of commutative dimonoids and examples of commutative dimonoids, to introduce the notion of the diband of dimonoids, to construct different samples of dimonoids, to demonstrate that dimonoids are embedded into some dimonoid formed by a semigroup isomorphically, to set a free commutative dimonoid [5, 6, 8, 9].

This paper introduces a dimodule as a new algebraic structure on a semigroup and a dimonoid. This structure inspires by the algebraic form of modules. The dimodules are an algebraic expansion by processing with the dimonoid and semigroups under certain conditions. In this paper, there are studies of some algebraic properties of dimodule concepts and some dimodule examples. We have the definition of a distributive dimonoid. We show with examples that a distributive dimonoid does not have to be a commutative or an idempotent dimonoid. We also have some

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examples of dimonoid and dimonoid homomorphism.

2. Preliminaries

This section contains basic definitions of the semigroups and the modules [1–3]. In this section, there are definitions of the dimonoids and some concepts of them [4–6, 8, 9]. Moreover, this section includes the definition of a distributive dimonoid and some new examples of dimonoids.

2.1. Semigroups

Let S be a nonempty set and "$\cdot$" be a binary operation on S. Then the algebraic structure $(S,\cdot)$ is called a semigroup if and only if, for all $k,l,m\in S$, $k\cdot(l\cdot m)=(k\cdot l)\cdot m$. Let $P(S)$ denote the set of all the subsets of S and $K,M\in P(S)$. If $K=\varnothing$ or $M=\varnothing$, then $K\cdot M=\varnothing$. If otherwise, $K\cdot M$ is the set $\{k\cdot m\mid k\in K, m\in M\}$.

If, for all $s\in S$, $0\cdot s=0$ ($s\cdot 0=0$), then an element $0\in S$ is a left (right) zero element. If an element $0\in S$ is both the left and right zero elements, it is a zero element. A semigroup S in which each element is a left (right) zero element is a left (right) zero semigroup. Let there is an element $0\in S$ in a semigroup $(S,\cdot)$ such that $x\cdot y=0$ for all $x,y\in S$. Then the semigroup is a zero semigroup. Let $(S,\cdot)$ and $(Y,*)$ be semigroups. Then a mapping $f:S\to Y$ is a homomorphism of semigroups if, for all $k,l\in S$, $f(k\cdot l)=f(k)*f(l)$. Let $\{S_i\mid i\in I\}$ be a family of the semigroups. Then $\prod_{i\in I}S_i$ denotes the Cartesian product of the family $\{S_i\mid i\in I\}$ and $\prod_{i\in I}S_i$ is a semigroup.

2.2. Dimonoids

Jean-Louis Loday presented the concept of dimonoid in 2001.

Definition 2.1 *[4] An arbitrary set $D\neq\varnothing$ on which there are two associative operations "$*$" and "$\circ$" is a dimonoid if, for all $k,l,m\in D$, provide the axioms in below:*

(1) $(k*m)*l=k*(m\circ l)$,

(2) $(k\circ m)*l=k\circ(m*l)$,

(3) $(k*m)\circ l=k\circ(m\circ l)$.

Example 2.2 *[4] Let D be a nonempty set and let two binary operations "$*$" and "$\circ$" be defined by, respectively, $k*l=k$ and $k\circ l=l$ for all $k,l\in D$. Then $(D,*,\circ)$ is a dimonoid.*

Example 2.3 *Let* $D = \{k, l\}$. *Then* $(D, *, \circ)$ *is a dimonoid with the following binary operations* "$*$" *and* "$\circ$":

$*$	$\boldsymbol{k}$	$\boldsymbol{l}$
$\boldsymbol{k}$	k	k
$\boldsymbol{l}$	k	k

$\circ$	$\boldsymbol{k}$	$\boldsymbol{l}$
$\boldsymbol{k}$	k	l
$\boldsymbol{l}$	k	l

Definition 2.4 [8] *If, for all* $k \in D$, $k * k = k = k \circ k$, *then a dimonoid* $(D, *, \circ)$ *is an idempotent dimonoid (or diband).*

Example 2.5 *Let* $D = \{k, l\}$. *Then* $(D, *, *)$ *is an idempotent dimonoid with the* "$*$" *binary operation:*

$*$	$\boldsymbol{k}$	$\boldsymbol{l}$
$\boldsymbol{k}$	k	k
$\boldsymbol{l}$	l	l

Example 2.6 *Let* $D = \{k, l\}$. *Then* $(D, *, \circ)$ *is an idempotent dimonoid with the binary operations* "$*$" *and* "$\circ$" *which are defined by the following table:*

$*$	$\boldsymbol{k}$	$\boldsymbol{l}$
$\boldsymbol{k}$	k	k
$\boldsymbol{l}$	l	l

$\circ$	$\boldsymbol{k}$	$\boldsymbol{l}$
$\boldsymbol{k}$	k	l
$\boldsymbol{l}$	k	l

Example 2.7 [5] *Let* $(D, *)$ *be a zero semigroup including fixed elements with* $a \neq b, b \neq 0$ *and for all* $k, l \in D$, *a binary relation* "$\circ$" *on* D *be defined by*

$$k \circ l = \begin{cases} a, & k = l = b \\ 0, & otherwise. \end{cases}$$

Then $(D, *, \circ)$ *is a dimonoid.*

Example 2.8 [9] *Let* $(S, \cdot)$ *be a semigroup with zero and* A *be a nonempty set. Then* A *is both a left* S*-act and a right* S*-act with the following commutative actions:*

$$S \times A \longrightarrow A : (s, l) = s \odot l = l,$$

$$A \times S \longrightarrow A : (l, s) = l \circledcirc s = l.$$

Consider the S*-act morphism* $\psi : A \longrightarrow S, x \longmapsto 0$. *Then* $(A, *, \circ)$ *is a dimonoid with the following binary operations:*

$$m * n := m \circledcirc \psi(n),$$

$$m \circ n := \psi(m) \odot n.$$

Theorem 2.9 *Let* $(D,*,\circ)$ *be a dimonoid and* S *be a nonempty set. If* $\vartheta : D \to S$ *is a bijective function, then* $(S,*_1,\circ_1)$ *is a dimonoid with binary operations defined as follows:*

$$s *_1 v = \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(v)),$$

$$s \circ_1 v = \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(v))$$

for all $s, v \in S$.

Proof For all $s,p,z \in S$,

$$\begin{aligned}(s *_1 p) *_1 z &= \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p)) *_1 z = \vartheta(\vartheta^{-1}(\vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p))) * \vartheta^{-1}(z)) \\ &= \vartheta((\vartheta^{-1}(s) * \vartheta^{-1}(p)) * \vartheta^{-1}(z)) = \vartheta(\vartheta^{-1}(s) * (\vartheta^{-1}(p) \circ \vartheta^{-1}(z))) \\ &= \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(\vartheta(\vartheta^{-1}(p) \circ \vartheta^{-1}(z)))) = \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p \circ_1 z)) = s *_1 (p \circ_1 z),\end{aligned}$$

$$\begin{aligned}(s *_1 p) *_1 z &= \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(p)) *_1 z = \vartheta(\vartheta^{-1}(\vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(p))) * \vartheta^{-1}(z)) \\ &= \vartheta((\vartheta^{-1}(s) \circ \vartheta^{-1}(p)) * \vartheta^{-1}(z)) = \vartheta(\vartheta^{-1}(s) \circ (\vartheta^{-1}(p) * \vartheta^{-1}(z))) \\ &= \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(\vartheta(\vartheta^{-1}(p) * \vartheta^{-1}(z)))) = \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(p *_1 z)) = s \circ_1 (p *_1 z),\end{aligned}$$

$$\begin{aligned}(s *_1 p) \circ_1 z &= \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p)) \circ_1 z = \vartheta(\vartheta^{-1}(\vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p))) \circ \vartheta^{-1}(z)) \\ &= \vartheta((\vartheta^{-1}(s) * \vartheta^{-1}(p)) \circ \vartheta^{-1}(z)) = \vartheta(\vartheta^{-1}(s) \circ (\vartheta^{-1}(p) \circ \vartheta^{-1}(z))) \\ &= \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(\vartheta(\vartheta^{-1}(p) \circ \vartheta^{-1}(z)))) = \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(p \circ_1 z)) = s \circ_1 (p \circ_1 z),\end{aligned}$$

$$\begin{aligned}(s *_1 p) *_1 z &= \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p)) *_1 z = \vartheta(\vartheta^{-1}(\vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p))) * \vartheta^{-1}(z)) \\ &= \vartheta((\vartheta^{-1}(s) * \vartheta^{-1}(p)) * \vartheta^{-1}(z)) = \vartheta(\vartheta^{-1}(s) * (\vartheta^{-1}(p) * \vartheta^{-1}(z))) \\ &= \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(\vartheta(\vartheta^{-1}(p) * \vartheta^{-1}(z)))) = \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p *_1 z)) = s *_1 (p *_1 z),\end{aligned}$$

$$\begin{aligned}(s \circ_1 p) \circ_1 z &= \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(p)) \circ_1 z = \vartheta(\vartheta^{-1}(\vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(p))) \circ \vartheta^{-1}(z)) \\ &= \vartheta((\vartheta^{-1}(s) \circ \vartheta^{-1}(p)) \circ \vartheta^{-1}(z)) = \vartheta(\vartheta^{-1}(s) \circ (\vartheta^{-1}(p) \circ \vartheta^{-1}(z))) \\ &= \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(\vartheta(\vartheta^{-1}(p) \circ \vartheta^{-1}(z))))\vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(p \circ_1 z)) = s \circ_1 (p \circ_1 z).\end{aligned}$$

□

Definition 2.10 *[5] Let* $(D_1,*_1,\circ_1),(D_2,*_2,\circ_2)$ *be dimonoids. Then a mapping* $f : D_1 \to D_2$ *is called a homomorphism of dimonoids if, for all* $k,l \in D_1$, $f(k *_1 l) = f(k) *_2 f(l)$ *and* $f(k \circ_1 l) = f(k) \circ_2 f(l)$.

Example 2.11 *Let* D_1 *and* D_2 *be dimonoids in Example 2.5 and Example 2.6, respectively. Then all the homomorphisms of dimonoids from* D_1 *to* D_2 *are the functions* $f(a) = k$ *and* $g(a) = l$ *for all* $a \in D_1$.

Definition 2.12 *[8] Let* $\varnothing \neq T \subseteq D$. *Then* T *is called a subdimonoid, if for all* $k, l \in T$ *implies* $k * l \in T$, $k \circ l \in T$.

Definition 2.13 *[5] Let* $(D, *, \circ)$ *be a dimonoid. Then* D *is called a commutative dimonoid if both operations are commutative.*

Example 2.14 *[5] Let* A *be an arbitrary set such that* $0, k, l, m, n \in A$ *and* $k \neq l$, $l \neq m$, $m \neq n$, $n \neq k$. *The operations "*$*$*" and "*$\circ$*" on the set* A *be defined as follows:*

$$x * y = \begin{cases} l, & x=y=k \\ 0, & otherwise \end{cases}, \qquad x \circ y = \begin{cases} n, & x=y=m \\ 0, & otherwise \end{cases}$$

for all $x, y \in A$. *So* $(A, *, \circ)$ *is a commutative dimonoid.*

Theorem 2.15 *[5] In a commutative dimonoid* $(D, *, \circ)$, *for all* $k, l, m \in D$, *the following equalities hold:*

$$(k * l) * m = k * (l \circ m) = (k \circ l) * m = k \circ (l * m) = (k * l) \circ m = k \circ (l \circ m).$$

Theorem 2.16 *[5] Let* $(D, *, \circ)$ *be a commutative dimonoid with an idempotent operation "*$*$*". Then its operations coincide.*

Definition 2.17 $(D, *, \circ)$ *is a distributive dimonoid if and only if*

$$k \circ (l * m) = (k \circ l) * (k \circ m),$$

$$(l * m) \circ k = (l \circ k) * (m \circ k)$$

for all $k, l, m \in D$.

Example 2.18 *Let* $(D, *, \circ)$ *be the dimonoid in Example 2.2. Then* $(D, *, \circ)$ *is a distributive dimonoid.*

Theorem 2.19 *If* $(D, *, \circ)$ *is a commutative idempotent dimonoid, then it is a distributive dimonoid.*

Proof Let $(D, *, \circ)$ is a commutative idempotent dimonoid. Then according to Theorem 2.16, "$*$" and "$\circ$" are the same operations. So $(k \circ l) * (k \circ m) = (k * l) * (k * m) = (k * k) * (l * m) =$

$k*(l*m)=k\circ(l*m)$ for all $k,l,m\in D$. Since $(D,*,\circ)$ is a commutative dimonoid, then $(D,*,\circ)$ is distributive dimonoid. □

The dimonoid $(D,*,\circ)$ in Example 2.2 is a distributive and non-commutative. In Example 2.7, the dimonoid $(D,*,\circ)$ is a distributive and commutative dimonoid but not idempotent since $b*b=0\neq b$.

Example 2.20 *Let $D=\{k,l,m\}$ be the commutative dimonoid with the operation "$*$" defined by the following table:*

$*$	$\boldsymbol{k}$	$\boldsymbol{l}$	$\boldsymbol{m}$
$\boldsymbol{k}$	k	k	k
$\boldsymbol{l}$	k	l	m
$\boldsymbol{m}$	k	m	l

Then $(D,*,*)$ is not distributive since $m*(l*l)\neq(m*l)*(m*l)$. Also $(D,*,*)$ is not idempotent since $m*m\neq m$.

Example 2.21 *Let $D=\{k,l,m\}$ be an arbitrary set. $(D,*,*)$ is a commutative with the operation "$*$"defined in table. Although $(D,*,*)$ commutative dimonoid is distributive dimonoid, it is not idempotent since $m*m=l\neq m$.*

$*$	$\boldsymbol{k}$	$\boldsymbol{l}$	$\boldsymbol{m}$
$\boldsymbol{k}$	k	k	k
$\boldsymbol{l}$	k	l	l
$\boldsymbol{m}$	k	l	l

Theorem 2.22 *Let $(D,*,\circ)$ be an arbitrary dimonoid, and let S be the dimonoid generated from D as in the Theorem 2.9. If $(D,*,\circ)$ is distributive, then S is so.*

Proof Let $k,l,m\in S$. Then $k\circ_1(l*_1m)=k\circ_1(\vartheta(\vartheta^{-1}(l)*\vartheta^{-1}(m)))=\vartheta(\vartheta^{-1}(k)\circ(\vartheta^{-1}(l)*\vartheta^{-1}(m)))=\vartheta((\vartheta^{-1}(k)\circ\vartheta^{-1}(l))*(\vartheta^{-1}(k)\circ\vartheta^{-1}(m)))$. Let $(\vartheta^{-1}(k)\circ\vartheta^{-1}(l)):=\vartheta^{-1}(a)$ and $(\vartheta^{-1}(k)\circ\vartheta^{-1}(m)):=\vartheta^{-1}(b)$. Then $k\circ_1(l*_1m)=\vartheta(\vartheta^{-1}(a)*\vartheta^{-1}(b))=a*_1b=\vartheta(\vartheta^{-1}(k)\circ\vartheta^{-1}(l))*_1\vartheta(\vartheta^{-1}(k)\circ\vartheta^{-1}(l))=(k\circ_1 l)*_1(k\circ_1 m)$. Thus S is left distributive since $k\circ_1(l*_1m)=(k\circ_1l)*_1(k\circ_1m)$. Similarly, S is right distributive. □

Theorem 2.23 *[7] Let $\{D_i\mid i\in I\}$ be a family of dimonoids. Then the Cartesian product of the family $\{D_i\mid i\in I\}$, $\prod_{i\in I}D_i$, is a dimonoid.*

3. Dimodules

Let $(D, *, \circ)$ be a dimonoid. A (left) D-dimodule is a semigroup $(S, \cdot)$ together with a function $D \times S \to S$ (the image of (u, x) being denoted by ux) such that for all $u, v \in D$ and for all $x, y \in S$:

(1) $u(x \cdot y) = ux \cdot uy$,

(2) $(u * v)x = ux \cdot vx$,

(3) $u(vx) = (u \circ v)x$.

A right D-dimodule is defined similarly via function $S \times D \to S$ denoted $(x, u) \mapsto xu$ and satisfying the obvious of $(1)-(3)$. In this paper, unless specified otherwise, a D-dimodule means a left D-dimodule. All theorems about left D-dimodules also hold for right D-dimodules.

Example 3.1 *Let $(D, *, \circ)$ be a dimonoid and $(S, \cdot)$ be a semigroup with an idempotent element a. Then S is a D-dimodule with the operation*

$$D \times S \longrightarrow S$$

$$(x, y) \longmapsto a$$

Example 3.2 *Let $D = S = \{a, b\}$. Then $(D, *, \circ)$ is dimonoid and $(S, \cdot)$ is a semigroup for the operations "$*, \circ, \cdot$" in the following tables:*

$*$	a	b
a	a	a
b	a	a

$\circ$	a	b
a	a	b
b	a	b

$\cdot$	a	b
a	a	b
b	b	b

(i) Let a function $D \times S \to S$ be defined as $(d, s) \to ds = s$. Then S is a D-dimodule.

(ii) Let a function $D \times S \to S$ be defined as $(d, s) \to ds = d$. Then S is not a D-dimodule since $(a * b)a = a \neq b = aa \cdot ba$.

Example 3.3 *Let $(D, *, \circ)$ be the dimonoid and let $(\mathbb{N}, \cdot)$ be the semigroup of natural numbers with the multiplication. Let a function $D \times \mathbb{N} \to \mathbb{N}$ be defined as follows:*

$$dn = \begin{cases} 0, & 2 \mid n \\ 1, & 2 \nmid n. \end{cases}$$

Then $\mathbb{N}$ is a D-dimodule.

Example 3.4 *Let $(D, *)$ be the semigroup in Example 2.20. If the function $D \times D \to D$ is defined as $(d, s) \longmapsto ds = d * s$, then D is not a D-dimodule since $(m * m)m = m$ and $mm * mm = l * l = l$.*

Example 3.5 *Let* $(D,*,\circ)$ *be a dimonoid in which* $(D,*)$ *is an idempotent semigroup and let a function* $D\times D\longrightarrow D$ *defined as* $(x,y)\longmapsto xy=y$. *Then* D *is a* D*-dimodule.*

Proposition 3.6 *Let* S_1,S_2 *be semigroups and* f *be a homomorphism of semigroup from* S_1 *to* S_2. *Then* S_1 *is a* D*-dimodule if* S_2 *is* D*-dimodule.*

Proof Let the semigroup S_2 be D-dimodule with the mapping $D\times S_2\to S_2$, $(u,y)\longmapsto uy$. Thus consider the mapping $D\times S_1\to S_1$, $(u,x)\longmapsto ux=uf(x)$. Then S_1 is a D-dimodule. □

Proposition 3.7 *Let* $(D,*,\circ)$ *be a distributive dimonoid and a function* $D\times D\longrightarrow D$ *be defined as* $(x,y)\longmapsto xy=x\circ y$. *Then* $(D,*)$ *is a* D*-dimodule.*

Proof Straightforward. □

Example 3.8 shows that the Proposition 3.7 may not be correct if $(D,*,\circ)$ is not a distributive dimonoid, in general.

Example 3.8 *Consider the dimonoid* D *in Example 2.20. Thus* $(D,*)$ *is not a* D*-dimodule since* $m*(l*l)=m\neq l=(m*l)*(m*l)$.

Proposition 3.9 *Let* $\{S_iD_i\text{-module}\,|\,i\in I\}$. *Then* $\prod_{i\in I}S_i$ *is a* $\prod_{i\in I}D_i$*-module.*

Proof Consider the mapping $\prod_{i\in I}D_i\times\prod_{i\in I}S_i\to\prod_{i\in I}S_i$, $((d_i)_{i\in I},(s_i)_{i\in I})\longmapsto(d_i)_{i\in I}.(s_i)_{i\in I}=(d_is_i)_{i\in I}$. Then $\prod_{i\in I}S_i$ is a $\prod_{i\in I}D_i$-module. □

Proposition 3.10 *Let* $(D,*,\circ)$ *be a dimonoid and a semigroup* S *be a* D*-dimodule with a bijective mapping* $D\times S\to S$. *Then* D *is a distributive dimodule.*

Proof Let $k,l,m\in D$ and $x\in S$. Thus $[k\circ(l*m)]x=k[(l*m)x]=k((lx)(mx))=(k(lx))(k(mx))=((k\circ l)x)((k\circ m)x)=[(k\circ l)*(k\circ m)]x$ and $[(l*m)\circ k]x=[(l\circ k)*(m\circ k)]x$ similarly. Hence D is distributive via bijectivity. □

Definition 3.11 *Let* $(S,\cdot)$ *be a* D*-dimodule and* $\varnothing\neq E\subseteq S$. *Then* E *is called a* D*-subdimodule of* S *if, for all* $x,y\in E$ *and* $u\in D$, $x\cdot y,ux\in E$.

Example 3.12 *Listed below are some examples of subdimodules:*

(i) Each dimodule is a subdimodule of itself.

(ii) Let D *be the* D*-dimodule in Example 3.5. Then each subsemigroup of* D *is a subdimodule of* D.

(iii) Let $(S,\cdot)$ be the D-dimodule in Example 3.2-(i) and $E=\{a\}$. Then E is a D-subdimodule of S.

Proposition 3.13 *Let S be a D-dimodule and $\{E_i \mid i\in I\}$ be a family of the D-subdimodules of S. Then $\bigcap_{i\in I} E_i$ is a D-subdimodule of S if $\bigcap_{i\in I} E_i \neq \varnothing$.*

Proof Let $x,y\in\bigcap_{i\in I} E_i$ and $u\in D$. Thus $x,y\in E_i$ for all $i\in I$. Hence, for all $i\in I$, $x\cdot y\in E_i$ and $ux\in E_i$ since E_i is a D-subdimodule. Then $x\cdot y, ux\in\bigcap_{i\in I} E_i$. Therefore $\bigcap_{i\in I} E_i$ is a D-dimodule of S. □

Example 3.14 shows that Proposition 3.13 may not be correct for the union of the families of subdimodules.

Example 3.14 *Let $D=\{a,b,c\}$ and $(D,*)$ be the semigroup with the table below. If the function $D\times D\longrightarrow D$ is defined as $(u,x)\longmapsto ux=x$, then D is a D-dimodule.*

$*$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
$\boldsymbol{a}$	a	a	a
$\boldsymbol{b}$	a	b	a
$\boldsymbol{c}$	a	a	c

The subsets $A=\{b\}$ and $B=\{c\}$ of D are D-subdimodules. However, $A\cup B=\{b,c\}$ is not a D-subdimodule since $b*c=a\notin A\cup B$.

Proposition 3.15 *Let S be a D-dimodule and $A\subseteq S$.*

(i) Let $a\in A$ be idempotent element and $(A:^D S)_a$ be the set $\{u\in D \mid ux=a \text{ for all } x\in A\}$. Then $(A:^D S)_a$ is a subdimonoid of D if it is nonempty.

(ii) Let A is a subsemigroup of S and $(A:_D S)=\{u\in D \mid uS\subseteq A\}$. Then $(A:_D S)$ is a subdimonoid of D if it is nonempty.

Proof Straightforward. □

Proposition 3.16 *Let $\{S_i \mid i\in I\}$ be a family of the D-dimodules. Then $\prod_{i\in I} S_i$ is a D-dimodule and it is called direct product of the family $\{S_i \mid i\in I\}$.*

Proof Let the mapping $D\times\prod_{i\in I} S_i\to\prod_{i\in I} S_i$, $(d,(s_i)_{\in I})\longmapsto d(s_i)_{i\in I}=(ds_i)_{i\in I}$. Then $\prod_{i\in I} S_i$ is a D-dimodule. □

Definition 3.17 *Let S_1,S_2 be D-dimodules. A function $f:S_1\longrightarrow S_2$ is called a homomorphism of D-dimodules if, for all $x,y\in S_1$ and $u\in D$, $f(x\cdot y)=f(x)\cdot f(y)$ and $f(ux)=uf(x)$.*

Example 3.18 *Let* S_1 *be a* D*-dimodule and* S_1 *be the* D*-dimodule in Example 3.1. Let a function* $f : S_1 \longrightarrow S_2$ *be defined by* $f(x) = a$ *for all* $x \in S_1$. *Then* f *is a homomorphism of* D*-dimodules.*

Example 3.19 *Let two binary operations on* $\mathbb{Z}_5$ *be defined as follows:*

$$\bar{x} * \bar{y} = \begin{cases} \bar{2}, & \bar{x}=\bar{y}=\bar{1} \\ \bar{0}, & otherwise \end{cases}, \qquad \bar{x} \circ \bar{y} = \begin{cases} \bar{4}, & \bar{x}=\bar{y}=\bar{3} \\ \bar{0}, & otherwise. \end{cases}$$

$(\mathbb{Z}_5, *, \circ)$ *is a dimonoid [5]. The semigroup* $(\mathbb{Z}_2, \cdot)$ *is a* $\mathbb{Z}_5$*-dimodule with the operation* $\mathbb{Z}_5 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$, $(\bar{u}, \bar{x}) \longmapsto \bar{1}$ *and the semigroup* $(\mathbb{Z}_4, +)$ *is a* $\mathbb{Z}_5$*-dimodule with the operation* $\mathbb{Z}_5 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4$, $(\bar{u}, \bar{x}) \longmapsto \bar{0}$. *Then a function* $f : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$, $\bar{x} \longmapsto f(\bar{x}) = \bar{1}$ *is a homomorphism of* $\mathbb{Z}_5$*-dimodules.*

Example 3.20 *Let* D *be the dimonoid in Example 3.2 and* S *be the* D*-dimodule in the case (i). Then* $\mathbb{N}$ *is also a* D*-dimodule since* D *is an arbitrary dimonoid in Example 3.3. Consider* $f : \mathbb{N} \to S$,

$$f(n) = \begin{cases} b, & 2 \mid n \\ a, & 2 \nmid n. \end{cases}$$

Then f *is a homomorphism of* D*-dimodules.*

Theorem 3.21 *Let* S *and* Y *be* D*-dimodules, and* $f : S \longrightarrow Y$ *be a homomorphism of* D*-dimodules. If* E *is a subdimodule of* S, *then* $f(E)$ *is a subdimodule of* Y.

Proof $\varnothing \neq f(E) \subseteq Y$ since E is a subdimodule of S. Let $u \in D$ and $a, b \in f(E)$. There exist $x, y \in E$ such that $a = f(x), b = f(y)$ since $a, b \in f(E)$. $a \cdot b = f(x) \cdot f(y) = f(x \cdot y)$ and $ua = uf(x) = f(ux)$ since f is a homomorphism of D-dimodules. Hence $a \cdot b, ua \in f(E)$ since $x \cdot y, ux \in E$. Thus $f(E)$ is a subdimodule of Y. □

Theorem 3.22 *Let* S *and* Y *be* D*-dimodules,* $f : S \longrightarrow Y$ *be a homomorphism of* D*-dimodules and* X *be a subdimodule of* Y. *Then* $f^{-1}(X)$ *is a subdimodule of* S *if* $f^{-1}(X) \neq \varnothing$.

Proof $\varnothing \neq f^{-1}(X) \subseteq S$ since X is a subdimodule of Y. Let $u \in D$ and $x, y \in f^{-1}(X)$. Thus $f(x), f(y) \in X$. $f(x) \cdot f(y) = f(x \cdot y) \in X$ and $uf(x) = f(ux) \in X$ since f is a homomorphism of D-dimodule. Hence $x \cdot y, ux \in f^{-1}(X)$. Thus $f^{-1}(X)$ is a subdimodule of S. □

Corollary 3.23 *Let* S *and* Y *be* D*-dimodules,* $f : S \longrightarrow Y$ *be a surjective homomorphism of* D*-dimodule and* X *be a subdimodule of* Y. *Then* $f^{-1}(X)$ *is a subdimodule of* S.

Proof $X \neq \varnothing$ since X is a subdimodule of Y. Thus there exists $y \in X$. Hence there exists $x \in S$ such that $f(x) = y$ since f is a surjective function. Hence $f^{-1}(X) \neq \varnothing$. Thus $f^{-1}(X)$ is a subdimodule of S as per Theorem 3.22. □

Theorem 3.24 *Let S be a D-dimodule, $a \in S$ and $Da = \{da \mid d \in D\}$. Then Da is a subdimodule of S.*

Proof $\varnothing \neq Da \subseteq S$. Thus let $x, y \in Da$ and $u \in D$. Hence there exist $d_1, d_2 \in D$ such that $x = d_1 a, y = d_2 a$. $x \cdot y = (d_1 a) \cdot (d_2 a) = (d_1 * d_2) a \in Da$ since $d_1 * d_2 \in D$ and $ux = u(d_1 a) = (u \circ d_1) a \in Da$ since $u \circ d_1 \in D$. Therefore Da is a subdimodule of S. □

Theorem 3.25 *Let $(D, *, \circ)$ be a distributive dimonoid, S be a D-dimodule, $a \in S$ and $Da = \{da \mid d \in D\}$. Then the map $f : D \longrightarrow Da, f(d) = da$ is a surjective homomorphism of D-dimodule.*

Proof The surjective map f is a homomorphism of D-dimodule since $f(u) \cdot f(v) = (ua) \cdot (va) = (u * v)a = f(u * v)$ and $f(d \circ u) = (d \circ u)a = d(ua) = df(u)$ for all $u, v, d \in D$. □

Theorem 3.26 *Let D_1 and D_2 be two dimonoids and let $f : D_1 \to D_2$ be a dimonoid homomorphism. Then S is a D_1-dimodule if S is a D_2-dimodule.*

Proof Consider $D_1 \times S \to S$, $(u, x) \longmapsto f(u)x$. Let $u, v \in D_1$ and $x, y \in S$. Then

$$\begin{aligned} u(x \cdot y) &= f(u)(x \cdot y) = (f(u)x) \cdot (f(u)y) = (ux) \cdot (uy), \\ (u * v)x &= f(u * v)x = (f(u) * f(v))x = (f(u)x) \cdot (f(v)x) = (ux) \cdot (vx), \\ u(vx) &= u(f(v)x) = f(u)(f(v)x) = (f(u) \circ f(v))x = f(u \circ v) = (u \circ v)x \end{aligned}$$

since S is a D_2-dimodule and $f : D_1 \to D_2$ be a dimonoid homomorphism. □

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Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Ertuğrul Akçay]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%50).
Author [Canan Akın]: Thought and designed the research/problem, contributed to completing the research and solving the problem (%50).

Conflicts of Interest

The authors declare no conflict of interest.

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