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Fekete-Szegö Inequality for Certain Subclasses of Analytic Functions Defined by The Combination of Differential and Integral Operators

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Highlights:
- The combination of linear multiplier differential and Noor integral operators is defined
- Fekete-Szegö inequality and coefficient estimates for these new subclasses

Keywords:
- Fekete-Szegö inequality
- Starlike and convex functions of complex order
- Noor integral operator
- Analytic functions

ABSTRACT:
In this paper, we introduced certain general new subclasses of analytic functions defined by the combination of two special operator which one of them derivative (Deniz-Orhan derivative operator) and other integral (Noor integral operators). For these classes coefficient estimates and the Fekete–Szegö inequality is completely solved.

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INTRODUCTION

The class A is well-known family of analytic functions $\hbar$ of the form

$$\hbar(\sigma) = \sigma + \sum_{\kappa=2}^{\infty} a_\kappa \sigma^\kappa \tag{1}$$

in the open unit disk $\mathrm{U} = \{\sigma \in \mathbb{C} : |\sigma| < 1\}$. Also, let S be the class of univalent functions in A. It is common knowledge that for $\hbar \in \mathrm{S}$, $|a_3 - a_2^2| \le 1$. A traditional theorem of Fekete-Szegö (Fekete and Szegö, 1933) expresses that for $\hbar \in \mathrm{S}$ given by (1).

$$|a_3 - \delta a_2^2| \le \begin{cases} 3 - 4\delta & \text{if } \delta \le 0, \\ 1 + 2\exp\left(\dfrac{-2\delta}{1-\delta}\right) & \text{if } 0 < \delta < 1, \\ 4\delta - 3 & \text{if } \delta \ge 1. \end{cases}$$

This inequality is sharp because there is a function in S that ensures equality for each δ. Pfluger (Pfluger, 1984) proved this inequality for the complex δ values as follows:

$$|a_3 - \delta a_2^2| \le 1 + 2\left|\exp\left(\frac{-2\delta}{1-\delta}\right)\right|.$$

Till now, a number of authors have sought to apply the forementioned inequality to broader classes of analytical functions.

The classes of starlike and convex functions of order α given by, respectively

$$\mathrm{S}^*(\alpha) = \left\{\hbar \in \mathrm{S} : \Re\left(\frac{\sigma\hbar'(\sigma)}{\hbar(\sigma)}\right) > \alpha, \quad 0 \le \alpha < 1, \ \sigma \in \mathrm{U}\right\}$$

and

$$\mathrm{C}(\alpha) = \left\{\hbar \in \mathrm{S} : \Re\left(1 + \frac{\sigma\hbar''(\sigma)}{\hbar'(\sigma)}\right) > \alpha, \quad 0 \le \alpha < 1, \ \sigma \in \mathrm{U}\right\}.$$

In particular, the classes $\mathrm{S}^* = \mathrm{S}^*(0)$ and $\mathrm{C} = \mathrm{C}(0)$ are the familiar classes of starlike and convex functions in U, respectively. Nasr and Aouf (Nasr and Aouf, 1985), Wiatrowski (Wiatrowski, 1971), Nasr and Aouf (Nasr and Aouf, 1982) defined these classes for complex order α.

MATERIALS AND METHODS

Let $\hbar(\sigma) = \sigma + \sum_{\kappa=2}^{\infty} a_\kappa \sigma^\kappa$ and $\lambda\!\!\!^{-}(\sigma) = \sigma + \sum_{\kappa=2}^{\infty} b_\kappa \sigma^\kappa$ be analytic functions in U. The Hadamard product of $\hbar$ and $\lambda\!\!\!^{-}$, denoted by $\hbar * \lambda\!\!\!^{-}$ is defined by

$$(\hbar * \lambda\!\!\!^{-})(\sigma) = \sigma + \sum_{\kappa=2}^{\infty} a_\kappa b_\kappa \sigma^\kappa = (\lambda\!\!\!^{-} * \hbar)(\sigma) \quad (\sigma \in \mathrm{U}).$$

Deniz and Orhan (Deniz and Orhan, 2010) introduced the following linear multiplier differential operator for $\mathrm{T}_{\lambda,\mu}^{m}\hbar$ as follows

$$\mathrm{T}^{0}_{\eta,\xi}\hbar(\sigma)=\hbar(\sigma)$$
$$\mathrm{T}^{1}_{\eta,\xi}\hbar(\sigma)=\eta\xi\sigma^{2}\hbar''(\sigma)+(\eta-\xi)\sigma\hbar'(\sigma)+(1-\eta-\xi)\hbar(\sigma)=\mathrm{T}_{\eta,\xi}\hbar(\sigma)$$
$$\mathrm{T}^{2}_{\eta,\xi}\hbar(\sigma)=\mathrm{T}_{\eta,\xi}\left(\mathrm{T}^{1}_{\eta,\xi}\hbar(\sigma)\right)$$
$$\vdots$$
$$\mathrm{T}^{m}_{\eta,\xi}\hbar(\sigma)\ =\mathrm{T}_{\eta,\xi}\left(\mathrm{T}^{m-1}_{\eta,\xi}\hbar(\sigma)\right)\ \left(m\in\mathbb{N}=\{1,2,3,\ldots\}\right),$$

where $\eta\geq\xi\geq0$. We note that

$$\mathrm{T}^{m}_{\eta,\xi}\hbar(\sigma)=\sigma+\sum_{\kappa=2}^{\infty}\left[1+(\eta\xi\kappa+\eta-\xi)(\kappa-1)\right]^{m}a_{\kappa}\sigma^{\kappa}\ \left(m\in\mathbb{N}_{0}=\mathbb{N}\cup\{0\}\right), \tag{2}$$

with $\mathrm{T}^{m}_{\eta,\xi}\hbar(0)=0$.

It should be noted that the operator $\mathrm{T}^{m}_{\eta,\xi}$ is a generalization of many other operators discussed previously. We have the following for $\hbar\in\mathrm{A}$ in particular:

(i) $\mathrm{T}^{m}_{\eta,0}\hbar(\sigma)=\mathrm{T}^{m}_{\eta}\hbar(\sigma)$ the operator investigated by Al-Oboudi (Al-Oboudi, 2004).

(ii) $\mathrm{T}^{m}_{1,0}\hbar(\sigma)=\mathrm{T}^{m}\hbar(\sigma)$ the operator investigated by Sălăgean (Sălăgean, 1983).

(iii) $\mathrm{T}^{m}_{\eta,\xi}\hbar(\sigma)$ the operator considered for $0\leq\mu\leq\lambda\leq1$ by Răducanu and Orhan (Răducanu and Orhan, 2010).

Denote by

$$\mathrm{R}^{\varsigma}:=\frac{\sigma}{(1-\sigma)^{\varsigma+1}}*\hbar(\sigma)\ \ (\varsigma\in\mathbb{N}_{0}).$$

Then implies that

$$\mathrm{R}^{\varsigma}\hbar(\sigma)=\frac{\sigma\left(\sigma^{\varsigma-1}\hbar(\sigma)\right)^{(\varsigma)}}{\varsigma!}\quad(\varsigma\in\mathbb{N}_{0}).$$

The operator $\mathrm{R}^{\varsigma}\hbar$ is called Ruscheweyh derivative operator (Ruscheweyh, 1975). Noor (Noor, 1999) defined and investigated an integral operator $\mathrm{N}^{\varsigma}:\mathrm{A}\to\mathrm{A}$ analogous to $\mathrm{R}^{\varsigma}\hbar$ as follows.

Let $\hbar_{\varsigma}(\sigma)=\dfrac{\sigma}{(1-\sigma)^{\varsigma+1}}$, $\varsigma\in\mathbb{N}_{0}$, and let $\hbar^{(-1)}_{\varsigma}$ be defined such that

$$\hbar_{\varsigma}(\sigma)*\hbar^{(-1)}_{\varsigma}(\sigma)=\frac{\sigma}{1-\sigma}.$$

Then

$$\mathrm{N}^{\varsigma}\hbar(\sigma)=\hbar^{(-1)}_{\varsigma}(\sigma)*\hbar(\sigma)=\left[\frac{\sigma}{(1-\sigma)^{\varsigma+1}}\right]^{(-1)}*\hbar(\sigma)=\sigma+\sum_{\kappa=2}^{\infty}\frac{\Gamma(\varsigma+1)\kappa!}{\Gamma(\varsigma+\kappa)}a_{\kappa}\sigma^{\kappa}:=\zeta(\sigma). \tag{3}$$

For the function $\zeta(\sigma)$ given by (3), we define the following convolution operator:

$$\mathrm{K}^{0}_{\eta,\xi}\zeta(\sigma)=\zeta(\sigma),$$

$$\mathrm{K}^{1}_{\eta,\xi}\zeta(\sigma)=\mathrm{K}_{\eta,\xi}\zeta(\sigma)=\eta\xi\sigma^{2}\zeta''(\sigma)+(\eta-\xi)\sigma\zeta'(\sigma)+(1-\eta-\xi)\zeta(\sigma)$$

$$=\sigma+\sum_{\kappa=2}^{\infty}\left[1+(\eta\xi\kappa+\eta-\xi)(\kappa-1)\right]\frac{\Gamma(\varsigma+1)\kappa!}{\Gamma(\varsigma+\kappa)}a_{\kappa}\sigma^{\kappa},$$

$$\vdots$$

$$\mathrm{K}^{m}_{\eta,\xi}\zeta(\sigma)=\mathrm{K}_{\eta,\xi}(\mathrm{K}^{m-1}_{\eta,\xi}\zeta(\sigma))\quad(m\in\mathbb{N}).$$

It can be easily seen that

$$\mathrm{K}^{m}_{\eta,\xi}\zeta(\sigma)=\sigma+\sum_{\kappa=2}^{\infty}\left[1+(\eta\xi\kappa+\eta-\xi)(\kappa-1)\right]^{m}\frac{\Gamma(\varsigma+1)\kappa!}{\Gamma(\varsigma+\kappa)}a_{\kappa}\sigma^{\kappa}, \tag{4}$$

where $m,\varsigma\in\mathbb{N}_0$ and $\eta\geq\xi\geq 0$.

Here the letters m and ς are related to the linear multiplier differential operator and the Noor integral operator, respectively.

We now define new subclasses of analytic functions using the operator $\mathrm{K}^{m}_{\eta,\xi}\zeta(\sigma)$, as follows.

Definition 1. Let $\vartheta\in\mathbb{C}\setminus\{0\}$, and let $\hbar$ be an univalent function of the form (1). We say that $\hbar$ belongs to $\mathrm{S}^{m}_{\eta,\xi}(\vartheta,\varsigma)$ if

$$\Re\left(1+\frac{1}{\vartheta}\left(\frac{\sigma\left(\mathrm{K}^{m}_{\eta,\xi}\zeta(\sigma)\right)'}{\mathrm{K}^{m}_{\eta,\xi}\zeta(\sigma)}-1\right)\right)>0\quad\left(m,\varsigma\in\mathbb{N}_0,\eta\geq\xi\geq 0,\ \sigma\in\mathrm{U}\right),$$

where $\zeta(\sigma):=\mathrm{N}^{\varsigma}\hbar(\sigma)$ is given by (3).

Definition 2. Let $\vartheta\in\mathbb{C}\setminus\{0\}$, and let $\hbar$ be an univalent function of the form (1). We say that $\hbar$ belongs to $\mathrm{C}^{m}_{\eta,\xi}(\vartheta,\varsigma)$ if

$$\Re\left(1+\frac{1}{\vartheta}\frac{\sigma\left(\mathrm{K}^{m}_{\eta,\xi}\zeta(\sigma)\right)''}{\left(\mathrm{K}^{m}_{\eta,\xi}\zeta(\sigma)\right)'}\right)>0\quad\left(m,\varsigma\in\mathbb{N}_0,\eta\geq\xi\geq 0,\ \sigma\in\mathrm{U}\right),$$

where $\zeta(\sigma):=\mathrm{N}^{\varsigma}\hbar(\sigma)$ is given by (3).

The following significant subclasses have been examined by numerous writers in earlier publications, taking precise values to the parameters ϑ,ς,η and ξ, for example, $\mathrm{S}^{m}_{\eta,\xi}(\vartheta,1)=\mathrm{S}^{m}(\vartheta,\eta,\xi)$, $\mathrm{C}^{m}_{\eta,\xi}(\vartheta,1)=\mathrm{C}^{m}(\vartheta,\eta,\xi)$ (Orhan, Deniz and Çağlar (see: Orhan et al, 2012)), $\mathrm{S}^{0}(1,\varsigma)=\mathrm{M}(n,0)$ (Sokol and Bansal (see: Sokol and Bansal, 2012)), $\mathrm{S}^{0}(\vartheta,\varsigma=n)=N^{*}_{(n)}$ Noor (see: Noor, 1999)).

In fact, many authors have studied the Fekete-Szegö inequality for various a variety of sublasses of A, the upper bound for $\left|a_3-\delta a_2^2\right|$ is studied by a variety of authors (see: Abdel-Gawad and Thomas, 1992; Chonweerayoot et al, 1992; Darus and Thomas, 1996; Darus and Thomas, 1998; Keogh and Merkes, 1969; Koepf, 1987; London, 1993; Ma and Minda, 1994) and (see also recent research on this subject by (Çağlar and Orhan, 2021; Deniz et al, 2012; Kanas and Darwish, 2010; Kazımoğlu and Deniz, 2020; Kazımoğlu and Mustafa, 2020; Orhan et al, 2010; Orhan and Răducanu, 2009)).

We focus on the coefficient estimates and the Fekete-Szegö inequality for the subclasses $S_{\eta,\xi}^{m}(\vartheta,\varsigma)$ and $C_{\eta,\xi}^{m}(\vartheta,\varsigma)$ in this paper.

RESULTS AND DISCUSSION

We denote by P a class of analytic function in U with $p(0)=1$ and $\Re p(\sigma)>0$. The following lemma is required to prove our main results.

Lemma 1. (Pommerenke, 1975) Let $p\in P$ with $p(\sigma)=1+c_1\sigma+c_2\sigma^2+\cdots$, then $|c_n|\leq 2$, for $n\geq 1$. If $|c_1|=2$ then $p(\sigma)\equiv p_1(\sigma)=(1+\gamma_1\sigma)/(1-\gamma_1\sigma)$ with $\gamma_1=c_1/2$. Inversely, if $p(\sigma)\equiv p_1(\sigma)$ for some $|\gamma_1|=1$, then $c_1=2\gamma_1$ and $|c_1|=2$. Additionally, we have

$$\left|c_2-\frac{c_1^2}{2}\right|\leq 2-\frac{|c_1|^2}{2}.$$

If $|c_1|<2$ and $\left|c_2-\frac{c_1^2}{2}\right|=2-\frac{|c_1|^2}{2}$, then $p(\sigma)\equiv p_2(\sigma)$, where

$$p_2(\sigma)=\frac{1+\sigma\dfrac{\gamma_2\sigma+\gamma_1}{1+\bar{\gamma}_1\gamma_2\sigma}}{1-\sigma\dfrac{\gamma_2\sigma+\gamma_1}{1+\bar{\gamma}_1\gamma_2\sigma}},$$

and $\gamma_1=c_1/2$, $\gamma_2=\dfrac{2c_2-c_1^2}{4-|c_1|^2}$. Conversely, if $p(\sigma)\equiv p_2(\sigma)$ for some $|\gamma_1|<1$ and $|\gamma_2|=1$ then $\gamma_1=c_1/2$, $\gamma_2=\dfrac{2c_2-c_1^2}{4-|c_1|^2}$ and $\left|c_2-\frac{c_1^2}{2}\right|\text{„}\ 2-\frac{|c_1|^2}{2}$.

Then, we present the result that follows.

Theorem 1. Let $m,\varsigma\in\mathbb{N}_0$, $\eta\geq\xi\geq 0$ and $\vartheta\in\mathbb{C}\setminus\{0\}$. If $\hbar$ of the form (1) is in $S_{\lambda,\mu}^{m}(\vartheta,\varsigma)$, then

$$|a_2|\leq\frac{|\vartheta|(\varsigma+1)}{[2\eta\xi+\eta-\xi+1]^m}, \tag{5}$$

and

$$|a_3|\leq\frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{6[6\eta\mu+2\eta-2\xi+1]^m}\max\{1,|1+2\vartheta|\}. \tag{6}$$

Consider the functions

$$\frac{\sigma\left(K_{\eta,\xi}^{m}\zeta(\sigma)\right)'}{K_{\eta,\xi}^{m}\zeta(\sigma)}=1+\vartheta\left(p_1(\sigma)-1\right) \tag{7}$$

and

$$\frac{\sigma\left(K_{\eta,\xi}^{m}\zeta(\sigma)\right)'}{K_{\eta,\xi}^{m}\zeta(\sigma)}=1+\vartheta\left(p_2(\sigma)-1\right), \tag{8}$$

where p_1 and p_2 are given in Lemma 1. In (5) and (6) equalities are satisfied for the functions (7) and (8), respectively.

Proof. Denote $\mathrm{K}_{\eta,\xi}^{m}\zeta(\sigma)=\sigma+\Delta_2\sigma^2+\Delta_3\sigma^3+\cdots$, then

$$\Delta_2=\frac{2\left[2\eta\xi+\eta-\xi+1\right]^m}{\varsigma+1}a_2 \quad \text{and} \quad \Delta_3=\frac{6\left[6\eta\xi+2\eta-2\xi+1\right]^m}{(\varsigma+1)(\varsigma+2)}a_3. \tag{9}$$

According to the definition of the class $\mathrm{S}_{\eta,\xi}^{m}(\vartheta,\varsigma)$ there exists

$p\in\mathrm{P}$ such that $\dfrac{\sigma\left(\mathrm{K}_{\eta,\xi}^{m}\zeta(\sigma)\right)'}{\mathrm{K}_{\eta,\xi}^{m}\zeta(\sigma)}=1+\vartheta\left(p(\sigma)-1\right)$, so that

$$\frac{\sigma\left(1+2\Delta_2\sigma+3\Delta_3\sigma^2+\cdots\right)}{\sigma+\Delta_2\sigma^2+\Delta_3\sigma^3+\cdots}=1-\vartheta+\vartheta\left(1+c_1\sigma+c_2\sigma^2+\cdots\right).$$

We get by equating the coefficients of both sides

$$\Delta_2=\vartheta c_1 \quad \text{and} \quad \Delta_3=\frac{\vartheta^2c_1^2}{2}+\frac{\vartheta c_2}{2}, \tag{10}$$

so that, on account of (9) and (10)

$$a_2=\frac{\vartheta c_1(\varsigma+1)}{2\left[2\eta\xi+\eta-\xi+1\right]^m} \quad \text{and} \quad a_3=\frac{\vartheta\left(\vartheta c_1^2+c_2\right)(\varsigma+1)(\varsigma+2)}{12\left[6\eta\xi+2\eta-2\xi+1\right]^m}. \tag{11}$$

Taking (11) and Lemma 1 into account, we get

$$|a_2|=\left|\frac{\vartheta c_1(\varsigma+1)}{2\left[2\eta\xi+\eta-\xi+1\right]^m}\right| ,, \frac{|\vartheta|(\varsigma+1)}{\left[2\eta\xi+\eta-\xi+1\right]^m}, \tag{12}$$

and

$$\begin{aligned}|a_3|&=\left|\frac{\vartheta(\varsigma+1)(\varsigma+2)}{12\left[6\eta\xi+2\eta-2\xi+1\right]^m}\left[c_2-\frac{c_1^2}{2}+\frac{(1+2\vartheta)c_1^2}{2}\right]\right|\\&\leq\frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{12\left[6\eta\xi+2\eta-2\xi+1\right]^m}\left[2-\frac{|c_1|^2}{2}+\frac{|1+2\vartheta||c_1|^2}{2}\right]\\&=\frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{12\left[6\eta\xi+2\eta-2\xi+1\right]^m}\left[2+\frac{\left(|1+2\vartheta|-1\right)|c_1|^2}{2}\right]\\&\leq\frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{6\left[6\eta\xi+2\eta-2\xi+1\right]^m}\max\left\{1,\left[1+|1+2\vartheta|-1\right]\right\}.\end{aligned}$$

Thus, we have

$$|a_3| ,, \frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{6\left[6\eta\xi+2\eta-2\xi+1\right]^m}\max\left\{1,|1+2\vartheta|\right\}.$$

We can now calculate the sharpness of the estimates in (5) and (6).

Firstly, in (5) the equality holds if $c_1=2$. Alternatively, we have $p(\sigma)\equiv p_1(\sigma)=(1+\sigma)/(1-\sigma)$.

As a result,, the extremal function in $\mathrm{S}_{\eta,\xi}^{m}(\vartheta,\varsigma)$ is given by

$$\frac{\sigma\left(\mathrm{K}_{\eta,\xi}^{m}\zeta(\sigma)\right)'}{\mathrm{K}_{\eta,\xi}^{m}\zeta(\sigma)}=\frac{1+(2b-1)\sigma}{1-\sigma}. \tag{13}$$

Next, in (6), for first case, the equality holds if $c_1=c_2=2$. Therefore, the extremal functions in $\mathrm{S}_{\eta,\xi}^{m}(\vartheta,\varsigma)$ is given by (13) and for second case, the equality holds if $c_1=0, c_2=2$. Equivalently, we have $p(\sigma)\equiv p_2(\sigma)=(1+\sigma^2)/(1-\sigma^2)$. Therefore, the extremal function in $\mathrm{S}_{\eta,\xi}^{m}(\vartheta,\varsigma)$ is given by

$$\frac{\sigma\left(\mathrm{K}_{\eta,\xi}^{m}\zeta(\sigma)\right)'}{\mathrm{K}_{\eta,\xi}^{m}\zeta(\sigma)}=\frac{1+(2\vartheta-1)\sigma^2}{1-\sigma^2}.$$

Putting $\varsigma=1$ in Theorem 1, we get the following result from Orhan, Deniz and Çağlar (Orhan et al, 2012, Theorem 1).

Corollary 1. Let $m\in\mathbb{N}_0$, $\eta\geq\xi\geq0$ and $\vartheta\in\mathbb{C}\setminus\{0\}$. If $\hbar$ of the form (1) is in $\mathrm{S}_m(\vartheta,\eta,\xi)$, then

$$|a_2|\leq\frac{2|\vartheta|}{[2\eta\xi+\eta-\xi+1]^m},$$

and

$$|a_3|\leq\frac{|\vartheta|}{[6\eta\xi+2\eta-2\xi+1]^m}\max\{1,|1+2\vartheta|\}.$$

Firstly, we think functional $|a_3-\delta a_2^2|$ for $\vartheta\in\mathbb{C}\setminus\{0\}$ and $\delta\in\mathbb{C}$.

Theorem 2. Let $m,r\in\mathbb{N}_0$, $\eta\geq\xi\geq0$, $\vartheta\in\mathbb{C}\setminus\{0\}$ and $\hbar\in\mathrm{S}_{\eta,\xi}^{m}(b,\varsigma)$. Then for $\delta\in\mathbb{C}$

$$|a_3-\delta a_2^2|\leq\frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{6[6\eta\xi+2\eta-2\xi+1]^m}\max\left\{1,\left|1+2\vartheta-\frac{6\delta\vartheta(\varsigma+1)[6\eta\xi+2\eta-2\xi+1]^m}{(\varsigma+2)[2\eta\xi+\eta-\xi+1]^{2m}}\right|\right\}.$$

There is a function $\mathrm{S}_{\eta,\xi}^{m}(\vartheta,\varsigma)$ that ensures equality for each δ.

Proof. From (11), we have

$$\begin{aligned}a_3-\delta a_2^2&=\frac{\vartheta(\varsigma+1)(\varsigma+2)}{12[6\eta\xi+2\eta-2\xi+1]^m}\left[c_2+\vartheta c_1^2\right]-\delta\frac{\vartheta^2c_1^2(\varsigma+1)^2}{4[2\eta\xi+\eta-\xi+1]^{2m}}\\&=\frac{\vartheta(\varsigma+1)(\varsigma+2)}{12[6\eta\xi+2\eta-2\xi+1]^m}\left[c_2+\vartheta c_1^2-\frac{3\delta\vartheta(\varsigma+1)[6\eta\xi+2\eta-2\xi+1]^m}{(\varsigma+2)[2\eta\xi+\eta-\xi+1]^{2m}}c_1^2\right]\\&=\frac{\vartheta(\varsigma+1)(\varsigma+2)}{12[6\eta\xi+2\eta-2\xi+1]^m}\left[c_2-\frac{c_1^2}{2}+\frac{c_1^2}{2}\left(1+2\vartheta-\frac{6\delta\vartheta(\varsigma+1)[6\eta\xi+2\eta-2\xi+1]^m}{(\varsigma+2)[2\eta\xi+\eta-\xi+1]^{2m}}\right)\right].\end{aligned}$$

Then, with the help of Lemma 1, we get

$$\left|a_3-\delta a_2^2\right| \le \frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{12\left[6\eta\xi+2\eta-2\xi+1\right]^m}\left[2-\frac{\left|c_1^2\right|}{2}+\frac{\left|c_1^2\right|}{2}\left|1+2\vartheta-\frac{6\delta\vartheta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^m}{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}\right|\right]$$

$$=\frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{12\left[6\eta\xi+2\eta-2\xi+1\right]^m}\left[2+\frac{\left|c_1^2\right|}{2}\left(\left|1+2\vartheta-\frac{6\delta\vartheta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^m}{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}\right|-1\right)\right]$$

$$\le\frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{6\left[6\eta\xi+2\eta-2\xi+1\right]^m}\max\left\{1,\left|1+2\vartheta-\frac{6\delta\vartheta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^m}{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}\right|\right\}.$$

For $\varsigma=1$ in Theorem 2, we get the following result obtained by Orhan, Deniz and Çağlar (Orhan et al, 2012, Theorem 1).

Corollary 2. Let $m\in\mathbb{N}_0$, $\eta\ge\xi\ge0$ and $\vartheta\in\mathbb{C}\setminus\{0\}$. If $\hbar$ of the form (1) is in $\mathrm{S}_m(\vartheta,\eta,\xi)$, then for $\delta\in\mathbb{C}$

$$\left|a_3-\delta a_2^2\right|\le\frac{|\vartheta|}{\left[6\eta\xi+2\eta-2\xi+1\right]^m}\max\left\{1,\left|1+2\vartheta-\frac{4\delta\vartheta\left[6\eta\xi+2\eta-2\xi+1\right]^m}{\left[2\eta\xi+\eta-\xi+1\right]^{2m}}\right|\right\}.$$

There is a function $\mathrm{S}_m(\vartheta,\eta,\xi)$ that ensures equality for each δ.

We consider the case where δ and ϑ are real. Then we have:

Theorem 3. Let $m,\varsigma\in\mathbb{N}_0$, $\eta\ge\xi\ge0$, $\vartheta>0$ and $\hbar\in\mathrm{S}_{\eta,\xi}^m(\vartheta,\varsigma)$. Then for $\delta\in\mathbb{R}$ we have

$$\left|a_3-\delta a_2^2\right|\le\begin{cases}\dfrac{\vartheta(\varsigma+1)(\varsigma+2)}{6\left[6\eta\xi+2\eta-2\xi+1\right]^m}\left[1+2\vartheta\left(1-\dfrac{3\delta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^m}{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}\right)\right] & \text{if } \delta\le A\le B,\\[2ex] \dfrac{\vartheta(\varsigma+1)(\varsigma+2)}{6\left[6\eta\xi+2\eta-2\xi+1\right]^m} & \text{if } A<\delta<B,\\[2ex] \dfrac{\vartheta(\varsigma+1)(\varsigma+2)}{6\left[6\eta\xi+2\eta-2\xi+1\right]^m}\left[2\vartheta\left(\dfrac{3\delta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^m}{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}-1\right)-1\right] & \text{if } \delta\ge B,\end{cases}$$

where $A=\dfrac{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}{3(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^m}$ and $B=\dfrac{(1+2\vartheta)(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}{6\vartheta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^m}$. There is a function $\mathrm{S}_{\eta,\xi}^m(\vartheta,\varsigma)$ such that equality holds for each δ.

Proof. First, let $\delta\le\dfrac{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}{3(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^m}\le\dfrac{(1+2\vartheta)(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}{6\vartheta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^m}$. In this case, (11) and Lemma 1 give

$$\left|a_3-\delta a_2^2\right|\le\frac{\vartheta(\varsigma+1)(\varsigma+2)}{12\left[6\eta\xi+2\eta-2\xi+1\right]^m}\left[2-\frac{\left|c_1^2\right|}{2}+\frac{\left|c_1^2\right|}{2}\left(1+2\vartheta-\frac{6\delta\vartheta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^m}{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}\right)\right]$$

$$\le\frac{\vartheta(\varsigma+1)(\varsigma+2)}{6\left[6\eta\xi+2\eta-2\xi+1\right]^m}\left[1+2\vartheta\left(1-\frac{3\delta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^m}{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}\right)\right].$$

Now, let $\frac{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}{3(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}<\delta<\frac{(1+2\vartheta)(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}{6\vartheta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}$. Then, using the above calculations, we obtain

$$\left|a_3-\delta a_2^2\right|\leq\frac{\vartheta(\varsigma+1)(\varsigma+2)}{6\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}.$$

Finally, if $\delta\geq\frac{(1+2\vartheta)(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}{6\vartheta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}$, then

$$\begin{aligned}\left|a_3-\delta a_2^2\right|&\leq\frac{\vartheta(\varsigma+1)(\varsigma+2)}{12\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}\left[2-\frac{\left|c_1^2\right|}{2}+\frac{\left|c_1^2\right|}{2}\left(\frac{6\delta\vartheta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}-2\vartheta-1\right)\right]\\&\leq\frac{\vartheta(\varsigma+1)(\varsigma+2)}{6\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}\left[2\vartheta\left(\frac{3\delta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}-1\right)-1\right].\end{aligned}$$

Taking $\varsigma=1$ in Theorem 3, we get the following result from Orhan, Deniz and Çağlar (Orhan et al, 2012, Theorem 2).

Corollary 3. Let $m\in\mathbb{N}_0$, $\eta\geq\xi\geq0$ and $\vartheta>0$. If $\hbar$ of the form (1) is in $\mathrm{S}_m(\vartheta,\eta,\xi)$, then for $\delta\in\mathbb{R}$

$$\left|a_3-\delta a_2^2\right|\leq\begin{cases}\frac{\vartheta}{\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}\left[1+2\vartheta\left(1-\frac{2\delta\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}{\left[2\eta\xi+\eta-\xi+1\right]^{2m}}\right)\right] & \text{if } \delta\leq A\leq B,\\ \frac{\vartheta}{\left[6\eta\xi+2\eta-2\xi+1\right]^{m}} & \text{if } A<\delta<B,\\ \frac{\vartheta}{\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}\left[2\vartheta\left(\frac{2\delta\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}{\left[2\eta\xi+\eta-\xi+1\right]^{2m}}-1\right)-1\right] & \text{if } \delta\geq B,\end{cases}$$

where $A=\frac{\left[2\eta\xi+\eta-\xi+1\right]^{2m}}{2\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}$ and $B=\frac{(1+2\vartheta)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}{4\vartheta\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}$. There is a function $\mathrm{S}_m(\vartheta,\eta,\xi)$ that ensures equality for each δ.

We now get a solution of the Fekete-Szegö inequality and bounds of coefficients in $\mathrm{C}_{\eta,\xi}^{m}(\vartheta,\varsigma)$.

Theorem 4. Let $m,\varsigma\in\mathbb{N}_0$, $\eta\geq\xi\geq0$, $\delta\in\mathbb{C}$ and $\vartheta\in\mathbb{C}\setminus\{0\}$. If $\hbar$ of the form (1) is in $\mathrm{C}_{\eta,\xi}^{m}(b,r)$, then

$$\left|a_2\right|\leq\frac{\left|\vartheta\right|(\varsigma+1)}{2\left[2\eta\xi+\eta-\xi+1\right]^{m}},\quad \left|a_3\right|\leq\frac{\left|\vartheta\right|(\varsigma+1)(\varsigma+2)}{18\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}\max\left\{1,\left|1+2\vartheta\right|\right\}.$$

and

$$\left|a_3-\delta a_2^2\right|\leq\frac{\left|\vartheta\right|(\varsigma+1)(\varsigma+2)}{18\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}\max\left\{1,\left|1+2\vartheta-\frac{9\delta\vartheta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^{m}}{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}\right|\right\}.$$

Proof. Denote $\mathrm{K}_{\eta,\xi}^{m}\zeta(\sigma)=\sigma+\Delta_2\sigma^2+\Delta_3\sigma^3+\cdots$, then

$$\Delta_2 = \frac{2\left[2\eta\xi + \eta - \xi + 1\right]^m}{\varsigma + 1} a_2 \text{ and } \Delta_3 = \frac{6\left[6\eta\xi + 2\eta - 2\xi + 1\right]^m}{(\varsigma + 1)(\varsigma + 2)} a_3. \quad (14)$$

According to the definition of the class $C^m_{\eta,\xi}(\vartheta, \varsigma)$, there exists

$p \in P$ such that $\dfrac{\sigma\left(K^m_{\eta,\xi}\zeta(\sigma)\right)''}{\left(K^m_{\eta,\xi}\zeta(\sigma)\right)'} = 1 + \vartheta\left(p(\sigma) - 1\right)$, so that

$$\frac{\sigma\left(2\Delta_2 + 6\Delta_3\sigma + \cdots\right)}{1 + 2\Delta_2\sigma + 3\Delta_3\sigma^2 + \cdots} = \vartheta\left(1 + c_1\sigma + c_2\sigma^2 + \cdots\right) - \vartheta.$$

We get by equating the coefficients of both sides

$$\Delta_2 = \frac{\vartheta c_1}{2} \text{ and } 6\Delta_3 - 4\Delta_2^2 = \vartheta c_2, \quad (15)$$

so that, on account of (14) and (15)

$$a_2 = \frac{\vartheta c_1(\varsigma + 1)}{4\left[2\eta\xi + \eta - \xi + 1\right]^m} \text{ and } a_3 = \frac{\vartheta\left(\vartheta c_1^2 + c_2\right)(\varsigma + 1)(\varsigma + 2)}{36\left[6\eta\xi + 2\eta - 2\xi + 1\right]^m}. \quad (16)$$

From (16) and Lemma 1, we get

$$|a_2| = \left|\frac{\vartheta c_1(\varsigma + 1)}{4\left[2\eta\xi + \eta - \xi + 1\right]^m}\right| ,, \frac{|\vartheta|(\varsigma + 1)}{2\left[2\eta\xi + \eta - \xi + 1\right]^m}, \quad (17)$$

and

$$|a_3| = \left|\frac{\vartheta(\varsigma + 1)(\varsigma + 2)}{36\left[6\eta\xi + 2\eta - 2\xi + 1\right]^m}\left[c_2 - \frac{c_1^2}{2} + \frac{(1 + 2\vartheta)c_1^2}{2}\right]\right|$$

$$\leq \frac{|\vartheta|(\varsigma + 1)(\varsigma + 2)}{36\left[6\eta\xi + 2\eta - 2\xi + 1\right]^m}\left[2 - \frac{|c_1|^2}{2} + \frac{|1 + 2\vartheta||c_1|^2}{2}\right]$$

$$= \frac{|\vartheta|(\varsigma + 1)(\varsigma + 2)}{36\left[6\eta\xi + 2\eta - 2\xi + 1\right]^m}\left[2 + \frac{\left(|1 + 2\vartheta| - 1\right)|c_1|^2}{2}\right]$$

$$\leq \frac{|\vartheta|(\varsigma + 1)(\varsigma + 2)}{18\left[6\eta\xi + 2\eta - 2\xi + 1\right]^m} \max\left\{1, \left[1 + |1 + 2\vartheta| - 1\right]\right\}.$$

Thus, we have

$$|a_3| ,, \frac{|\vartheta|(\varsigma + 1)(\varsigma + 2)}{18\left[6\eta\xi + 2\eta - 2\xi + 1\right]^m} \max\left\{1, |1 + 2\vartheta|\right\}.$$

Then, with the help of Lemma 1, we get

$$\left|a_3-\delta a_2^2\right| \le \frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{36\left[6\eta\xi+2\eta-2\xi+1\right]^m}\left[2-\frac{\left|c_1^2\right|}{2}+\frac{\left|c_1^2\right|}{2}\left|1+2\vartheta-\frac{9\delta\vartheta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^m}{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}\right|\right]$$

$$=\frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{36\left[6\eta\xi+2\eta-2\xi+1\right]^m}\left[2+\frac{\left|c_1^2\right|}{2}\left(\left|1+2\vartheta-\frac{9\delta\vartheta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^m}{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}\right|-1\right)\right]$$

$$\le\frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{18\left[6\eta\xi+2\eta-2\xi+1\right]^m}\max\left\{1,\left|1+2\vartheta-\frac{9\delta\vartheta(\varsigma+1)\left[6\eta\xi+2\eta-2\xi+1\right]^m}{(\varsigma+2)\left[2\eta\xi+\eta-\xi+1\right]^{2m}}\right|\right\}.$$

Putting $\varsigma=1$ in Theorem 4, we get the following result investigated by Orhan, Deniz and Çağlar (Orhan et al, 2012, Theorem 4).

Corollary 4. Let $m\in\mathbb{N}_0$, $\eta\ge\xi\ge0$, $\delta\in\mathbb{C}$ and $\vartheta\in\mathbb{C}\setminus\{0\}$. If $\hbar$ of the form (1) is in $C_m(\vartheta,\eta,\xi)$, then

$$|a_2|\le\frac{|\vartheta|}{\left[2\eta\xi+\eta-\xi+1\right]^m},\quad |a_3|\le\frac{|\vartheta|}{3\left[6\eta\xi+2\eta-2\xi+1\right]^m}\max\left\{1,|1+2\vartheta|\right\}.$$

and

$$\left|a_3-\delta a_2^2\right|\le\frac{|\vartheta|}{3\left[6\eta\xi+2\eta-2\xi+1\right]^m}\max\left\{1,\left|1+2\vartheta-\frac{6\delta\vartheta\left[6\eta\xi+2\eta-2\xi+1\right]^m}{\left[2\eta\xi+\eta-\xi+1\right]^{2m}}\right|\right\}.$$

CONCLUSION

In our present study, we have introduced and studied the coefficient problems related with each of the two new subclasses $S_{\eta,\xi}^m(\vartheta,\varsigma)$ and $C_{\eta,\xi}^m(\vartheta,\varsigma)$ of the class of analytic functions defined by the combination of factional differential and Noor integral operators in the open unit disk. We have studied some interesting results such as the Fekete-Szegö inequalities according to the case of δ. Also, for certain values of the parameters, we re-obtain some special classes studied earlier by various authors.

Conflict of Interest

There is no conflict of interest.

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