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On Isometric Immersions of Null Manifolds into Semi-Riemannian Space Forms of Arbitrary Index

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

A null manifold (M, g) is a differentiable manifold M endowed with a degenerate metric tensor g. In this work we provide sufficient conditions for a null manifold to be isometrically immersed as a hypersurface into a simple connected semi-Riemannian manifold $\mathbb{Q}_{c,q}$ of constant sectional curvature c and index q .

Keywords: *degenerate metrics; isometric immersions; Gauss-Codazzi-Ricci equations.*

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1. Introduction and main results

One of the most fundamental questions in Riemannian submanifold theory consists on determining the existence and uniqueness of isometric immersions of a specific Riemannian manifold (M, g) into some ambient space $(\bar{M}, \bar{g})$. Its origins can be traced back to a celebrated theorem of Bonnet, that established the existence of an immersed surface $\Phi : U \subset \mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ that realizes any two triples (E, F, G), (e, f, g) of differentiable functions as coefficients of its first and second fundamental forms, provided that the structure equations are satisfied. Furthermore, any two such immersions are related by a rigid motion in $\mathbb{R}^3$ [4]. In fact, the equations of Gauss, Codazzi and Ricci must hold on any submanifold of a given ambient space, thus they are necessary conditions for the existence of isometric immersions. The standard formulation of the Fundamental Theorem of Submanifolds states that these equations are sufficient conditions as well, up to isometries, for a Riemannian n-manifold to admit a local isometric immersion into spaces of constant sectional curvature with dimension greater than n. For example, refer to the classical results in [16, 19, 20, 29, 30], and [6, 9, 10] for recent developments.

This problem has also been addressed in the semi-Riemannian setting [18, 28] and to this day is an active area of research [7, 12, 23, 24, 25]. Furthermore, applications to general relativity theory arise naturally when considering the Lorentzian scenario [21, 28]. Recall that when the ambient manifold is Lorentzian the causal character of tangent spaces to a submanifold can be timelike, spacelike or null (lightlike). If all tangent spaces share a common causal character, then the submanifold is called timelike, spacelike or null, accordingly, being the latter case the main focus of our present work. Some authors have considered isometric immersions of manifolds with indefinite metric [1, 17, 26], and the problem of reduction of the codimension for lightlike isotropic submanifolds was studied in [5]. The problem of existence and uniqueness of isometric immersions of lightlike (or null) k-manifolds M into a semi-Euclidean space $\mathbb{R}^{k+1}_q$ of arbitrary index $q \geq 1$ was solved by K. L. Duggal and A. Bejancu (see [14, Theorem 4.1]), assuming the existence of a semi-Riemannian metric on a suitable vector bundle. In this paper, we strengthen the Fundamental Theorem of K. Duggal and A. Bejancu in two different ways. First, by providing an explicit construction of the semi-Riemannian metric whose existence is only assumed in the Fundamental Theorem in [14]. Furthermore, we generalize the Fundamental Theorem

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by proving the existence of isometric immersions of a null manifold M^k into a semi-Riemannian space form $\mathbb{Q}^{k+1}_{c,q}$ of arbitrary index $q \geq 1$ and constant sectional curvature c.

In order to state our main result, let us note that if M is a smooth submanifold of $\bar{M}$, then we can find suitable null versions of the Gauss-Codazzi equations, (refer for instance to Section 2 or to equations (2.1.9)-(2.1.10) in [15]). Conversely, it is natural to investigate, just as in the Euclidean setting, if such null structure equations give necessary and sufficient conditions for the existence of an isometric immersion $f\colon M \to \bar{M}$. Our main result reads as follows.

Theorem A (See Theorem 5.2)**.** *Let $(M, g, S(TM))$ be a null simply connected manifold of dimension $n+1$, with a screen distribution of dimension n and index $q-1$. Let ϵ be a vector bundle over M of dimension 1 and let $\overline{g}^\gamma$ be the metric over $\gamma = TM \oplus \epsilon$ given by (5.30). Further, let ∇^γ be a connection over γ which satisfies (5.31) - (5.37). Moreover, suppose that the Gauss-Codazzi-Ricci equations for $c = \sigma/r^2$ given by (5.38)-(5.41) hold for γ, where $\sigma = sign(c)$.*

Then there exists an isometric immersion $f : M \to \mathbb{Q}^{n+2}_{c,q}$, such that $\overline{f} = i \circ f$. Furthermore, there exist an isometry of vector bundles $\phi : \delta \oplus \epsilon \to \mathrm{tr}^f(TM)$, such that

$$\begin{aligned} h^f &= \phi h, \\ \nabla^{t,f}\phi &= \phi\nabla^t. \end{aligned}$$

Moreover, let $f, g : M \to \mathbb{Q}^{n+2}_{c,q}$ be two such isometric immersions of a null manifold and suppose there exists an isometry of vector bundles $\overline{\psi} : f^*T\mathbb{Q}^{n+2}_q \to f^*T\mathbb{Q}^{n+2}_q$ such that

$$\overline{\psi}(f_*) = g_*$$

and $\overline{\psi}|_{\mathrm{tr}^f(TM)} = \psi$ satisfies

$$\begin{aligned} \psi(\phi^f(N)) &= \phi^g(N), \\ \psi(\phi^f(u)) &= \phi^g(u). \end{aligned}$$

Then there exists an isometry $\tau : \mathbb{Q}^{n+2}_q \to \mathbb{Q}^{n+2}_q$, such that

$$\tau f = g \quad \textit{and} \quad \tau_*|_{\mathrm{tr}^f(TM)} = \psi$$

The paper is divided as follows: in Section 2 we discuss the basic structure equations for a null submanifold. Then in Section 3 we establish the appropriate data and compatibility conditions which allows us to give a proof in section 4 of the Fundamental Theorem for immersions in Lorentz-Minkowski spaces. Finally in section 5 we show our main result.

2. The null Gauss-Codazzi-Ricci equations

In this section we establish our notation; for a detailed account, refer to [11] and [15]. Let $\bar{M}^{n+2}$ be a Lorentzian manifold with metric $\bar{g}$ and M a $(n+1)$-dimensional manifold. If $f\colon M \to \bar{M}$ is an immersion, the *metric on M induced by f* is defined by

$$g(X, Y) = \bar{g}(f_*X, f_*Y)$$

for all $X, Y \in \Gamma(TM)$ and we say that f is an *isometric immersion*.

M is a *null* manifold if g is a degenerate metric. Equivalently, M is null if there exists a 1-dimensional distribution over M, the *radical distribution* $\mathrm{Rad}(TM) \subset TM$, such that $g(X, U) = 0$ for any $X \in \Gamma(TM)$ and $U \in \Gamma(\mathrm{Rad}(TM))$. We choose and fix hereafter a n-dimensional distribution over M, called a *screen distribution* $S(TM)$, which is complementary to the radical distribution at each point; that is,

$$TM = S(TM) \oplus \mathrm{Rad}(TM). \tag{2.1}$$

Let $f^*T\bar{M}$ be the bundle over M such that the fiber at a point p is $T_{f(p)}\bar{M}$. In this context, there is a well-defined 1-dimensional distribution $\mathrm{tr}^f(TM)$ over M, called the *transversal distribution*, such that

$$f^*T\bar{M} = f_*TM \oplus \mathrm{tr}^f(TM) \tag{2.2}$$

and

$$\bar{g}(f_*X, V) = 0, \quad \bar{g}(V, V) = 0, \quad \bar{g}(f_*U, V') \neq 0,$$

for any $X \in \Gamma(TM)$, any $V \in \Gamma(\mathrm{tr}^f(TM))$ and any nowhere vanishing sections $U \in \Gamma(\mathrm{Rad}(TM))$ and $V' \in \Gamma(\mathrm{tr}^f(TM))$.

The Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$ relative to $\bar{g}$ induces naturally a connection $\bar{\nabla}^f$ on $f^*T\bar{M}$ defined by $\bar{\nabla}^f_X f_*Y = \bar{\nabla}_{f_*X} f_*Y$. For $X, Y \in \Gamma(TM)$, we use decomposition (2.2) to write the Gauss formula as

$$\bar{\nabla}^f_X f_*Y = f_*\nabla_X Y + h^f(X, Y). \tag{2.3}$$

where $X, Y, \nabla_X Y \in \Gamma(TM)$ and $h^f(X, Y) \in \Gamma(\mathrm{tr}^f(TM))$. ∇ is a torsion-free connection on TM, while h^f is a symmetric section of $\mathrm{Hom}^2(TM, TM; \mathrm{tr}^f(TM))$. Note that ∇ is not a metric connection: If $X, Y, Z \in \Gamma(TM)$, then

$$\begin{aligned}
0 = (\bar{\nabla}^f_X \bar{g})(f_*Y, f_*Z) &= X(\bar{g}(f_*Y, f_*Z)) - \bar{g}(\bar{\nabla}^f_X f_*Y, f_*Z) - \bar{g}(f_*Y, \bar{\nabla}^f_X f_*Z) \\
&= X(g(Y, Z)) - \bar{g}(f_*\nabla_X Y + h^f(X, Y), f_*Z) - \bar{g}(f_*Y, f_*\nabla_X Z + h^f(X, Z)) \\
&= X(g(Y, Z)) - \bar{g}(f_*\nabla_X Y, f_*Z) - \bar{g}(f_*Y, f_*\nabla_X Z) \\
&\quad - \bar{g}(h^f(X, Y), f_*Z) - \bar{g}(f_*Y, h^f(X, Z)) \\
&= X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\
&\quad - \bar{g}(h^f(X, Y), f_*Z) - \bar{g}(f_*Y, h^f(X, Z))
\end{aligned}$$

therefore,

$$(\nabla_X g)(Y, Z) = \bar{g}(h^f(X, Y), f_*Z) + \bar{g}(f_*Y, h^f(X, Z)).$$

Given $X \in \Gamma(TM)$ and $V \in \Gamma(\mathrm{tr}^f(TM))$, we use again (2.2) to write the Weingarten formula

$$\bar{\nabla}^f_X V = -f_*A^f_V X + \nabla^{f,t}_X V; \tag{2.4}$$

here A^f_V is the shape operator of M in $\bar{M}$, while $\nabla^{f,t}$ is a connection on $\mathrm{tr}^f(TM)$.

Let P denote the projection of TM onto $S(TM)$ relative to decomposition (2.1). We have the following Gauss-Weingarten equations in TM:

$$\begin{aligned}
\nabla_X PY &= \nabla^*_X PY + h^*(X, PY), \\
\nabla_X U &= -A^*_U X + \nabla^{*t}_X U,
\end{aligned}$$

where ∇^* and ∇^{*t} are connections on $S(TM)$ and $\mathrm{Rad}(TM)$, respectively. A^*_U is the *screen shape operator* and $h^* \in \mathrm{Hom}^2(TM, S(TM), \mathrm{Rad}(TM))$. In fact, ∇^* is a metric connection associated to $g|_{S(TM)}$.

Consider a vector field $U \in \Gamma(\mathrm{Rad}(TM))$. Since $\bar{\nabla}^f$ is a metric connection, using (2.3) with $Y = U$ and taking the scalar product with U we obtain $h^f(X, U) = 0$. Then,

$$\begin{aligned}
\bar{g}(h^f(X, Y), f_*U) = \bar{g}(\bar{\nabla}^f_X f_*Y, f_*U) &= -\bar{g}(f_*Y, \bar{\nabla}^f_X f_*U) \\
&= -\bar{g}(f_*Y, f_*\nabla_X U + h^f(X, U)) \\
&= -g(Y, \nabla_X U) = g(Y, A^*_U X).
\end{aligned}$$

From (2.4) it is easy to see that $\bar{g}(f_*A^f_V X, V) = 0$ for any $V \in \Gamma(\mathrm{tr}^f(TM))$, which means that A^f_V is $S(TM)$-valued; therefore to determine it we calculate its product with any vector field of the form PY, where $Y \in \Gamma(TM)$:

$$\begin{aligned}
g(PY, A^f_V X) = \bar{g}(f_*PY, f_*A^f_V X) &= -\bar{g}(f_*PY, \bar{\nabla}^f_X V) \\
&= \bar{g}(\bar{\nabla}^f_X f_*PY, V) = \bar{g}(f_*\nabla_X PY + h^f(X, Y), V) \\
&= \bar{g}(f_*(\nabla^*_X PY + h^*(X, PY)), V) = \bar{g}(f_*h^*(X, PY), V).
\end{aligned}$$

In short, we have the following relations between the shape operators and the second fundamental forms h^f and h^*:

$$\bar{g}(h^f(X, Y), f_*U) = g(Y, A^*_U X), \quad g(PY, A^f_V X) = \bar{g}(f_*h^*(X, PY), V). \tag{2.5}$$

Remark 2.1. Since locally any immersion is an embedding, for the sake of clarity we will work locally, identify M with $f(M)$, consider f as the inclusion map and omit the reference to it. We will return to use f only in the statement and proof of Theorem 4.1.

Denote by $\bar{R}$, R, R^*, R^t and R^{*t} the curvature tensors of $\bar{\nabla}$, ∇, ∇^*, ∇^t and ∇^{*t}, respectively, with the sign convention in [27]; for example,

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X.Y]} Z.$$

Also, let

$$(\nabla_X h)(Y,Z) = \nabla^t_X (h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$
$$(\nabla_X h^*)(Y,PZ) = \nabla^{*t}_X (h^*(Y,PZ)) - h^*(\nabla_X Y, PZ) - h^*(Y, \nabla^*_X PZ).$$

Proposition 2.1 (see [2] and [14]). *Let $(\bar{M}^{n+2}, \bar{g})$ be a semi-Riemannian manifold and M a null hypersurface of $\bar{M}$. For any $X,Y,Z \in \Gamma(TM)$, $U \in \Gamma(\mathrm{Rad}(TM))$ and $V \in \Gamma(\mathrm{tr}(TM))$ we have*

1. *The Gauss-Codazzi equations*

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + A_{h(Y,Z)}X - A_{h(X,Z)}Y \\ &\quad + (\nabla_Y h)(X,Z) - (\nabla_X h)(Y,Z), \\ R(X,Y)PZ &= R^*(X,Y)PZ + A^*_{h^*(Y,PZ)}X - A^*_{h^*(X,PZ)}Y \\ &\quad + (\nabla_Y h^*)(X,PZ) - (\nabla_X h^*)(Y,PZ). \end{aligned} \tag{2.6}$$

2. *The Ricci equations*

$$\begin{aligned} \bar{R}(X,Y)U &= \nabla^*_X A^*_U Y - \nabla^*_Y A^*_U X + A^*_{\nabla^{*t}_Y U}X - A^*_{\nabla^{*t}_X U}Y \\ &\quad + h^*(X, A^*_U Y) - h^*(Y, A^*_U X) + R^{*t}(X,Y)U \\ &\quad + h(X, A^*_U Y) - h(Y, A^*_U X), \\ \bar{R}(X,Y)V &= \nabla^*_X A_V Y - \nabla^*_Y A_V X + A_{\nabla^t_Y V}X - A_{\nabla^t_X V}Y \\ &\quad + h^*(X, A_V Y) - h^*(Y, A_V X) \\ &\quad + h(X, A_V Y) - h(Y, A_V X) + R^t(X,Y)V. \end{aligned} \tag{2.7}$$

We arranged the above Gauss-Codazzi-Ricci equations so that each line is a component in $S(TM)$, $\mathrm{Rad}(TM)$ or $\mathrm{tr}(TM)$, depending on the case. In particular, if $\bar{M}$ has zero constant curvature, each line in the right hand side of (2.6) and (2.7) vanishes, implying

$$\begin{aligned} R(X,Y)Z &= A_{h(X,Z)}Y - A_{h(Y,Z)}X, \\ (\nabla_X h)(Y,Z) &= (\nabla_Y h)(X,Z), \\ \nabla^*_X A^*_U Y - \nabla^*_Y A^*_U X &= A^*_{\nabla^{*t}_X U}Y - A^*_{\nabla^{*t}_Y U}X, \\ h^*(X, A^*_U Y) &= h^*(Y, A^*_U X) - R^{*t}(X,Y)U, \\ h(X, A^*_U Y) &= h(Y, A^*_U X), \\ \nabla^*_X A_V Y - \nabla^*_Y A_V X &= A_{\nabla^t_X V}Y - A_{\nabla^t_Y V}X, \\ h^*(X, A_V Y) &= h^*(Y, A_V X), \\ h(X, A_V Y) &= h(Y, A_V X) - R^t(X,Y)V. \end{aligned} \tag{2.8}$$

3. Compatibility conditions

In order to set up the general framework for our constructions, we consider the following data:

- M is a null $(n+1)$-dimensional manifold with metric g.
- $S(TM)$ is an n-dimensional screen distribution on M; note that g restricted to $S(TM)$ is a Riemannian metric.
- $\mathrm{Rad}(TM)$ is the 1-dimensional radical distribution, with

$$TM = S(TM) \oplus \mathrm{Rad}(TM). \tag{3.1}$$

- ∇ is a torsion-free connection on M, such that restricted to $S(TM)$ is a metric connection relative to g.

- Decomposition (3.1) gives rise to the following Gauss-Weingarten equations on M:

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY),$$
$$\nabla_X U = -A^*_U X + \nabla^{*t}_X U,$$

  where P is the projection of TM onto $S(TM)$, $X, Y \in \Gamma(TM)$ and $U \in \Gamma(\mathrm{Rad}(TM))$.
- $\mathcal{E}$ is a 1-dimensional vector bundle over M, playing the role of $\mathrm{tr}(TM)$.

Following [11, p. 37] we define the bundle

$$\bar{\mathcal{E}} = TM \oplus \mathcal{E} = S(TM) \oplus \mathrm{Rad}(TM) \oplus \mathcal{E}.$$

We will define a metric $\bar{g}^{\bar{\mathcal{E}}}$ on this bundle as follows. Let P, P_R and $P_{\mathcal{E}}$ denote the projections of $\bar{\mathcal{E}}$ onto $S(TM)$, $\mathrm{Rad}(TM)$ and $\mathcal{E}$, respectively; also, let $g_R, g_{\mathcal{E}}$ be Riemannian metrics on $\mathrm{Rad}(TM)$ and $\mathcal{E}$, respectively, and consider the product metric

$$\bar{g} = P^*g + P^*_R g_R + P^*_{\mathcal{E}} g_{\mathcal{E}}$$

on $\bar{\mathcal{E}}$. Take $\{\xi, N\}$ a $\bar{g}$-orthonormal 2-frame with respect to this metric, such that $\mathrm{span}(\xi) = \mathrm{Rad}(TM)$ and $\mathrm{span}(N) = \mathcal{E}$, respectively.

Definition 3.1. Let $X, Y \in \Gamma(\bar{\mathcal{E}})$. Then

$$\bar{g}^{\bar{\mathcal{E}}}(X, Y) = \bar{g}(X, Y) - (\bar{g}(X, \xi) - \bar{g}(X, N))(\bar{g}(Y, \xi) - \bar{g}(Y, N)). \tag{3.2}$$

Proposition 3.1. *$\bar{g}^{\bar{\mathcal{E}}}$ is a Lorentzian metric on $\bar{\mathcal{E}}$. Moreover, if $X, Y \in \Gamma(S(TM))$, then*

1. $\bar{g}^{\bar{\mathcal{E}}}(X, Y) = g(X, Y)$;
2. $\bar{g}^{\bar{\mathcal{E}}}(X, \xi) = \bar{g}^{\bar{\mathcal{E}}}(X, N) = 0$;
3. $\bar{g}^{\bar{\mathcal{E}}}(\xi, \xi) = \bar{g}^{\bar{\mathcal{E}}}(N, N) = 0$;
4. $\bar{g}^{\bar{\mathcal{E}}}(\xi, N) = 1$.

Proof. Straightforward calculations give the enumerated properties of $\bar{g}^{\bar{\mathcal{E}}}$. It is clear from the definition that $\bar{g}^{\bar{\mathcal{E}}}(X, Y)$ is bilinear and symmetric in X, Y. Since $\bar{g}^{\bar{\mathcal{E}}} \mid_{S(TM)} = g \mid_{S(TM)}$, $\bar{g}^{\bar{\mathcal{E}}}$ is Riemannian on $S(TM)$. On the other hand, properties 3 and 4 imply $\bar{g}^{\bar{\mathcal{E}}}$ restricted to $\mathrm{Rad}(TM) \oplus \mathcal{E}$ is Lorentzian; therefore it is also a Lorentzian metric on $\bar{\mathcal{E}}$. □

We define a symmetric operator $h^{\mathcal{E}} \in \mathrm{Hom}^2(TM, TM, \mathcal{E})$ playing the role of the second fundamental form h, by

$$\bar{g}^{\bar{\mathcal{E}}}(h^{\mathcal{E}}(X, Y), U) = g(Y, A^*_U X), \tag{3.3}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(\mathrm{Rad}(TM))$.

Given a connection $\nabla^{\mathcal{E}} : \Gamma(TM) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ on $\mathcal{E}$, we define a connection on $\bar{\mathcal{E}}$, analogous to (2.3) and (2.4):

$$\bar{\nabla}^{\bar{\mathcal{E}}}_X Y = \nabla_X Y + h^{\mathcal{E}}(X, Y),$$
$$\bar{\nabla}^{\bar{\mathcal{E}}}_X V = -A_V X + \nabla^{\mathcal{E}}_X V.$$

where $X, Y \in \Gamma(TM)$ and $V \in \Gamma(\mathcal{E})$; we define the $S(TM)$-valued operator A_V by

$$g(PY, A_V X) = \bar{g}^{\bar{\mathcal{E}}}(h^*(X, PY), V). \tag{3.4}$$

Our next result establishes necessary and sufficient conditions for the metric compatibility of the connection $\bar{\nabla}^{\bar{\mathcal{E}}}$.

Proposition 3.2. *$\bar{\nabla}^{\bar{\mathcal{E}}}$ is a metric connection on $\bar{\mathcal{E}}$ relative to $\bar{g}^{\bar{\mathcal{E}}}$ if and only if*

$$\begin{aligned} \bar{g}^{\bar{\mathcal{E}}}(h^*(X, Y), V) &= \bar{g}^{\bar{\mathcal{E}}}(Y, A_V X), \\ X(\bar{g}^{\bar{\mathcal{E}}}(U, V)) &= \bar{g}^{\bar{\mathcal{E}}}(\nabla^{*t}_X U, V) + \bar{g}^{\bar{\mathcal{E}}}(U, \nabla^{\mathcal{E}}_X V), \end{aligned} \tag{3.5}$$

for any $X \in \Gamma(TM)$, $Y \in \Gamma(S(TM))$, $U \in \Gamma(\mathrm{Rad}(TM))$, $V \in \Gamma(\mathcal{E})$.

Proof. We have to prove that

$$(\bar{\nabla}^{\bar{\mathcal{E}}}_X \bar{g}^{\bar{\mathcal{E}}})(Y,Z) = X(\bar{g}^{\bar{\mathcal{E}}}(Y,Z)) - \bar{g}^{\bar{\mathcal{E}}}(\bar{\nabla}^{\bar{\mathcal{E}}}_X Y, Z) - \bar{g}^{\bar{\mathcal{E}}}(Y, \bar{\nabla}^{\bar{\mathcal{E}}}_X Z)$$

vanishes for any $X \in \Gamma(TM)$ and $Y, Z \in \Gamma(\bar{\mathcal{E}})$. Let us separate in several cases:

1. Let $Y, Z \in \Gamma(S(TM))$. Then

$$\begin{aligned}(\bar{\nabla}^{\bar{\mathcal{E}}}_X \bar{g}^{\bar{\mathcal{E}}})(Y,Z) &= X(\bar{g}^{\bar{\mathcal{E}}}(Y,Z)) - \bar{g}^{\bar{\mathcal{E}}}(\bar{\nabla}^{\bar{\mathcal{E}}}_X Y, Z) - \bar{g}^{\bar{\mathcal{E}}}(Y, \bar{\nabla}^{\bar{\mathcal{E}}}_X Z)\\ &= X(g(Y,Z)) - g(\nabla^*_X Y, Z) - g(Y, \nabla^*_X Z),\end{aligned}$$

   which vanishes since ∇^* is a metric connection relative to g.

2. Let $Y \in \Gamma(S(TM))$ and $U \in \Gamma(\mathrm{Rad}(TM))$. Then

$$\begin{aligned}(\bar{\nabla}^{\bar{\mathcal{E}}}_X \bar{g}^{\bar{\mathcal{E}}})(Y,U) &= X(\bar{g}^{\bar{\mathcal{E}}}(Y,U)) - \bar{g}^{\bar{\mathcal{E}}}(\bar{\nabla}^{\bar{\mathcal{E}}}_X Y, U) - \bar{g}^{\bar{\mathcal{E}}}(Y, \bar{\nabla}^{\bar{\mathcal{E}}}_X U)\\ &= -\bar{g}^{\bar{\mathcal{E}}}(h^{\mathcal{E}}(X,Y),U) + g(Y, A^*_U X),\end{aligned}$$

   which vanishes because of definition (3.3).

3. Take $Y \in \Gamma(S(TM))$ and $V \in \Gamma(\mathcal{E})$. Now,

$$\begin{aligned}(\bar{\nabla}^{\bar{\mathcal{E}}}_X \bar{g}^{\bar{\mathcal{E}}})(Y,V) &= X(\bar{g}^{\bar{\mathcal{E}}}(Y,V)) - \bar{g}^{\bar{\mathcal{E}}}(\bar{\nabla}^{\bar{\mathcal{E}}}_X Y, V) - \bar{g}^{\bar{\mathcal{E}}}(Y, \bar{\nabla}^{\bar{\mathcal{E}}}_X V)\\ &= -\bar{g}^{\bar{\mathcal{E}}}(h^*(X,Y),V) + \bar{g}^{\bar{\mathcal{E}}}(Y, A_V X),\end{aligned}$$

   which vanishes because of our hypothesis.

4. If $U, U' \in \Gamma(\mathrm{Rad}(TM))$, then

$$\begin{aligned}(\bar{\nabla}^{\bar{\mathcal{E}}}_X \bar{g}^{\bar{\mathcal{E}}})(U,U') &= X(\bar{g}^{\bar{\mathcal{E}}}(U,U')) - \bar{g}^{\bar{\mathcal{E}}}(\bar{\nabla}^{\bar{\mathcal{E}}}_X U, U') - \bar{g}^{\bar{\mathcal{E}}}(U, \bar{\nabla}^{\bar{\mathcal{E}}}_X U')\\ &= -\bar{g}^{\bar{\mathcal{E}}}(\nabla_X U + h^{\mathcal{E}}(X,U), U') - \bar{g}^{\bar{\mathcal{E}}}(U, \nabla_X U' + h^{\mathcal{E}}(X,U'))\\ &= -\bar{g}^{\bar{\mathcal{E}}}(\nabla_X U, U') - \bar{g}^{\bar{\mathcal{E}}}(U, \nabla_X U') = 0,\end{aligned}$$

   where we used (3.3) in the form

$$\bar{g}^{\bar{\mathcal{E}}}(h^{\mathcal{E}}(X,U),U') = g(U, A^*_{U'} X) = 0.$$

5. If $V, V' \in \Gamma(\mathcal{E})$, then

$$\begin{aligned}(\bar{\nabla}^{\bar{\mathcal{E}}}_X \bar{g}^{\bar{\mathcal{E}}})(V,V') &= X(\bar{g}^{\bar{\mathcal{E}}}(V,V')) - \bar{g}^{\bar{\mathcal{E}}}(\bar{\nabla}^{\bar{\mathcal{E}}}_X V, V') - \bar{g}^{\bar{\mathcal{E}}}(V, \bar{\nabla}^{\bar{\mathcal{E}}}_X V')\\ &= -\bar{g}^{\bar{\mathcal{E}}}(-A_V X + \nabla^{\mathcal{E}}_X V, V') - \bar{g}^{\bar{\mathcal{E}}}(V, -A_{V'} X + \nabla^{\mathcal{E}}_X V')\\ &= \bar{g}^{\bar{\mathcal{E}}}(\nabla^{\mathcal{E}}_X V, V') + \bar{g}^{\bar{\mathcal{E}}}(V, \nabla^{\mathcal{E}}_X V') = 0;\end{aligned}$$

   recall that the operators A_V, $A_{V'}$ are required to take values in $S(TM)$.

6. Finally, if $U \in \Gamma(\mathrm{Rad}(TM))$ and $V \in \Gamma(\mathcal{E})$,

$$\begin{aligned}(\bar{\nabla}^{\bar{\mathcal{E}}}_X \bar{g}^{\bar{\mathcal{E}}})(U,V) &= X(\bar{g}^{\bar{\mathcal{E}}}(U,V)) - \bar{g}^{\bar{\mathcal{E}}}(\bar{\nabla}^{\bar{\mathcal{E}}}_X U, V) - \bar{g}^{\bar{\mathcal{E}}}(U, \bar{\nabla}^{\bar{\mathcal{E}}}_X V)\\ &= X(\bar{g}^{\bar{\mathcal{E}}}(U,V)) - \bar{g}^{\bar{\mathcal{E}}}(\nabla^{*t}_X U, V) - \bar{g}^{\bar{\mathcal{E}}}(U, \nabla^{\mathcal{E}}_X V).\end{aligned}$$

The claim follows. □

The proof of the following result is an straightforward calculation but we include it for completeness.

Proposition 3.3. *The curvature tensor $\bar{R}^{\bar{\mathcal{E}}}$ of $\bar{\nabla}^{\bar{\mathcal{E}}}$ is identically zero if and only if equations (2.8) hold.*

Proof. As usual, we may suppose that all Lie brackets vanish identically. First we calculate $\bar{R}^{\bar{\mathcal{E}}}(X,Y)Z$ for $X,Y,Z \in \Gamma(TM)$. Note that

$$\bar{\nabla}^{\bar{\mathcal{E}}}_Y \bar{\nabla}^{\bar{\mathcal{E}}}_X Z = \nabla_Y \nabla_X Z + h^{\mathcal{E}}(Y, \nabla_X Z) - A_{h^{\mathcal{E}}(X,Z)} Y + \nabla^{\mathcal{E}}_Y (h^{\mathcal{E}}(X,Z))$$

and analogously for $\bar{\nabla}^{\bar{\mathcal{E}}}_X \bar{\nabla}^{\bar{\mathcal{E}}}_Y Z$. Therefore,

$$\begin{aligned}
\bar{R}^{\bar{\mathcal{E}}}(X,Y)Z &= R(X,Y)Z + A_{h^{\mathcal{E}}(Y,Z)}X - A_{h^{\mathcal{E}}(X,Z)}Y \\
&\quad - h^{\mathcal{E}}(X, \nabla_Y Z) - \nabla^{\mathcal{E}}_X (h^{\mathcal{E}}(Y,Z)) \\
&\quad + h^{\mathcal{E}}(Y, \nabla_X Z) + \nabla^{\mathcal{E}}_Y (h^{\mathcal{E}}(X,Z)) \\
&\quad + h^{\mathcal{E}}(\nabla_X Y, Z) - h^{\mathcal{E}}(\nabla_Y X, Z) \\
&= R(X,Y)Z + A_{h^{\mathcal{E}}(Y,Z)}X - A_{h^{\mathcal{E}}(X,Z)}Y \\
&\quad + (\nabla^{\mathcal{E}}_Y h^{\mathcal{E}})(X,Z) - (\nabla^{\mathcal{E}}_X h^{\mathcal{E}})(Y,Z) \\
&= 0;
\end{aligned}$$

note we added two terms in order to obtain $\nabla^{\mathcal{E}} h^{\mathcal{E}}$.

Analogously, if $U \in \Gamma(\mathrm{Rad}(TM))$,

$$\begin{aligned}
\bar{R}^{\bar{\mathcal{E}}}(X,Y)U &= \nabla^*_X A^*_U Y - \nabla^*_Y A^*_U X + A_{\nabla^{*t}_Y U} X - A_{\nabla^{*t}_X U} Y \\
&\quad - h^*(X, A^*_U Y) + h^*(Y, A^*_U X) + R^{*t}(X,Y)U \\
&\quad + h^{\mathcal{E}}(X, A^*_U Y) - h^{\mathcal{E}}(Y, A^*_U X) \\
&= 0.
\end{aligned}$$

Finally, if $V \in \Gamma(\mathrm{tr}(TM))$,

$$\begin{aligned}
\bar{R}^{\bar{\mathcal{E}}}(X,Y)V &= \nabla^*_X A_V Y - \nabla^*_Y A_V X + A_{\nabla^{\mathcal{E}}_Y V} X - A_{\nabla^{\mathcal{E}}_X V} Y \\
&\quad + h^*(X, A_V Y) - h^*(Y, A_V X) \\
&\quad + h^{\mathcal{E}}(X, A_V Y) - h^{\mathcal{E}}(Y, A_V X) + R^{\mathcal{E}}(X,Y)V \\
&= 0,
\end{aligned}$$

which proves our claim. □

4. Isometric immersions into $\mathbb{R}^{n+2}_1$

We now prove a stronger version of the Fundamental Theorem for null hypersurfaces isometrically immersed into Lorentz-Minkowski space (cfr. [14, Theorem 4.1]). In the next section we extend this result for isometric immersions of a null manifold M^k into a simply connected semi-Riemannian manifold $\mathbb{Q}^{k+1}_{c,q}$ with constant sectional curvature c and index $q \geq 1$.

Theorem 4.1. *Let $(M, g, S(TM))$ be a null, simply connected $(n+1)$-dimensional manifold with an n-dimensional screen distribution $S(TM)$. Let $\mathcal{E}$ be a 1-dimensional vector bundle over M, $\bar{g}^{\bar{\mathcal{E}}}$ a metric on $\bar{\mathcal{E}} = TM \oplus \mathcal{E}$ defined by (3.2) and a connection $\bar{\nabla}^{\bar{\mathcal{E}}}$ on $\bar{\mathcal{E}}$ satisfying (3.5). Suppose further the Gauss-Codazzi-Ricci equations (2.8) hold on $\bar{\mathcal{E}}$. Then there exists an isometric immersion $f \colon M \to \mathbb{R}^{n+2}_1$ and a vector bundle isometry $\phi \colon \mathcal{E} \to \mathrm{tr}(TM)$ such that*

$$h^f = \phi h^{\mathcal{E}} \quad \text{and} \quad \nabla^t \phi = \phi \nabla^{\mathcal{E}}. \tag{4.1}$$

Moreover, let $f, g \colon M^{n+1} \to \mathbb{R}^{n+2}_1$ be isometric immersions of a null manifold. Suppose there is a vector bundle isometry $\psi \colon \mathrm{tr}^f(TM) \to \mathrm{tr}^g(TM)$ such that

$$\psi h^f = h^g \quad \text{and} \quad \psi \nabla^{f,t} = \nabla^{g,t} \psi.$$

Then there is an isometry $\tau \colon \mathbb{R}^{n+2}_1 \to \mathbb{R}^{n+2}_1$ such that

$$\tau f = g \quad \text{and} \quad \tau_*|_{\mathrm{tr}^f(TM)} = \psi.$$

Proof. The argument follows closely [11, p. 37] and [8]. By our previous work, the curvature tensor $\bar{R}^{\bar{\mathcal{E}}}$ of $\bar{\mathcal{E}}$ vanishes identically. Since the connection is flat, by standard results (see [11, Corollary A.5]) there exists a global parallel orthonormal frame $\{E_i\}$, $i = 1, \ldots, n+2$, relative to $\bar{\nabla}^{\bar{\mathcal{E}}}$. Let $(x_1, \ldots, x_{n+1})$ be local coordinates in a simply connected neighborhood U of $p \in M$; we write

$$\frac{\partial}{\partial x_i} = \sum_{k=1}^{n+2} a_{ik} E_k$$

for some functions a_{ik}. Since $\{E_i\}$ is an orthonormal frame, the coefficients of the metric $\bar{g}^{\bar{\mathcal{E}}}$ are given by

$$g_{ij} = \bar{g}^{\bar{\mathcal{E}}}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \sum_{k=1}^{n+2} a_{ik} a_{jk};$$

since E_i is parallel, we have

$$\bar{\nabla}^{\bar{\mathcal{E}}}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n+2} \frac{\partial a_{jk}}{\partial x_i} E_k \quad \text{and} \quad \bar{\nabla}^{\bar{\mathcal{E}}}_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = \sum_{k=1}^{n+2} \frac{\partial a_{ik}}{\partial x_j} E_k,$$

which gives

$$\frac{\partial a_{jk}}{\partial x_i} = \frac{\partial a_{ik}}{\partial x_j},$$

implying that the forms

$$\omega_k = a_{1k} dx_1 + \cdots + a_{n+1,k} dx_{n+1}, \quad k, = 1, \ldots, n+2,$$

are closed; since U is simply connected, for each k there is a function f_k such that

$$\frac{\partial f_k}{\partial x_i} = a_{ik}.$$

Let $f \colon U \to \mathbb{R}^{n+2}_1$ given by $f = (f_1, \ldots, f_{n+2})$, so that

$$f_*\left(\frac{\partial}{\partial x_i}\right) = \left(\frac{\partial f_1}{\partial x_i}, \ldots, \frac{\partial f_{n+2}}{\partial x_i}\right) = (a_{i1}, \ldots, a_{i,n+2}),$$

and therefore

$$g^0\left(f_*\left(\frac{\partial}{\partial x_i}\right), f_*\left(\frac{\partial}{\partial x_j}\right)\right) = \sum_{k=1}^{n+2} a_{ik} a_{jk} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right),$$

where ∇^0 is the canonical flat connection in $\mathbb{R}^{n+2}_1$. This means that f is an isometric immersion. In particular, $f(U)$ is a null hypersurface in $\mathbb{R}^{n+2}_1$ and $f_*(S(TM)|_U)$ is an screen distribution defined in $f(U)$. We have also a well defined transversal distribution $T(f(U))^t$. Since the tangent bundle $T\mathbb{R}^{n+2}_1$ is trivial, in the remaining of the proof we use the standard identification of the pullback bundle $f^*T\mathbb{R}^{n+2}_1$ with $M \times \mathbb{R}^{n+2}_1$ (refer to [31, Prop. 20.5]).

We define a vector bundle morphism $\bar{\phi} \colon TU \oplus E \to U \times \mathbb{R}^{n+2}_1$ by

$$\bar{\phi}(E_k) = e_k,$$
$$\bar{\phi}\left(\frac{\partial}{\partial x_i}\right) = \sum_{k=1}^{n+1} a_{ik} \bar{\phi}(E_k) = \sum_{k=1}^{n+1} a_{ik} e_k = f_*\left(\frac{\partial}{\partial x_i}\right).$$

In particular, $\bar{\phi}|_{TU} = f_*$ is an isomorphism onto $T(f(U))$. Since $\bar{\phi}$ is an isometry on the fibers, it sends E isomorphically onto $T(f(U))^t$. Also, since $\bar{\phi}$ maps the parallel orthonormal frame $\{E_i\}$ into the parallel orthonormal frame $\{e_i\}$, we have for any $Y \in \Gamma(TM)$ and $V \in \Gamma(E)$,

$$\bar{\phi}(\bar{\nabla}^{\bar{\mathcal{E}}}_X Y) = \nabla^0_{f_* X} \bar{\phi} Y,$$
$$\bar{\phi}(\bar{\nabla}^{\bar{\mathcal{E}}}_X V) = \nabla^0_{f_* X} \bar{\phi} V;$$

in short, $\bar\phi$ is a parallel vector bundle isometry.

Let $\phi = \bar\phi|_{\mathcal{E}}$. For $X, Y \in \Gamma(TM)$ we have

$$\bar\phi \bar\nabla^{\bar{\mathcal{E}}}_X Y = \bar\phi(\nabla_X Y + h^{\mathcal{E}}(X,Y)) = f_* \nabla_X Y + \phi h^{\mathcal{E}}(X,Y)$$
$$\nabla^0_X \bar\phi Y = \nabla^0_X f_* Y = f_* \nabla_X Y + h^f(X,Y).$$

Taking the transversal components, we have $h^f = \phi h^{\mathcal{E}}$, the first claim in (4.1).

From (2.5) and (3.4), for $V \in \Gamma(\mathrm{tr}(TM))$ we have

$$\begin{aligned} g(PY, A_V X) &= \bar g^{\bar{\mathcal{E}}}(h^*(X,PY), V) = g^0(\bar\phi h^*(X,PY), \bar\phi V) \\ &= g^0(f_* h^*(X,PY), \phi V) = g(PY, A^f_{\phi V} X); \end{aligned}$$

that is, $A_V = A^f_{\phi V}$. Therefore,

$$\begin{aligned} \bar\phi \bar\nabla^{\bar{\mathcal{E}}}_X V &= \bar\phi(-A_V X + \nabla^{\mathcal{E}}_X V) = -f_* A_V X + \phi \nabla^{\mathcal{E}}_X V \\ &= -f_* A^f_{\phi V} X + \phi \nabla^{\mathcal{E}}_X V, \end{aligned}$$

while (2.4) implies

$$\nabla^0_X \bar\phi V = \nabla^0_X \phi V = -f_* A^f_{\phi V} X + \nabla^t_X \phi V;$$

implying in turn $\nabla^t \phi = \phi \nabla^{\mathcal{E}}$, the second claim in (4.1).

Since M is simply connected, the vector bundle morphism $\bar\phi$ may be extended globally. For simplicity we will denote this extension again by $\bar\phi$.

To prove the uniqueness part, following again [11], we define

$$\bar\psi \colon f^* T\mathbb{R}^{n+2}_1 \to g^* T\mathbb{R}^{n+2}_1$$

by

$$\bar\psi f_* = g_* \quad \text{and} \quad \bar\psi|_{\mathrm{tr}^f(TM)} = \psi.$$

We prove that $\bar\psi$ is parallel relative to the pullbacks $\bar\nabla^{f,0}$ and $\bar\nabla^{g,0}$ of the standard connection ∇^0 of $\mathbb{R}^{n+2}_1$ on $f^* T\mathbb{R}^{n+2}_1$ and $g^* T\mathbb{R}^{n+2}_1$, respectively. For $X, Y \in \Gamma(TM)$ we have

$$\begin{aligned} \bar\nabla^{g,0}_X \bar\psi f_* Y &= \bar\nabla^{g,0}_X g_* Y \\ &= g_* \nabla_X Y + h^g(X,Y) \\ &= \bar\psi f_* \nabla_X Y + \psi h^f(X,Y) \\ &= \bar\psi(f_* \nabla_X Y + h^f(X,Y)) \\ &= \bar\psi \bar\nabla^{f,0}_X f_* Y. \end{aligned}$$

On the other hand, if $V \in \Gamma(\mathrm{tr}^f(TM))$,

$$\begin{aligned} \bar\nabla^{g,0}_X \bar\psi V &= \bar\nabla^{g,0}_X \psi V = -g_* A^g_{\psi V} X + \nabla^{g,t}_X \psi V \\ &= -\bar\psi f_* A^f_V X + \psi \nabla^{f,t}_X V \\ &= \bar\psi(-f_* A^f_V X + \nabla^{f,t}_X V) \\ &= \bar\psi \bar\nabla^{f,0}_X V; \end{aligned}$$

here we used the fact that $A^g_{\psi V} = A^f_V$, which is proved using (2.5):

$$\begin{aligned} g(PY, A^g_{\psi V} X) &= \bar g(g_* h^*(X,PY), \psi V) = \bar g(\bar\psi f_* h^*(X,PY), \bar\psi V) \\ &= \bar g(f_* h^*(X,PY), V) = g(PY, A^f_V X). \end{aligned}$$

Then $\bar\psi$ is parallel. Since $T\mathbb{R}^{n+2}_1 \simeq \mathbb{R}^{2n+4}$ is flat, its transition functions are locally constant (see Theorem 5.5 in [3] and Corollary 9.2 in [22]). Thus $\bar\psi$ defines an orthogonal transformation B on $\mathbb{R}^{n+2}_1$. Since $B f_* = \bar\psi f_* = g_*$, there is an isometry τ on $\mathbb{R}^{n+2}_1$ such that $\tau_* = B$ and $\tau f = g$. Therefore,

$$\tau_*|_{\mathrm{tr}^f(TM)} = B|_{\mathrm{tr}^f(TM)} = \bar\psi|_{\mathrm{tr}^f(TM)} = \psi. \qquad \square$$

5. Isometric immersions into $\mathbb{Q}_{c,q}^{n+2}$

We prove in this section a Fundamental Theorem for isometric immersions for null manifolds in a simple connected semi-Riemannian space form $\mathbb{Q}_{c,q}^{n+2}$ of constant sectional curvature c and arbitrary index $q \geq 1$. We take advantage of the fact that manifolds $\mathbb{Q}_{c,q}^{n+2}$ can be isometrically immersed as hypersurfaces of a semi-Euclidean space and follow an approach inspired in [11]. To accomplish this, we rely on certain codimension two null submanifolds, named *half-lightlike submanifolds* in [13]. As a first step, we prove an immersion result for this kind of submanifolds in semi-Euclidean spaces.

Let $(\bar{M}, \bar{g})$ be an $(m+2)$-dimensional semi-Riemannian manifold of index $q \geq 1$ and (M, g) a lightlike submanifold of codimension two of $\bar{M}$. If $\dim(\mathrm{Rad}(TM)) = 1$ then M is called *half-lightlike submanifold*. Observe that $T_xM^{\perp}$ of T_xM in $T_x\bar{M}$ for each $x \in M$ is a degenerate 2-dimensional subspace of $T_x\bar{M}$ and there exists a complementary non-degenerate distribution $S(TM)$ to $\mathrm{Rad}(TM)$ in TM, called a *screen distribution* of M, with the orthogonal decomposition

$$TM = \mathrm{Rad}(TM) \perp S(TM). \tag{5.1}$$

In this case, $(TM)^{\perp}$ is also half-lightlike since $\mathrm{Rad}(TM)$ is a 1-dimensional vector sub-bundle of $(TM)^{\perp}$. Thus there exists a complementary distribution D to $\mathrm{Rad}(TM)$, which is called a *screen transversal bundle* of M. Thus there exists a rank 2 distribution $D^{\perp}$ such that

$$S(TM)^{\perp} = D \perp D^{\perp}$$

and a unique null rank 1 vector bundle $\mathrm{ltr}(TM)$ complementary to $\mathrm{Rad}(TM)$ in $D^{\perp}$, called *lightlike transversal bundle* of M. Finally, the *transversal vector bundle* is defined as (see [15, p. 158]):

$$\mathrm{tr}(TM) = D \perp \mathrm{ltr}(TM),$$

Following the approach given in [15] we can write the decomposition

$$T\bar{M} = S(TM) \perp D \perp (\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)),$$

which gives us the following *Gauss-Weingarten* formulae. If $\bar{\nabla}$ is the metric connection on $\bar{M}$, $N \in \mathrm{ltr}(TM)$ and $u \in D$,

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + h(X,Y), \\ \bar{\nabla}_X N &= -A_N X + \nabla_X N, \\ \bar{\nabla}_X u &= -A_N X + \nabla_X u, \end{aligned} \tag{5.2}$$

for any $X, Y \in \Gamma(TM)$, where $\nabla_X Y$, $A_N X$ and $A_u X$ are in $\Gamma(TM)$, while $h(X,Y)$, $\nabla_X N$ and $\nabla_X u$ are in $\Gamma(\mathrm{tr}(TM))$. The connection ∇ is torsion-free on M and $h(X,Y)$ is a symmetric C^{∞}-bilinear form with values in $\Gamma(\mathrm{tr}(TM))$. Now, let $\{\xi, N\}$ be a pair of locally lightlike sections on a neighborhood $\mathcal{U} \subset M$ with $\xi \in \mathrm{Rad}(TM)$ and $N \in \mathrm{ltr}(TM)$. Then we can define symmetric smooth bilinear forms $D_1,\ D_2$ and 1-forms $\rho_1,\ \rho_2,\ \varepsilon_1,\ \varepsilon_2$ on $\mathcal{U}$ by

$$\begin{aligned} D_1(X,Y) &= \bar{g}(h(X,Y), \xi), \\ D_2(X,Y) &= \mu\bar{g}(h(X,Y), u), \\ \rho_1(X) &= \bar{g}(\nabla_X N, \xi), \quad \rho_2(X) = \mu\bar{g}(\nabla_X N, u), \\ \varepsilon_1(X) &= \bar{g}(\nabla_X u, \xi), \quad \varepsilon_2(X) = \mu\bar{g}(\nabla_X u, u), \end{aligned}$$

where $\mu = \bar{g}(u,u) = \pm 1$, depending on the causal character of the unit vector field u and $X, Y \in \Gamma(TM)$. Consequently, we have the relations

$$\begin{aligned} h(X,Y) &= D_1(X,Y)N + D_2(X,Y)u, \\ \nabla_X N &= \rho_1(X)N + \rho_2(X)u, \\ \nabla_X u &= \varepsilon_1(X)N + \varepsilon_2(X)u, \end{aligned}$$

which imply corresponding modifications in the Gauss-Weingarten formulae (5.2). The curvature tensors $\bar{R}$ of $\bar{\nabla}$ and R of ∇ satisfy corresponding structure equations in terms of all the objects previously defined. For details, see section 4.1 of [15]. The geometry of half-lightlike submanifolds was developed by K. L. Duggal and A. Bejancu in [13], which we use here to establish the following setting, generalizing the previous section with a similar approach.

- M is a null $(n+1)$-manifold with metric g.
- $S(TM)$ is a screen distribution in M of dimension n and index $q-1$. Observe that the restriction of g to $S(TM)$ is semi-Riemannian.
- ∇ is a torsion-free connection of M, which restricted to $S(TM)$ is a g-metric connection.
- The decomposition (5.1) gives rise to the following Gauss-Weingarten equations of M:

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \tag{5.3}$$

$$\nabla_X U = -A^*_U X + \nabla^{\perp}_X U,$$

where P is the projection of TM over $S(TM)$, $X, Y \in \Gamma(TM)$ and $U \in \Gamma(\mathrm{Rad}(TM))$, with $\nabla^*_X PY, A^*_U X \in \Gamma(S(TM))$, $h^*(X, PY), \nabla^{\perp}_X U \in \Gamma(\mathrm{Rad}(TM))$. Besides, ∇^* is a metric connection on $S(TM)$, $\nabla^{\perp}$ is a linear connection, while A^* and h^* are $C^\infty(M)$-bilinear forms.

- δ is another rank-1 vector bundle over M, playing the role of the screen transversal bundle D.
- ϵ is a rank 1 vector bundle over M, playing the role of the lightlike transversal bundle $\mathrm{ltr}(TM)$.

Now, let $\bar{\epsilon}$ the vector bundle over M defined by

$$\bar{\epsilon} = TM \oplus \epsilon \oplus \delta = S(TM) \oplus \mathrm{Rad}(TM) \oplus \epsilon \oplus \delta,$$

and define a Riemannian metric over $\bar{\epsilon}$ in the following way: let g_R, g_ϵ Riemannian metrics over $\mathrm{Rad}(TM)$ and ϵ, respectively, and let g_δ a semi-Riemannian metric over δ. Consider the product metric

$$\bar{g} = P^* g + P^*_{\mathrm{Rad}(TM)} g_R + P^*_\epsilon g_\epsilon + + P^*_\delta g_\delta$$

over $\bar{\epsilon}$ and let $\{\xi, N, u\}$ an orthonormal frame such that

$$span(\xi) = \mathrm{Rad}(TM), \quad span(N) = \epsilon, \quad span(u) = \delta.$$

Finally, let

$$\lambda = g_\delta(u, u).$$

Definition 5.1. Let $X, Y \in \Gamma(\bar{\epsilon})$. We define

$$\bar{g}^{\bar{\epsilon}}(X, Y) = \bar{g}(X, Y) - (\bar{g}(X, \xi) - \bar{g}(X, N))(\bar{g}(Y, \xi) - \bar{g}(Y, N)). \tag{5.4}$$

Proposition 5.1. *$\bar{g}^{\bar{\epsilon}}$ is a Lorentzian metric over $\bar{\epsilon}$. Moreover, if $X, Y \in \Gamma(S(TM)))$, then*

1. $\bar{g}^{\bar{\epsilon}}(X, Y) = g(X, Y)$;
2. $\bar{g}^{\bar{\epsilon}}(X, \xi) = \bar{g}^{\bar{\epsilon}}; (X, N) = \bar{g}^{\bar{\epsilon}}(X, u) = 0$;
3. $\bar{g}^{\bar{\epsilon}}(u, u) = \lambda$, $\bar{g}^{\bar{\epsilon}}(N, N) = 0$, $\bar{g}^{\bar{\epsilon}}(\xi, \xi) = 0$;
4. $\bar{g}^{\bar{\epsilon}}(u, \xi) = 0$, $\bar{g}^{\bar{\epsilon}}(u, N) = 0$, $\bar{g}^{\bar{\epsilon}}(\xi, N) = 1$.

Corollary 5.1. *With respect to the metric $\bar{g}^{\bar{\epsilon}}$, we have the following decomposition*

$$\bar{\epsilon} = S(TM) \perp \delta \perp (Rad(TM) \oplus \epsilon). \tag{5.5}$$

Proposition 5.2. *If the index of δ is d respect to the metric $\bar{g}^{\bar{\epsilon}}$, then $\bar{\epsilon}$ has index $q + d$.*

Proof. Let $\zeta = (\mathrm{Rad}(TM) \oplus \epsilon)$. It is a rank two vector bundle with $0 \neq \xi \in \mathrm{Rad}(TM)$ and $0 \neq N \in \epsilon$ independent non null elements in ζ. Consequently, ζ is timelike and therefore have index 1.

The decomposition (5.5) together with the fact that $\bar{\epsilon}$ is a semi-Riemannian bundle with respect to $\bar{g}^{\bar{\epsilon}}$, imply that

$$\mathrm{ind}(\bar{\epsilon}) = \mathrm{ind}(S(TM)) + \mathrm{ind}(\delta) + \mathrm{ind}(\gamma) = q - 1 + d + 1 = q + d.$$

□

Now we define the connection $\overline{\nabla}^{\overline{\epsilon}}$ on $\overline{\epsilon}$ by the following equations (compare to [15, Eqs. 4.1.7-4.1.9]):

$$\overline{\nabla}^{\overline{\epsilon}}_X Y = \nabla_X Y + D_1(X,Y)N + D_2(X,Y)u, \tag{5.6}$$

$$\overline{\nabla}^{\overline{\epsilon}}_X N = -A_N X + \rho_1(X)N + \rho_2(X)u,$$

and

$$\overline{\nabla}_X u = -A_u X + \epsilon_1(X)N,$$

for all $X \in \Gamma(TM)$, where D_1, D_2 are bilinear $C^\infty(M)$ forms, ρ_1, ρ_2 and ϵ_1 are 1-forms and A_u y A_N are $C^\infty(M)$ linear operators on $\Gamma(TM)$.

Proposition 5.3. *$\overline{\nabla}^{\overline{\epsilon}}$ is a metric connection over $\overline{\epsilon}$ with respect to $\overline{g}^{\epsilon}$ if and only if the following equations hold (compare to [15, Eqs. 4.1.10-4.1.21]).*

$$\overline{g}^{\epsilon}(h^*(X,PY),N) = \overline{g}^{\epsilon}(A_N X, PY), \tag{5.7}$$

$$D_1(X,PY) = g(A^*_\xi X, PY), \tag{5.8}$$

$$D_1(X,\xi) = 0, \tag{5.9}$$

$$\overline{g}^{\epsilon}(A_N X, N) = 0, \tag{5.10}$$

$$\rho_1(X) = -\overline{g}^{\epsilon}(\nabla^{\perp}_X \xi, N), \tag{5.11}$$

$$\rho_2(X) = \lambda \overline{g}^{\epsilon}(A_u X, N), \tag{5.12}$$

$$\epsilon_1(X) = -\lambda D_2(X,\xi), \tag{5.13}$$

$$\overline{g}^{\epsilon}(A_u X, Y) = \lambda D_2(X,Y) + \epsilon_1(X)\overline{g}^{\epsilon}(Y,N). \tag{5.14}$$

Proof. We must prove that

$$(\overline{\nabla}^{\overline{\epsilon}}_X \overline{g}^{\epsilon})(Y,Z) = X(\overline{g}^{\epsilon}(Y,Z)) - \overline{g}^{\epsilon}(\overline{\nabla}^{\overline{\epsilon}}_X Y, Z) - \overline{g}^{\epsilon}(Y, \overline{\nabla}^{\overline{\epsilon}}_X Z)$$

vanishes for all $X \in \Gamma(TM)$ and $Y, Z \in \Gamma(\overline{\epsilon})$. We are going to consider each of the ten possible cases.

(1) Let $Y, Z \in \Gamma(S(TM))$. The decompositions (5.3), (5.6) together with the fact that ∇^* is a metric connection on $S(TM)$, give us

$$\begin{aligned}(\overline{\nabla}^{\overline{\epsilon}}_X \overline{g}^{\epsilon})(Y,Z) =& X(g(Y,Z)) - \overline{g}^{\epsilon}(\nabla_X Y + D_1(X,Y)N + D_2(X,Y)u, Z),\\ & - \overline{g}^{\epsilon}(\nabla_X Z + D_1(X,Z)N + D_2(X,Z)u, Y),\\ =& X(g(Y,Z)) - g(\nabla_X Y, Z) - g(\nabla_X Z, Y),\\ =& X(g(Y,Z)) - g(\nabla^*_X Y + h^*(X,Y), Z) - g(\nabla^*_X Z + h^*(X,Z), Y),\\ =& X(g(Y,Z)) - g(\nabla^*_X Y, Z) - g(\nabla^*_X Z, Y) = 0.\end{aligned}$$

(2) Let $Y \in \Gamma(S(TM))$, $r\xi \in \Gamma(\mathrm{Rad}(TM))$. By decomposition (5.6) we have

$$\begin{aligned}(\overline{\nabla}^{\overline{\epsilon}}_X \overline{g}^{\epsilon})(Y,r\xi) =& X(\overline{g}^{\epsilon}(Y,r\xi)) - \overline{g}^{\epsilon}(\overline{\nabla}^{\overline{\epsilon}}_X Y, r\xi) - \overline{g}^{\epsilon}(Y, \overline{\nabla}^{\overline{\epsilon}}_X r\xi),\\ =& - \overline{g}^{\epsilon}(\nabla_X Y + D_1(X,Y)N + D_2(X,Y)u, r\xi)\\ & - \overline{g}^{\epsilon}(\nabla_X r\xi + D_1(X,r\xi)N + D_2(X,r\xi)u, Y),\\ =& - \overline{g}^{\epsilon}(D_1(X,Y)N, r\xi) - g(\nabla_X r\xi, Y),\\ =& - rD_1(X,Y) - g(-A^*_{r\xi}X + \nabla^{\perp}_X r\xi, Y),\\ =& - r[D_1(X,Y) - g(A^*_\xi X, Y)],\end{aligned}$$

which vanishes if and only if (5.8) holds.

(3) Let $Y \in \Gamma(S(TM))$ and $vN \in \Gamma(\epsilon)$. By the structure equations,

$$\begin{aligned}(\overline{\nabla}^{\epsilon}_X \overline{g}^{\epsilon})(Y, vN) =& X(\overline{g}^{\epsilon}(Y, vN)) - \overline{g}^{\epsilon}(\overline{\nabla}^{\epsilon}_X Y, vN) - \overline{g}^{\epsilon}(Y, \overline{\nabla}^{\epsilon}_X vN),\\ =& - \overline{g}^{\epsilon}(\overline{\nabla}^{\epsilon}_X Y, vN) - \overline{g}^{\epsilon}(Y, X(v)N + v\overline{\nabla}^{\epsilon}_X N),\\ =& - \overline{g}^{\epsilon}(\nabla_X Y + D_1(X,Y)N + D_2(X,Y)u, vN)\\ & - v\overline{g}^{\epsilon}(-A_N X + \rho_1(X)N + \rho_2(X)u, Y),\\ =& - v[\overline{g}^{\epsilon}(\nabla_X Y, N) - \overline{g}^{\epsilon}(A_N X, Y)],\\ =& - v[\overline{g}^{\epsilon}(h^*(X,Y), N) - \overline{g}^{\epsilon}(A_N X, Y)],\end{aligned}$$

which vanishes if and only if (5.7) holds.

(4) Let $Y \in \Gamma(S(TM))$, $wu \in \Gamma(\delta)$. Using the structure equations again,

$$\begin{aligned}(\overline{\nabla}^{\epsilon}_X \overline{g}^{\epsilon})(Y, wu) =& X(\overline{g}^{\epsilon}(Y, wu)) - \overline{g}^{\epsilon}(\overline{\nabla}^{\epsilon}_X Y, wu) - \overline{g}^{\epsilon}(Y, \overline{\nabla}^{\epsilon}_X wu),\\ =& - \overline{g}^{\epsilon}(\overline{\nabla}^{\epsilon}_X Y, wu) - \overline{g}^{\epsilon}(Y, X(w)u + w\overline{\nabla}^{\epsilon}_X u),\\ =& - \overline{g}^{\epsilon}(\nabla_X Y + D_1(X,Y)N + D_2(X,Y)u, wu)\\ & - w\overline{g}^{\epsilon}(-A_u X + \epsilon_1(X)N, Y),\\ =& - w[\lambda D_2(X,Y) - g(A_u X, Y)],\\ =& - w[\lambda D_2(X,Y) - \lambda D_2(X,Y) - \epsilon_1(X)\overline{g}^{\epsilon}(Y, N)],\end{aligned}$$

which vanishes if and only if (5.14) holds.

(5) Now let $r\xi, r'\xi \in \Gamma(\mathrm{Rad}(TM))$. Using again the structure equations, we have

$$\begin{aligned}(\overline{\nabla}^{\epsilon}_X \overline{g}^{\epsilon})(r\xi, r'\xi) =& X(\overline{g}^{\epsilon}(r\xi, r'\xi)) - \overline{g}^{\epsilon}(\overline{\nabla}^{\epsilon}_X r\xi, r'\xi) - \overline{g}^{\epsilon}(r\xi, \overline{\nabla}^{\epsilon}_X r'\xi),\\ =& - \overline{g}^{\epsilon}(\nabla_X r\xi + D_1(X, r\xi)N + D_2(X, r\xi)u, r'\xi)\\ & - \overline{g}^{\epsilon}(\nabla_X r'\xi + D_1(X, r'\xi)N + D_2(X, r'\xi)u, r\xi),\\ =& - 2rr' D_1(X, \xi),\end{aligned}$$

which vanishes if and only if (5.9) holds.

(6) For $r\xi \in \Gamma(\mathrm{Rad}(TM)$ and $vN \in \Gamma(\epsilon)$, the structure equations imply that

$$\begin{aligned}(\overline{\nabla}^{\epsilon}_X \overline{g}^{\epsilon})(r\xi, vN) =& X(\overline{g}^{\epsilon}(r\xi, vN)) - \overline{g}^{\epsilon}(\overline{\nabla}^{\epsilon}_X r\xi, vN) - \overline{g}^{\epsilon}(r\xi, \overline{\nabla}^{\epsilon}_X vN),\\ =& X(rv) - \overline{g}^{\epsilon}(\nabla_X r\xi + D_1(X, r\xi)N + D_2(X, r\xi)u, vN)\\ & - \overline{g}^{\epsilon}(X(v)N + v\overline{\nabla}^{\epsilon}_X N, r\xi),\\ =& vX(r) + rX(v) - \overline{g}^{\epsilon}(X(r)\xi + r\nabla_X \xi, vN) - rX(v)\\ & - rv\overline{g}^{\epsilon}(-A_N X + \rho_1(X)N + \rho_2(X)u, \xi),\\ =& vX(r) - vX(r) - rv\overline{g}^{\epsilon}(-A^*_{\xi} X + \nabla^{\perp}_X \xi, N) - rv\rho_1(X),\\ =& - rv[\overline{g}^{\epsilon}(\nabla^{\perp}_X \xi, N) + \rho_1(X)],\end{aligned}$$

and it vanishes if and only if (5.11) holds.

(7) Now, letting $r\xi \in \Gamma(\mathrm{Rad}(TM))$, $wu \in \Gamma(\delta)$ we have

$$\begin{aligned}(\overline{\nabla}^{\epsilon}_X \overline{g}^{\epsilon})(r\xi, wu) =& X(\overline{g}^{\epsilon}(r\xi, wu)) - \overline{g}^{\epsilon}(\overline{\nabla}^{\epsilon}_X r\xi, wu) - \overline{g}^{\epsilon}(r\xi, \overline{\nabla}^{\epsilon}_X wu),\\ =& - \overline{g}^{\epsilon}(X(r)\xi + r\overline{\nabla}^{\epsilon}_X \xi, wu) - \overline{g}^{\epsilon}(r\xi, X(w)u + w\overline{\nabla}^{\epsilon}_X u),\\ =& - rw\overline{g}^{\epsilon}(\overline{\nabla}^{\epsilon}_X \xi, u) - rw\overline{g}^{\epsilon}(\xi, \overline{\nabla}^{\epsilon}_X u),\\ =& - rw\overline{g}^{\epsilon}(\nabla_X \xi + D_1(X,\xi)N + D_2(X,\xi)u, u)\\ & - rw\overline{g}^{\epsilon}(\xi, -A_u(X) + \epsilon_1(X)N),\\ =& - rw[\lambda D_2(X,\xi) + \epsilon_1(X)],\end{aligned}$$

which vanishes if and only if (5.13) holds.

(8) For $vN, v'N \in \Gamma(\epsilon)$ we have

$$\begin{aligned}(\overline{\nabla}^{\epsilon}_X\overline{g}^{\epsilon})(vN, v'N) =&X(\overline{g}^{\epsilon}(vN, v'N)) - \overline{g}^{\epsilon}(\overline{\nabla}^{\epsilon}_X vN, v'N) - \overline{g}^{\epsilon}(vN, \overline{\nabla}^{\epsilon}_X v'N),\\ =& - \overline{g}^{\epsilon}(X(v)N + v\overline{\nabla}^{\epsilon}_X N, v'N)\\ &- \overline{g}^{\epsilon}(X(v')N + v'\overline{\nabla}^{\epsilon}_X N, vN),\\ =& - 2vv'\overline{g}^{\epsilon}(-A_N X + \rho_1(X)N + \rho_2(X)u, N),\\ =&2vv'\overline{g}^{\epsilon}(A_N X, N),\end{aligned}$$

which vanishes if and only if (5.10) holds.

(9) For $vN \in \Gamma(\epsilon), wu \in \Gamma(\delta)$,

$$\begin{aligned}(\overline{\nabla}^{\epsilon}_X\overline{g}^{\epsilon})(vN, wu) =&X(\overline{g}^{\epsilon}(vN, wu)) - \overline{g}^{\epsilon}(\overline{\nabla}^{\epsilon}_X vN, wu) - \overline{g}^{\epsilon}(vN, \overline{\nabla}^{\epsilon}_X wu),\\ =& - \overline{g}^{\epsilon}(X(v)N + v\overline{\nabla}^{\epsilon}_X N, wu)\\ &- \overline{g}^{\epsilon}(X(w)u + w\overline{\nabla}^{\epsilon}_X u, vN),\\ =& - vw\overline{g}^{\epsilon}(-A_N X + \rho_1(X)N + \rho_2(X)u, u)\\ &- vw\overline{g}^{\epsilon}(-A_u X + \epsilon_1(X)N, N),\\ =& - vw[\lambda\rho_2(X) - \overline{g}^{\epsilon}(A_u X, N)],\end{aligned}$$

which vanishes if and only if (5.12) holds.

(10) Finally, for $wu, w'u \in \Gamma(\delta)$,

$$\begin{aligned}(\overline{\nabla}^{\epsilon}_X\overline{g}^{\epsilon})(wu, w'u) =&X(\overline{g}^{\epsilon}(wu, w'u)) - \overline{g}^{\epsilon}(\overline{\nabla}^{\epsilon}_X wu, w'u) - \overline{g}^{\epsilon}(wu, \overline{\nabla}^{\epsilon}_X w'u),\\ =&X(\sigma ww') - \overline{g}^{\epsilon}(X(w)u + w\overline{\nabla}^{\epsilon}_X u, w'u)\\ &- \overline{g}^{\epsilon}(X(w')u + w'\overline{\nabla}^{\epsilon}_X u, wu),\\ =&\sigma wX(w') + \sigma w'X(w) - w'\overline{g}^{\epsilon}(X(w)u - wA_u X\\ &+ w\epsilon_1(X)N, u) - w\overline{g}^{\epsilon}(X(w')u - w'A_u X + w'\epsilon_1(X)N, u)\\ =&\sigma wX(w') + \sigma w'X(w) - \sigma w'X(w) - \sigma wX(w') = 0.\end{aligned}$$

□

Proposition 5.4. *The curvature tensor $\overline{R}^{\epsilon}$ vanishes identically if and only the following relations hold (compare to [15, Eqs. 4.1.24-4.1.26]).*

$$\begin{aligned}0 =&R(X,Y)Z + D_1(X,Z)A_N Y - D_1(Y,Z)A_N X + D_2(X,Z)A_u Y - D_2(Y,Z)A_u X\\ &+ [(\nabla_X D_1)(Y,Z) - (\nabla_Y D_1)(X,Z) + \rho_1(X)D_1(Y,Z) - \rho_1(Y)D_1(X,Z)\\ &+ \epsilon_1(X)D_2(Y,Z) - \epsilon_1(Y)D_2(X,Z)]N\\ &+ [(\nabla_X D_2)(Y,Z) - (\nabla_Y D_2)(X,Z) + \rho_2(X)D_1(Y,Z) - \rho_2(Y)D_1(X,Z)]u,\end{aligned} \tag{5.15}$$

$$\begin{aligned}0 =& - \nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X,Y]\\ &+ \rho_1(X)A_N Y - \rho_1(Y)A_N X + \rho_2(X)A_u Y - \rho_2(Y)A_u X\\ &+ [D_1(Y, A_N X) - D_1(X, A_N Y) + 2d\rho_1(X,Y) + \epsilon_1(X)\rho_2(Y) - \epsilon_1(Y)\rho_2(X)]N\\ &+ [D_2(Y, A_N X) - D_2(X, A_N Y) + 2d\rho_2(X,Y) + \rho_1(Y)\rho_2(X) - \rho_1(X)\rho_2(Y)]u,\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}0 =& - \nabla_X(A_u Y) + \nabla_Y(A_u X) + A_u[X,Y] + \epsilon_1(X)A_N Y - \epsilon_1(Y)A_N X\\ &+ [D_1(Y, A_u X) - D_1(X, A_u Y) + 2d\epsilon_1(X,Y) + \rho_1(X)\epsilon_1(Y) - \rho_1(Y)\epsilon_1(X)]N\\ &+ [D_2(Y, A_u X) - D_2(X, A_u Y) + \epsilon_1(Y)\rho_2(X) - \epsilon_1(X)\rho_2(Y)]u.\end{aligned} \tag{5.17}$$

Proof. Let $X, Y \in \Gamma(TM)$ and assume, as usual, that the Lie bracket $[X, Y]$ is identically zero. We make the calculations for each of the three cases.

(a) Let $Z \in \Gamma(TM)$ and observe that

$$\begin{aligned}
\overline{\nabla}^{\overline{\epsilon}}_X \overline{\nabla}^{\overline{\epsilon}}_Y Z =& \overline{\nabla}^{\overline{\epsilon}}_X [\nabla_Y Z + D_1(Y,Z)N + D_2(Y,Z)u] \\
=& \overline{\nabla}^{\overline{\epsilon}}_X \nabla_Y Z + X(D_1(Y,Z))N + D_1(Y,Z)\overline{\nabla}^{\overline{\epsilon}}_X N \\
&+ X(D_2(Y,Z))u + D_2(Y,Z)\overline{\nabla}^{\overline{\epsilon}}_X u \\
=& \nabla_X \nabla_Y Z + D_1(X, \nabla_Y Z)N + D_2(X, \nabla_Y Z)u \\
&+ X(D_1(Y,Z))N + D_1(Y,Z)[-A_N X + \rho_1(X)N + \rho_2(X)u] \\
&+ X(D_2(Y,Z))u + D_2(Y,Z)[-A_u X + \epsilon_1(X)N] \\
=& \nabla_X \nabla_Y Z - A_N X D_1(Y,Z) - D_2(Y,Z)A_u X \\
&+ [X(D_1(Y,Z)) + D_1(X, \nabla_Y Z) + \rho_1(X)D_1(Y,Z) \\
&+ \epsilon_1(X)D_2(Y,Z)]N \\
&+ [X(D_2(Y,Z)) + D_2(X, \nabla_Y Z) + \rho_2(X)D_1(Y,Z)]u.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{R}^{\overline{\epsilon}}(X,Y)Z =& \overline{\nabla}^{\overline{\epsilon}}_X \overline{\nabla}^{\overline{\epsilon}}_Y Z - \overline{\nabla}^{\overline{\epsilon}}_Y \overline{\nabla}^{\overline{\epsilon}}_X Z \\
=& \overline{\nabla}^{\overline{\epsilon}}_X \overline{\nabla}^{\overline{\epsilon}}_Y Z - \overline{\nabla}^{\overline{\epsilon}}_Y \overline{\nabla}^{\overline{\epsilon}}_X Z - D_1(Z,[X,Y])N - D_2(Z,[X,Y])u \\
=& \nabla_X \nabla_Y Z - A_N X D_1(Y,Z) - D_2(Y,Z)A_u X \\
&+ [X(D_1(Y,Z)) + D_1(X, \nabla_Y Z) + \rho_1(X)D_1(Y,Z) \\
&+ \epsilon_1(X)D_2(Y,Z)]N \\
&+ [X(D_2(Y,Z)) + D_2(X, \nabla_Y Z) + \rho_2(X)D_1(Y,Z)]u \\
&- \nabla_Y \nabla_X Z - A_N Y D_1(X,Z) - D_2(X,Z)A_u Y \\
&- [Y(D_1(X,Z)) + D_1(Y, \nabla_X Z) + \rho_1(Y)D_1(X,Z) \\
&+ \epsilon_1(Y)D_2(X,Z)]N \\
&- [Y(D_2(X,Z)) + D_2(Y, \nabla_X Z) + \rho_2(Y)D_1(X,Z)]u \\
&- D_1(Z, \nabla_X Y)N + D_1(Z, \nabla_Y X)N \\
&- D_2(Z, \nabla_X Y)N + D_2(Z, \nabla_Y X)u \\
=& R(X,Y)Z + D_1(X,Z)A_N Y - D_1(Y,Z)A_N X \\
&+ D_2(X,Z)A_u Y - D_2(Y,Z)A_u X \\
&+ [(\nabla_X D_1)(Y,Z) - (\nabla_Y D_1)(X,Z) + \rho_1(X)D_1(Y,Z) \\
&- \rho_1(Y)D_1(X,Z) + \epsilon_1(X)D_2(Y,Z) - \epsilon_1(Y)D_2(X,Z)]N \\
&+ [(\nabla_X D_2)(Y,Z) - (\nabla_Y D_2)(X,Z) \\
&+ \rho_2(X)D_1(Y,Z) - \rho_2(Y)D_1(X,Z)]u,
\end{aligned}$$

which vanishes if and only if (5.15) holds.

(b) Letting $vN \in \Gamma(\epsilon)$, $v \neq 0$ we have

$$\begin{aligned}
\overline{\nabla}^{\overline{\epsilon}}_X \overline{\nabla}^{\overline{\epsilon}}_Y N =& \overline{\nabla}^{\overline{\epsilon}}_X [-A_N Y + \rho_1(Y)N + \rho_2(Y)u] \\
=& -\overline{\nabla}^{\overline{\epsilon}}_X (A_N Y) + X(\rho_1(Y))N + \rho_1(Y)\overline{\nabla}^{\overline{\epsilon}}_X N \\
&+ X(\rho_2(Y))u + \rho_2(Y)\overline{\nabla}^{\overline{\epsilon}}_X u \\
=& -\nabla_X(A_N Y) - D_1(X, A_N Y)N - D_2(X, A_N Y)u \\
&+ X(\rho_1(Y))N - \rho_1(Y)A_N X + \rho_1(Y)\rho_1(X)N + \rho_1(Y)\rho_2(X)u \\
&+ X(\rho_2(Y))u - \rho_2(Y)A_u X + \rho_2(Y)\epsilon_1(X)N \\
=& -\nabla_X(A_N Y) - \rho_1(Y)A_N X - \rho_2(Y)A_u X \\
&+ [-D_1(X, A_N Y) + X(\rho_1(Y)) + \rho_1(Y)\rho_1(X) + \epsilon_1(X)\rho_2(Y)]N \\
&+ [-D_2(X, A_N Y) + X(\rho_2(Y)) + \rho_1(Y)\rho_2(X)]u.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
\overline{R}^{\epsilon}(X,Y)vN =&\overline{\nabla}^{\epsilon}_X\overline{\nabla}^{\epsilon}_Y vN - \overline{\nabla}^{\epsilon}_Y\overline{\nabla}^{\epsilon}_X vN\\
=&\overline{\nabla}^{\epsilon}_X[Y(v)N + v\overline{\nabla}^{\epsilon}_Y N] - \overline{\nabla}^{\epsilon}_Y[X(v)N + v\overline{\nabla}^{\epsilon}_X N]\\
=&X(Y(v))N + Y(v)\overline{\nabla}^{\epsilon}_X N + X(v)\overline{\nabla}^{\epsilon}_Y N + v\overline{\nabla}^{\epsilon}_X\overline{\nabla}^{\epsilon}_Y N\\
&-Y(X(v))N - X(v)\overline{\nabla}^{\epsilon}_Y N - Y(v)\overline{\nabla}^{\epsilon}_X N - v\overline{\nabla}^{\epsilon}_Y\overline{\nabla}^{\epsilon}_X N\\
=&[X,Y](v)N + v[\overline{\nabla}^{\epsilon}_X\overline{\nabla}^{\epsilon}_Y N - \overline{\nabla}^{\epsilon}_Y\overline{\nabla}^{\epsilon}_X N]\\
=&v[\overline{\nabla}^{\epsilon}_X\overline{\nabla}^{\epsilon}_Y N - \overline{\nabla}^{\epsilon}_Y\overline{\nabla}^{\epsilon}_X N]\\
=&v\{-\nabla_X(A_NY) - \rho_1(Y)A_NX - \rho_2(Y)A_uX\\
&+[-D_1(X,A_NY) + X(\rho_1(Y)) + \rho_1(Y)\rho_1(X) + \epsilon_1(X)\rho_2(Y)]N\\
&+[-D_2(X,A_NY) + X(\rho_2(Y)) + \rho_1(Y)\rho_2(X)]u\\
&+\nabla_Y(A_NX) + \rho_1(X)A_NY + \rho_2(X)A_uY\\
&-[-D_1(Y,A_NX) + Y(\rho_1(X)) + \rho_1(X)\rho_1(Y) + \epsilon_1(Y)\rho_2(X)]N\\
&-[-D_2(Y,A_NX) + Y(\rho_2(X)) + \rho_1(X)\rho_2(Y)]u\}\\
=&v\{-\nabla_X(A_NY) + \nabla_Y(A_NX)\\
&+\rho_1(X)A_NY - \rho_1(Y)A_NX + \rho_2(X)A_uY - \rho_2(Y)A_uX\\
&+[D_1(Y,A_NX) - D_1(X,A_NY) + 2d\rho_1(X,Y)\\
&+\epsilon_1(X)\rho_2(Y) - \epsilon_1(Y)\rho_2(X)]N\\
&+[D_2(Y,A_NX) - D_2(X,A_NY) + 2d\rho_2(X,Y)\\
&+\rho_1(Y)\rho_2(X) - \rho_1(X)\rho_2(Y)]u\},
\end{aligned}
$$

which vanishes if and only if (5.16) holds.

(c) For $wu \in \Gamma(\delta)$ and $u \neq 0$ we obtain

$$
\begin{aligned}
\overline{\nabla}^{\epsilon}_X\overline{\nabla}^{\epsilon}_Y u =&\overline{\nabla}^{\epsilon}_X[-A_uY + \epsilon_1(Y)N]\\
=&-\overline{\nabla}^{\epsilon}_X(A_uY) + X(\epsilon_1(Y))N + \epsilon_1(Y)\overline{\nabla}^{\epsilon}_X N\\
=&-\nabla_X(A_uY) - D_1(X,A_uY)N - D_2(X,A_uY)u\\
&+X(\epsilon_1(Y))N - \epsilon_1(Y)A_NX + \epsilon_1(Y)\rho_1(X)N + \epsilon_1(Y)\rho_2(X)u\\
=&-\nabla_X(A_uY) - \epsilon_1(Y)A_NX\\
&+[-D_1(X,A_uY) + X(\epsilon_1(Y)) + \epsilon_1(Y)\rho_1(X)]N\\
&+[-D_2(X,A_uY) + \epsilon_1(Y)\rho_2(X)]u.
\end{aligned}
$$

In this way,

$$\begin{aligned}\overline{R}^{\overline{\epsilon}}(X,Y)wu =&\overline{\nabla}^{\overline{\epsilon}}_X\overline{\nabla}^{\overline{\epsilon}}_Y wu - \overline{\nabla}^{\overline{\epsilon}}_Y\overline{\nabla}^{\overline{\epsilon}}_X wu\\ =&\overline{\nabla}^{\overline{\epsilon}}_X[Y(w)u + w\overline{\nabla}^{\overline{\epsilon}}_Y u] - \overline{\nabla}^{\overline{\epsilon}}_Y[X(w)u + w\overline{\nabla}^{\overline{\epsilon}}_X u]\\ =&X(Y(w))u + Y(w)\overline{\nabla}^{\overline{\epsilon}}_X u + X(w)\overline{\nabla}^{\overline{\epsilon}}_Y u + w\overline{\nabla}^{\overline{\epsilon}}_X\overline{\nabla}^{\overline{\epsilon}}_Y u\\ &- Y(X(w))u - X(w)\overline{\nabla}^{\overline{\epsilon}}_Y u - Y(w)\overline{\nabla}^{\overline{\epsilon}}_X u - w\overline{\nabla}^{\overline{\epsilon}}_Y\overline{\nabla}^{\overline{\epsilon}}_X u\\ =&[X,Y](w)u + w[\overline{\nabla}^{\overline{\epsilon}}_X\overline{\nabla}^{\overline{\epsilon}}_Y u - \overline{\nabla}^{\overline{\epsilon}}_Y\overline{\nabla}^{\overline{\epsilon}}_X u]\\ =&w[\overline{\nabla}^{\overline{\epsilon}}_X\overline{\nabla}^{\overline{\epsilon}}_Y u - \overline{\nabla}^{\overline{\epsilon}}_Y\overline{\nabla}^{\overline{\epsilon}}_X u]\\ =&w\{-\nabla_X(A_uY) - \epsilon_1(Y)A_NX\\ &+ [-D_1(X,A_uY) + X(\epsilon_1(Y)) + \epsilon_1(Y)\rho_1(X)]N\\ &+ [-D_2(X,A_uY) + \epsilon_1(Y)\rho_2(X)]u\\ &+ \nabla_Y(A_uX) + \epsilon_1(X)A_NY\\ &- [-D_1(Y,A_uX) + Y(\epsilon_1(X)) + \epsilon_1(X)\rho_1(Y)]N\\ &- [-D_2(Y,A_uX) + \epsilon_1(X)\rho_2(Y)]u\}\\ =&w\{-\nabla_X(A_uY) + \nabla_Y(A_uX) + \epsilon_1(X)A_NY - \epsilon_1(Y)A_NX\\ &+ [D_1(Y,A_uX) - D_1(X,A_uY) + 2d\epsilon_1(X,Y)\\ &+ \rho_1(X)\epsilon_1(Y) - \rho_1(Y)\epsilon_1(X)]N\\ &+ [D_2(Y,A_uX) - D_2(X,A_uY)\\ &+ \epsilon_1(Y)\rho_2(X) - \epsilon_1(X)\rho_2(Y)]u\},\end{aligned}$$

which vanishes if and only if (5.17) holds.

□

Now we can state and prove the following Fundamental Theorem.

Theorem 5.1. *Let $(M,g,S(TM))$ a simply connected null $(n+1)$-manifold, with a screen distribution of dimension n and index $q-1$. Let ϵ and δ rank 1 vector bundles over M. Let $\overline{g}^{\overline{\epsilon}}$ the metric over $\overline{\epsilon} = TM \oplus \delta \oplus \epsilon$ defined by (5.4). Let us denote by d the index of δ in this metric and let $\overline{\nabla}^{\overline{\epsilon}}$ a connection over $\overline{\epsilon}$ satisfying (5.7)-(5.14). Moreover, suppose the Gauss-Codazzi-Ricci equations (5.15)-(5.17) are satisfied for $\overline{\epsilon}$. Then there exists an isometric immersion $f : M \to \mathbb{R}^{n+3}_{q+d}$, and a vector bundle isometry $\phi : \epsilon \oplus \delta \to tr(TM)$ such that $D_1 = D_1^0$, $D_2 = D_2^0$, $\rho_1 = \rho_1^0$, $\rho_2 = \rho_2^0$ and $\epsilon_1 = \epsilon_1^0$.*

Moreover, let $f,g : M^{n+1} \to \mathbb{R}^{n+3}_{q+d}$ two isometric immersions and suppose there exists a vector bundle isometry $\overline{\psi} : f^*T\mathbb{R}^{n+3}_{q+d} \to g^*T\mathbb{R}^{n+3}_{q+d}$ such that

$$\overline{\psi} f_* = g_*,$$

and $\overline{\psi}|_{\mathrm{tr}^f(TM)} = \psi$ satisfy

$$\psi(\phi^f(N)) = \phi^g(N), \quad \psi(\phi^f(u)) = \phi^g(u).$$

Then there exists an isometry $\tau : \mathbb{R}^{n+3}_{q+d} \to \mathbb{R}^{n+3}_{q+d}$ such that

$$\tau f = g \quad \text{and} \quad \tau_*|_{\mathrm{tr}^f(TM)} = \psi.$$

Proof. Proposition 5.3 implies that the connection $\overline{\nabla}^{\overline{\epsilon}}$ is compatible with the metric $\overline{g}^{\overline{\epsilon}}$. Then, Proposition 5.4 establishes that the curvature tensor $\overline{R}^{\overline{\epsilon}}$ vanishes identically over $\overline{\epsilon}$. As in Theorem 4.1, let us consider a global parallel frame $\{E_i\}$ with respect to $\overline{\nabla}^{\overline{\epsilon}}$. Besides, by Proposition 5.2 we know that $\overline{\epsilon}$ have index $q+d$. Thus, without loss of generality,

$$\overline{g}^{\overline{\epsilon}}(E_i,E_i) = \begin{cases} -1 & \text{if} \quad i \in \{1,\dots,q+d\} \\ 1 & \text{if} \quad i \in \{q+d+1,\dots,n+3\} \end{cases}$$

Now, let $(x_1,\dots,x_{n+1})$ be local coordinates in a simply connected neighbourhood U of the point p. We write

$$\frac{\partial}{\partial x_i} = \sum_{k=1}^{n+3} a_{ik}E_k$$

for some functions a_{ik}. Because $\{E_i\}$ is an orthonormal frame, the metric coefficients satisfy

$$\begin{aligned}\overline{g}^{\epsilon}_{ij} =& \overline{g}^{\epsilon}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \overline{g}^{\epsilon}\left(\sum_{l=1}^{n+3} a_{jl}E_l, \sum_{k=1}^{n+3} a_{jk}E_k\right) = \sum_{l=1}^{n+3}\sum_{k=1}^{n+3} a_{jl}a_{ik}\overline{g}^{\epsilon}(E_l, E_k)\\ =& -\sum_{k=1}^{q+d} a_{ik}a_{jk} + \sum_{k=q+d+1}^{n+3} a_{ik}a_{jk}\end{aligned}$$

On the other hand, since $\{E_i\}$ is a parallel frame,

$$\begin{aligned}\overline{\nabla}^{\epsilon}_{\frac{\partial}{\partial x_i}}\left(\frac{\partial}{\partial x_j}\right) =& \overline{\nabla}^{\epsilon}_{\sum_{k=1}^{n+3} a_{ik}E_k} \sum_{l=1}^{n+3} a_{jl}E_l = \sum_{l=1}^{n+3}\sum_{k=1}^{n+3} a_{ik}\overline{\nabla}^{\epsilon}_{E_k} a_{jl}E_l\\ =& \sum_{l=1}^{n+3}\sum_{k=1}^{n+3} a_{ik}\left(E_k(a_{jl})E_l + a_{jl}\overline{\nabla}^{\epsilon}_{E_k}E_l\right) = \sum_{l=1}^{n+3}\sum_{k=1}^{n+3} a_{ik}E_k(a_{jl})E_l\\ =& \sum_{l=1}^{n+3}\frac{\partial a_{jl}}{\partial x_i}E_l\end{aligned}$$

Analogously,

$$\overline{\nabla}^{\epsilon}_{\frac{\partial}{\partial x_j}}\left(\frac{\partial}{\partial x_i}\right) = \sum_{l=1}^{n+3}\frac{\partial a_{il}}{\partial x_j}E_l$$

In this way, because ∇ is a torsion-free connection,

$$\overline{\nabla}^{\epsilon}_{\frac{\partial}{\partial x_i}}\left(\frac{\partial}{\partial x_j}\right) = \overline{\nabla}^{\epsilon}_{\frac{\partial}{\partial x_j}}\left(\frac{\partial}{\partial x_i}\right)$$

Consequently,

$$\frac{\partial a_{jk}}{\partial x_i} = \frac{\partial a_{ik}}{\partial x_j} \quad \forall i, j, k$$

Therefore, the 1-forms given by

$$w_k = a_{1k}dx_1 + ... + a_{(n+1)k}dx_{n+1}$$

are closed for all $k \in \{1, ..., n+3\}$. By Poincaré Lemma, w_k is exact for all $k \in \{1, ..., n+3\}$. Therefore, there exist functions f_k such that

$$\frac{\partial f_k}{\partial x_i} = a_{ik} \qquad \forall i \in \{1, ..., n+1\}$$

Now, let $f : U \to \mathbb{R}^{n+3}_{q+d}$ given by $f = (f_1, ..., f_{n+3})$ and note that

$$f_*\left(\frac{\partial}{\partial x_i}\right) = \left(\frac{\partial f_1}{\partial x_i}, ..., \frac{\partial f_{n+3}}{\partial x_i}\right) = (a_{i1}, ..., a_{i(n+3)}).$$

Consequently,

$$g^0\left(f_*\left(\frac{\partial}{\partial x_i}\right), f_*\left(\frac{\partial}{\partial x_j}\right)\right) = -\sum_{k=1}^{q+d} a_{ik}a_{jk} + \sum_{k=q+d+1}^{n+3} a_{ik}a_{jk}$$

where g^0 is the standard flat metric of $\mathbb{R}^{n+3}_{q+d}$. Therefore, f is an isometric immersion. In particular, $f(U)$ is a null submanifold of $\mathbb{R}^{n+3}_{q+d}$.

Since the vector bundle $T\mathbb{R}^{n+3}_{q+d}$ is trivial, for the remaining of the proof we use the standard identification of the pullback bundle $f^*\mathbb{R}^{n+3}_{q+d}$ with $M \times \mathbb{R}^{n+3}_{q+d}$.

We define now the vector bundle morphism

$$\overline{\phi} : TU \oplus \epsilon|_U \oplus \delta|_U \to U \times \mathbb{R}^{n+3}_{q+d}$$

given by

$$\overline{\phi}(E_k) = e^0_k.$$

In particular,

$$\overline{\phi}\left(\frac{\partial}{\partial x_i}\right) = \sum_{k=1}^{n+3} a_{ik}\overline{\phi}(E_k) = \sum_{k=1}^{n+3} a_{ik}e^0_k = f_*\left(\frac{\partial}{\partial x_i}\right),$$

which implies that $\overline{\phi}|_{TU} = f_*$ is an isomorphism over $T(f(U))$.

Note that given $X, Y \in \Gamma(\overline{\epsilon})$ we have the following:

$$\begin{aligned}\overline{g}^{\epsilon}(X,Y) =& \overline{g}^{\epsilon}\left(\sum_{k=1}^{n+3} x_k E_k, \sum_{l=1}^{n+3} y_l E_l\right) = \sum_{k=1}^{n+3}\sum_{l=1}^{n+3} x_k y_l \overline{g}^{\epsilon}(E_k, E_l)\\ =& \sum_{k=1}^{n+3}\sum_{l=1}^{n+3} x_k y_l g^0(e^0_k, e^0_l) = g^0\left(\sum_{k=1}^{n+3} x_k e^0_k, \sum_{l=1}^{n+3} y_l e^0_l\right)\\ =& g^0(\phi(X), \phi(Y)).\end{aligned} \tag{5.18}$$

Consequently,

$$\mathrm{Rad}(Tf(U)) = \overline{\phi}(\mathrm{Rad}(TU)),$$

and thus $\mathrm{Rad}(Tf(U))$ is generated by $\overline{\phi}(\xi)$.

Note that $\overline{\phi}(S(TU))$ is a screen distribution on $Tf(U)$ because it is a distribution on $Tf(U)$, and equation (5.18) implies that it is orthogonal to $\mathrm{Rad}(Tf(U))$. Given $Z \in \Gamma(Tf(U))$ there exists $Z' \in \Gamma(TU)$ such that $\overline{\phi}(Z') = Z$, but $Z' = Z_1 + Z_2$ with $Z_1 \in \Gamma(\mathrm{Rad}(TU))$ and $Z_2 \in \Gamma(S(TU))$. Therefore,

$$Z = \overline{\phi}(Z_1) + \overline{\phi}(Z_2),$$

where $\overline{\phi}(Z_1) \in \Gamma(\mathrm{Rad}(Tf(U)))$ and $\overline{\phi}(Z_2) \in \Gamma(\phi(S(TU)))$. Consequently $\phi(S(TU))$ is a screen distribution of TU.

Because $\overline{\phi}(u) \in \Gamma(Tf(U)^{\perp})$ and $g^0(\overline{\phi}(u), \overline{\phi}(u)) = \overline{g}^{\epsilon}(u,u) = \lambda \neq 0$, we have that $\overline{\phi}(u) \notin \Gamma(\mathrm{Rad}(Tf(U)))$ and it is linearly independent with $\overline{\phi}(\xi)$. Therefore, $\overline{\phi}(u)$ span a complementary vector bundle to $\mathrm{Rad}(Tf(U))$ in $Tf(U)^{\perp}$. Then $\overline{\phi}(\delta|_U)$ is a screen transversal distribution. Further, $\overline{\phi}(\delta|_U)$ is a subbundle of $\overline{\phi}(S(TU))^{\perp}$ and both are semi-Riemannian bundles. Then we have

$$\overline{\phi}(S(TU))^{\perp} = \overline{\phi}(\delta|_U) \perp \oplus \overline{\phi}(\delta|_U)^{\perp}.$$

Since $\overline{\phi}(N), \overline{\phi}(\xi) \in \overline{\phi}(\delta|_U)^{\perp}$ are linearly independent,

$$\overline{\phi}(N) = \frac{1}{g^0(\overline{\phi}(\xi), V)}\left(V - \frac{g^0(V,V)}{2g^0(\overline{\phi}(\xi), V)}\overline{\phi}(\xi)\right), V \in \Gamma(F)$$

where F is a complementary distribution of $\mathrm{Rad}(Tf(U))$ in $\overline{\phi}(\delta|_U)^{\perp}$, therefore $\overline{\phi}(\epsilon|_U)$ is a lightlike transversal bundle.

Then $(f(U), g^0, \overline{\phi}(S(TU)))$ is a half-lightlike submanifold of $\mathbb{R}^{n+3}_{q+d}$ with a lightlike transversal distribution $\overline{\phi}(\epsilon|_U)$ and a screen distribution $\overline{\phi}(\delta|_U)$, generated by $\phi(N)$ and $\phi(U)$, respectively.

Letting $\phi = \overline{\phi}|_{\epsilon|_U \oplus \delta|_U}$ the connection $\overline{\nabla}^0$ takes on $f(U)$ the following form:

$$\overline{\nabla}^0_X f_* Y = \nabla^0_X f_* Y + D^0_1(X,Y)\phi(N) + D^0_2(X,Y)\phi(u), \quad X, Y \in \Gamma(TU) \tag{5.19}$$

$$\overline{\nabla}^0_X\overline{\phi}(N) = -A^0_N X + \rho^0_1(X)\phi(N) + \rho^0_2(X)\phi(u), \quad X \in \Gamma(TU) \tag{5.20}$$

$$\overline{\nabla}^0_X\overline{\phi}(u) = -A^0_u X + \epsilon^0_1(X)\phi(N), \quad X \in \Gamma(TU) \tag{5.21}$$

Now, because $\overline{\phi}$ transforms the parallel orthonormal frame $\{E_i\}$ on the parallel orthonormal frame $\{e^0_i\}$, we have that for all $Y \in \Gamma(\overline{\epsilon}|_U)$.

$$\begin{aligned}
\overline{\phi}(\overline{\nabla}^{\epsilon}_X Y) =&\overline{\phi}\left(\overline{\nabla}^{\epsilon}_X \sum_{k=1}^{n+3} y_l E_l\right) = \overline{\phi}\left(\sum_{k=1}^{n+3} X(y_k)E_k + y_k\overline{\nabla}^{\epsilon}_X E_k\right) = \overline{\phi}\left(\sum_{k=1}^{n+3} X(y_k)E_k\right)\\
=&\sum_{k=1}^{n+3} X(y_k)\overline{\phi}(E_k) = \sum_{k=1}^{n+3} X(y_k)e^0_k = \sum_{k=1}^{n+3} X(y_k)e^0_k + y_k\overline{\nabla}^0_X e^0_l = \overline{\nabla}^0_X \sum_{k=1}^{n+3} y_l e^0_l\\
=&\overline{\nabla}^0_X\overline{\phi}(Y).
\end{aligned}$$

Consequently, $\forall X, Y \in \Gamma(TU)$

$$\overline{\phi}(\overline{\nabla}^{\epsilon}_X Y) = \overline{\nabla}^0_X f_* Y, \tag{5.22}$$

$$\overline{\phi}(\overline{\nabla}^{\epsilon}_X N) = \overline{\nabla}^0_X \phi(N), \tag{5.23}$$

$$\overline{\phi}(\overline{\nabla}^{\epsilon}_X u) = \overline{\nabla}^0_X \phi(u). \tag{5.24}$$

The equation

$$\begin{aligned}
\overline{\phi}(\overline{\nabla}^{\epsilon}_X Y) =&\overline{\phi}(\nabla_X Y + D_1(X,Y)N + D_2(X,Y)u)\\
=&f_*(\nabla_X Y) + D_1(X,Y)\phi(N) + D_2(X,Y)\phi(u)
\end{aligned}$$

together with (5.19) and (5.22) imply that $\overline{\phi}(\overline{\nabla}_X Y) = \nabla^0_X f_* Y$, $D_1 = D^0_1$ and $D_2 = D^0_2$.

On the other hand,

$$\begin{aligned}
\overline{\phi}(\overline{\nabla}^{\epsilon}_X N) =&\overline{\phi}(-A_N X + \rho_1(X)N + \rho_2(X)u)\\
=&-f_*(A_N X) + \rho_1(X)\phi(N) + \rho_2(X)\phi(u)
\end{aligned}$$

together with (5.20) and (5.23) yields $f_* A_N = A^0_N$, $\rho_1 = \rho^0_1$ and $\rho_2 = \rho^0_2$.

Now, the equation

$$\begin{aligned}
\overline{\phi}(\overline{\nabla}^{\epsilon}_X u) =&\overline{\phi}(-A_u X + \epsilon(X)N)\\
=&-f_*(A_u X) + \epsilon_1(X)\phi(N)
\end{aligned}$$

together with (5.21) and (5.24) imply that $f_* A_u = A^0_u$ and $\epsilon_1 = \epsilon^0_1$.

Because M is simply connected, the vector bundle morphism $\overline{\phi}$ can be globally extended. We denote this global extension by $\overline{\phi}$.

Now we focus on the uniqueness part of the proof. We thus proceed to show that $\overline{\psi}$ is parallel respect to the pullbacks $\overline{\nabla}^{f,0}$ and $\overline{\nabla}^{g,0}$ of the standard flat connection $\overline{\nabla}^0$ of $\mathbb{R}^{n+3}_{q+d}$ over $f^*T\mathbb{R}^{n+3}_{q+d}$ and $g^*T\mathbb{R}^{n+3}_{q+d}$, respectively. First we note that

$$\begin{aligned} D_1^{f,0} &= D_1^{g,0}, \\ D_2^{f,0} &= D_2^{g,0}, \\ \rho_1^{f,0} &= \rho_1^{g,0}, \\ \rho_2^{f,0} &= \rho_2^{g,0}, \\ \epsilon_1^{f,0} &= \epsilon_1^{g,0}, \\ \overline{\psi}(A_N^{f,0}X) &= A_N^{g,0}X, \\ \overline{\psi}(A_u^{f,0}X) &= A_u^{g,0}X. \end{aligned}$$

Then, for every $X, Y \in \Gamma(TM)$ we have

$$\begin{aligned} \overline{\nabla}_X^{g,0}\overline{\psi}f_*Y =& \overline{\nabla}_X^{g,0}g_*Y \\ =& g_*\nabla_XY + D_1^{g,0}(X,Y)\phi^g(N) + D_2^{g,0}(X,Y)\phi^g(u) \\ =& \overline{\psi}f_*\nabla_XY + D_1^{f,0}(X,Y)\psi(\phi^f(N)) + D_2^{f,0}(X,Y)\psi(\phi^f(u)) \\ =& \overline{\psi}(f_*\nabla_XY + D_1^{f,0}(X,Y)\phi^f(N) + D_2^{f,0}(X,Y)\phi^f(u)) \\ =& \overline{\psi}\overline{\nabla}_X^{f,0}f_*Y \end{aligned}$$

Consequently,

$$\begin{aligned} \overline{\nabla}_X^{g,0}\overline{\psi}\phi^f(N) =& \overline{\nabla}_X^{g,0}\phi^g(N) \\ =& -A_N^{g,0}X + \rho_1^{g,0}(X)\phi^g(N) + \rho_2^{g,0}(X)\phi^g(u) \\ =& -\overline{\psi}(A_N^{f,0}X) + \rho_1^{f,0}(X)\psi(\phi^f(N)) + \rho_2^{f,0}(X)\psi(\phi^f(u)) \\ =& \overline{\psi}(-A_N^{f,0}X + \rho_1^{f,0}(X)\phi^f(N) + \rho_2^{f,0}(X)\phi^f(u)) \\ =& \overline{\psi}\overline{\nabla}_X^{f,0}\phi^f N, \end{aligned}$$

and

$$\begin{aligned} \overline{\nabla}_X^{g,0}\overline{\psi}\phi^f(u) =& \overline{\nabla}_X^{g,0}\phi^g(u) \\ =& -A_u^{g,0}X + \epsilon_1^{g,0}(X)\phi^g(N) \\ =& -\overline{\psi}(A_u^{f,0}X) + (\epsilon_1^{f,0}(X)\psi(\phi^f(N)) \\ =& \overline{\psi}(-A_u^{f,0}X + \epsilon_1^{f,0}(X)\phi^f(N)) \\ =& \overline{\psi}\overline{\nabla}_X^{f,0}\phi^f u. \end{aligned}$$

Therefore $\overline{\psi}$ is parallel. The fact that $T\mathbb{R}^{n+3}_{q+d} \cong \mathbb{R}^{2n+6}$ is flat implies that its transition functions are locally constant. Hence $\overline{\psi}$ defines an orthogonal transformation B over $\mathbb{R}^{n+2}_{q+d}$. Further, since $Bf_* = \overline{\psi}f_* = g_*$, there exists an isometry τ on $\mathbb{R}^{n+3}_{q+d}$ such that $\tau_* = B$ y $\tau f = g$. Then

$$\tau_*|_{\mathrm{tr}^f(TM)} = B|_{\mathrm{tr}^f(TM)} = \overline{\psi}|_{\mathrm{tr}^f(TM)} = \psi.$$

□

Our next goal is to prove a Fundamental Theorem for isometric immersions of 1-lightlike manifolds in semi-Riemannian space forms of constant sectional curvature $c \neq 0$. To achieve this goal, we construct a vector bundle that satisfies the assumptions of Theorem 5.1 from a vector bundle that satisfies the conditions of Theorem 5.2. Throughout the process we will denote the analogous objects between both vector bundles without distinction.

The following will be assumed:

- M is a null manifold with metric g and dimension $n+1$.
- $S(TM)$ is a screen distribution over M of dimension n and index $q-1$. Note that the restriction of g to the screen $S(TM)$ is a semi-Riemannian metric.
- $\mathrm{Rad}(TM)$ is the radical distribution of M with dimension 1 which satisfies:

$$TM = S(TM) \perp \mathrm{Rad}(TM). \tag{5.25}$$

- ∇ is a torsion-free connection over M which restricted to the screen $S(TM)$ becomes metric respect to g.
- The decomposition (5.25) imply the following Gauss-Weingarten formulae for M (compare to [13, Eqs. 2.12-2.13]):

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \tag{5.26}$$

  and

$$\nabla_X U = -A^*_U X + \nabla^{\perp}_X U, \tag{5.27}$$

  where P is the projection of TM over the screen $S(TM)$, $X, Y \in \Gamma(TM)$ and $U \in \Gamma(\mathrm{Rad}(TM))$.
  With $\nabla^*_X PY, A^*_U X \in \Gamma(S(TM))$ and $h^*(X, PY), \nabla^{\perp}_X U \in \Gamma(\mathrm{Rad}(TM))$.
  Besides, ∇^* is a metric connection over the screen $S(TM)$ and $\nabla^{\perp}$ is a linear connection, while A^* and h^* are $C^\infty(M)$ bilinear forms.

- ϵ is a rank 1 vector bundle over M which plays the role of $\mathrm{tr}(TM)$.

Now let γ the vector bundle over M given by

$$\gamma = TM \oplus \epsilon = S(TM) \oplus \mathrm{Rad}(TM) \oplus \epsilon. \tag{5.28}$$

We define a Riemannian metric on this bundle as follows. First let g_R and g_ϵ Riemannian metrics over the one dimensional $\mathrm{Rad}(TM)$ and ϵ, respectively. Now consider the product metric

$$g' = P^* g + P^*_{\mathrm{Rad}(TM)} g_R + P^*_\epsilon g_\epsilon \tag{5.29}$$

over $\bar{\epsilon}$. Let $\{\xi, N\}$ an orthonormal frame such that $span(\xi) = \mathrm{Rad}(TM)$ and $span(N) = \epsilon$.

Definition 5.2. For $X, Y \in \Gamma(\gamma)$ we define

$$g^\gamma(X,Y) = g'(X,Y) - (g'(X,\xi) - g'(X,N))(g'(Y,\xi) - g'(Y,N)). \tag{5.30}$$

As before, a straightforward computation based on (5.28) and (5.29) yields the following result.

Proposition 5.5. *$\bar{g}^{\bar{\epsilon}}$ is a Lorentzian metric over $\bar{\epsilon}$. Moreover, for $X, Y \in \Gamma(S(TM))$ we have:*

1. $g^\gamma(X,Y) = g(X,Y)$;
2. $g^\gamma(X,\xi) = g^\gamma(X,N) = 0$;
3. $g^\gamma(N,N) = 0$, $g^\gamma(\xi,\xi) = 0$;
4. $g^\gamma(\xi,N) = 1$.

Corollary 5.2. *With the metric g^γ the bundle γ can be written in the following decomposition:*

$$\gamma = S(TM) \perp (Rad(TM) \oplus \epsilon).$$

We define next a new connection ∇^γ over γ by

$$\nabla^\gamma_X Y = \nabla_X Y + h(X,Y),$$

and

$$\nabla_X V = -A^\gamma_V X + \nabla^t_X V,$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(\epsilon)$, where $\nabla_X Y, A^\gamma_V X \in \Gamma(TM)$ and $h(X,Y), \nabla^t_X V \in \Gamma(\epsilon)$. $h(X,Y)$ is a symmetric bilinear $C^\infty(M)$ form and A is a $C^\infty(M)$ linear operator, ∇^t is a linear connection over ϵ.

To extend the bundle γ to a semi-Riemannian bundle of dimension $(n+3)$, which satisfies the hypothesis of Theorem 5.7, we make the following assumptions between some of the objects previously defined.

$$B(X,Y)N = h(X,Y), \tag{5.31}$$

$$\nabla^t_X N = g^\gamma(\nabla^t_X N, \xi)N = -g^\gamma(\nabla^\perp_X \xi, N)N \tag{5.32}$$

and

$$B(X,\xi) = 0, \tag{5.33}$$

$$g^\gamma(h^*(X,PY),N) = g(A^\gamma_N X, PY), \tag{5.34}$$

$$g^\gamma(A^\gamma_N X, N) = 0 \tag{5.35}$$

$$g(A^*_\xi X, PY) = B(X,PY), \tag{5.36}$$

$$(\nabla_X g)(Y,Z) - g^\gamma(Y,N)B(X,Z) - g^\gamma(Z,N)B(X,Y) = 0 \tag{5.37}$$

for all $X,Y,Z \in \Gamma(TM)$, where B is a $C^\infty(M)$ bilinear form.

We assume also the following structure equations for constant sectional curvature $c = \frac{\sigma}{r^2} \neq 0$.

$$0 = R(X,Y)Z + A^\gamma_{h(X,Z)}Y - A^\gamma_{h(Y,Z)}X - \frac{\sigma}{r^2}(g(Y,Z)X - g(X,Z)Y), \tag{5.38}$$

$$0 = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{5.39}$$

$$\begin{aligned} 0 = &-\nabla_X(A^\gamma_V Y) - A^\gamma_{\nabla^t_Y V}X + \nabla_Y(A^\gamma_V X) + A^\gamma_{\nabla^t_X V}V + A^\gamma_V[X,Y] \\ &- \frac{\sigma}{r^2}(g^\gamma(Y,V)X - g^\gamma(X,V)Y), \end{aligned} \tag{5.40}$$

$$0 = -h(X, A^\gamma_V Y) + \nabla^t_X \nabla^t_Y V + h(Y, A^\gamma_V X) - \nabla^t_Y \nabla^t_X V - \nabla^t_{[X,Y]}V. \tag{5.41}$$

Let δ a semi-Riemannian rank 1 bundle with metric g^δ and index d and let $u \in \Gamma(\delta)$ unit which satisfies

$$g^\delta(u,u) = \sigma.$$

Define a metric $\overline{g}^{\overline{\epsilon}}$ over $\overline{\epsilon} = \delta \oplus \gamma$ such that $\gamma \perp \delta$ and restricted to δ and γ coincides with g^δ and g^γ, respectively. The following results are immediate consequences:

Proposition 5.6. *The metric $\overline{g}^{\overline{\epsilon}}$ coincides with the metric defined in* (5.4).

Corollary 5.3. *The metric $\overline{g}^{\overline{\epsilon}}$ defined on the bundle $\overline{\epsilon}$ has index $q+d$.*

Now, we define the connection $\overline{\nabla}^{\overline{\epsilon}}$ over $\overline{\epsilon}$ by the equations

$$\overline{\nabla}^{\overline{\epsilon}}_X Y = \nabla_X Y + B(X,Y)N + \frac{\sigma}{r} g(X,Y)u, \tag{5.42}$$

$$\overline{\nabla}^{\overline{\epsilon}}_X N = -A^\gamma_N X + g^\gamma(\nabla^t_X N, \xi)N + \frac{\sigma}{r} g^\gamma(X,N)u, \tag{5.43}$$

and

$$\overline{\nabla}_X u = -\frac{1}{r}X. \tag{5.44}$$

Next we show that this connection on $\overline{\epsilon}$ satisfies the hypotheses of Theorem 5.7.

Proposition 5.7. *Suppose conditions (5.31)-(5.37) are satisfied. Then equations (5.7)-(5.14) hold.*

Proof. (i) $\overline{g}^{\overline{\epsilon}}(h^*(X,PY),N) = \overline{g}^{\overline{\epsilon}}(A_N X, PY)$. Using (5.34), we have

$$\overline{g}^{\overline{\epsilon}}(h^*(X,PY),N) = g^{\gamma}(h^*(X,PY),N) = g^{\gamma}(A_N^{\gamma} X, PY) = \overline{g}^{\overline{\epsilon}}(A_N X, PY).$$

(ii) $D_1(X,PY) = g(A_\xi^* X, PY)$. By (5.36) it follows that

$$D_1(X,PY) = B(X,Y) = g(A_\xi^* X, PY).$$

(iii) $\epsilon_2(X) = 0$. This is immediate from the definition of $\overline{\nabla}^{\overline{\epsilon}}$.

(iv) $D_1(X,\xi) = 0$. From (5.33) we obtain

$$D_1(X,\xi) = B(X,\xi) = 0.$$

(v) $\overline{g}^{\overline{\epsilon}}(A_N X, N) = 0$. The equation (5.35) implies that

$$\overline{g}^{\overline{\epsilon}}(A_N X, N) = g^{\gamma}(A_N^{\gamma} X, N) = 0.$$

(vi) $\rho_1(X) = -\overline{g}^{\overline{\epsilon}}(\nabla_X^{\perp}\xi, N)$. Using (5.32),

$$\rho_1(X) = g^{\gamma}(\nabla_X^t N, \xi) = -g^{\gamma}(\nabla_X^{\perp}\xi, N) = -\overline{g}^{\overline{\epsilon}}(\nabla_X^{\perp}\xi, N).$$

(vii) $\rho_2(X) = \sigma\overline{g}^{\overline{\epsilon}}(A_u X, N)$. This follows from the definitions of ρ_2 and $A_u X$:

$$\rho_2(X) = \frac{\sigma}{r} g(X,N), = \sigma\overline{g}^{\overline{\epsilon}}(A_u X, N).$$

(viii) $\epsilon_1(X) = -\sigma D_2(X,\xi)$. By the item (iv) above, we have

$$\epsilon_1 = -\sigma D_1(X,\xi)$$

(ix) $\overline{g}^{\overline{\epsilon}}(A_u X, Y) = \sigma D_2(X,Y) + \epsilon_1(X)\overline{g}^{\overline{\epsilon}}(Y,N)$. It follows from definitions of $A_u X$, ϵ_2 and D_2:

$$\overline{g}^{\overline{\epsilon}}(A_u X, Y) = \overline{g}^{\overline{\epsilon}}\left(\frac{1}{r}X, Y\right) = \sigma\frac{\sigma}{r} g(X,Y) + 0 = \sigma D_2(X,Y) + \epsilon_1(X)\overline{g}^{\overline{\epsilon}}(Y,N).$$

□

We now verify the validity of the structure equations for half-lightlike manifolds when $c = 0$.

Proposition 5.8. *Assume the hypotheses of Proposition 5.7, and that equations (5.38)-(5.41) hold. Then equations (5.15) - (5.17) are satisfied.*

Proof. (i)

$$R(X,Y)Z + D_1(X,Z)A_N Y - D_1(Y,Z)A_N X + D_2(X,Z)A_u Y - D_2(Y,Z)A_u X = 0$$

Denote by C_1 the left hand side of the previous equation. Applying (5.31), (5.38) and the linearity of A^{γ}.

$$\begin{aligned} C_1 =& R(X,Y)Z + B(X,Z)A_N^{\gamma} Y - B(Y,Z)A_N^{\gamma} X + \frac{\sigma}{r^2} g(X,Z)Y - \frac{\sigma}{r^2} g(Y,Z)X \\ =& R(X,Y)Z + A^{\gamma}_{B(X,Z)N} Y - A^{\gamma}_{B(Y,Z)N} X - \frac{\sigma}{r^2}(g(Y,Z)X - g(X,Z)Y) \\ =& R(X,Y)Z + A^{\gamma}_{h(X,Z)} Y - A^{\gamma}_{h(Y,Z)} X - \frac{\sigma}{r^2}(g(Y,Z)X - g(X,Z)Y) \\ =& 0. \end{aligned}$$

(ii)

$$\begin{aligned} 0 =& \{(\nabla_X D_1)(Y,Z) - (\nabla_Y D_1)(X,Z) + \rho_1(X)D_1(Y,Z) - \rho_1(Y)D_1(X,Z) \\ &+ \epsilon_1(X)D_2(Y,Z) - \epsilon_1(Y)D_2(X,Z)\}N \end{aligned}$$

Let C_2 be the right hand side of the previous equation. Applying (5.31), (5.32), the fact that ∇^t is a linear connection and (5.38),

$$\begin{aligned}C_2 =&\{X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z) - Y(B(X,Z)) + B(\nabla_Y X, Z)\\ &+ B(X, \nabla_Y Z)\} + g^\gamma(\nabla^t_X N, \xi)B(Y,Z) - g^\gamma(\nabla^t_Y N, \xi)B(X,Z)\}N\\ =&X(B(Y,Z))N - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) - Y(B(X,Z))N\\ &+ h(\nabla_Y X, Z) + h(X, \nabla_Y Z) + B(Y,Z)\nabla^t_X N - B(X,Z)\nabla^t_Y N\\ =&X(B(Y,Z))N + B(Y,Z)\nabla^t_X N - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)\\ &- Y(B(X,Z))N - B(X,Z)\nabla^t_Y N + h(\nabla_Y X, Z) + h(X, \nabla_Y Z)\\ =&[\nabla^t_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)]\\ &- [\nabla^t_Y h(X,Z) - h(\nabla_Y X, Z) - h(X, \nabla_Y Z)]\\ =&(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z)\\ =&0.\end{aligned}$$

(iii)

$$\{(\nabla_X D_2)(Y,Z) - (\nabla_Y D_2)(X,Z) + \rho_2(X)D_1(Y,Z) - \rho_2(Y)D_1(X,Z)\}u = 0,$$

Let C_3 be the right hand side of the previous equation. Applying (5.37) yields

$$\begin{aligned}C_3 =&\left\{X\left(\frac{\sigma}{r}g(Y,Z)\right) - \frac{\sigma}{r}g(\nabla_X Y, Z) - \frac{\sigma}{r}g(Y, \nabla_X Z) - Y\left(\frac{\sigma}{r}g(X,Z)\right)\right.\\ &+ \frac{\sigma}{r}g(\nabla_Y X, Z) + \frac{\sigma}{r}g(X, \nabla_X Z)Y + \frac{\sigma}{r}g^\gamma(X,N)B(Y,Z)\\ &\left.-\frac{\sigma}{r}g^\gamma(Y,N)B(X,Z)\right\}u\\ =&\frac{\sigma}{r}\{(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) + g^\gamma(X,N)B(Y,Z)\\ &- g^\gamma(Y,N)B(X,Z)\}u\\ =&\frac{\sigma}{r}\{(\nabla_X g)(Y,Z) - g^\gamma(Y,N)B(X,Z) - g^\gamma(Z,N)B(X,Y)\\ &- (\nabla_Y g)(X,Z) + g^\gamma(X,N)B(Y,Z) + g^\gamma(Z,N)B(X,Y)\}u\\ =&0.\end{aligned}$$

(iv)

$$\begin{aligned}0 =& - \nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X,Y]\\ &+ \rho_1(X)A_N Y - \rho_1(Y)A_N X + \rho_2(X)A_u Y - \rho_2(Y)A_u X\end{aligned}$$

Denote by C_4 the right hand side of the previous equation. Using (5.40), (5.32) and the linearity of A^γ we have

$$\begin{aligned}C_4 =& - \nabla_X(A^\gamma_N Y) + \nabla_Y(A^\gamma_N X) + A^\gamma_N[X,Y]\\ &+ g^\gamma(\nabla^t_X N, \xi)A^\gamma_N Y - g^\gamma(\nabla^t_Y N, \xi)A^\gamma_N X - \frac{\sigma}{r^2}(g^\gamma(Y,N)X - g^\gamma(X,N)Y)\\ =& - \nabla_X(A^\gamma_N Y) + \nabla_Y(A^\gamma_N X) + A^\gamma_N[X,Y] + g^\gamma(\nabla^t_X N, \xi)A^\gamma_N Y - g^\gamma(\nabla^t_Y N, \xi)A^\gamma_N X\\ &+ \nabla_X(A^\gamma_N Y) + A^\gamma_{\nabla^t_Y N}X - \nabla_Y(A^\gamma_N X) - A^\gamma_{\nabla^t_X N}N - A^\gamma_N[X,Y]\\ =&g^\gamma(\nabla^t_X N, \xi)A^\gamma_N Y - g^\gamma(\nabla^t_Y N, \xi)A^\gamma_N X + A^\gamma_{\nabla^t_Y N}X - A^\gamma_{\nabla^t_X N}N\\ =&A^\gamma_{g^\gamma(\nabla^t_X N, \xi)N}Y - A^\gamma_{g^\gamma(\nabla^t_Y N, \xi)N}X + A^\gamma_{\nabla^t_Y N}X - A^\gamma_{\nabla^t_X N}N\\ =&A^\gamma_{\nabla^t_X N}N - A^\gamma_{\nabla^t_Y N}X + A^\gamma_{\nabla^t_Y N}X - A^\gamma_{\nabla^t_X N}N\\ =&0.\end{aligned}$$

(v)

$$\begin{aligned}0 =&\{D_1(Y, A_N X) - D_1(X, A_N Y) + 2d\rho_1(X,Y)\\ &+ \epsilon_1(X)\rho_2(Y) - \epsilon_1(Y)\rho_2(X)\}N\end{aligned}$$

Let C_5 be the right hand side of the previous equation. Using (5.31), (5.41), (5.32) and that ∇^t is linear connection, it follows that

$$\begin{aligned}
C_5 =&\{B(Y, A_N^\gamma X) - B(X, A_N^\gamma Y) + [X(\rho_1(Y)) - Y(\rho_1(X)) - \rho_1([X,Y])]\}N \\
=&h(Y, A_N^\gamma X) - h(X, A_N^\gamma Y) + X(g^\gamma(\nabla_Y^t N, \xi))N - Y(g^\gamma(\nabla_X^t N, \xi))N \\
&- g^\gamma(\nabla_{[X,Y]}^t N, \xi)N \\
=&-\nabla_X^t \nabla_Y^t N + \nabla_Y^t \nabla_X^t N + \nabla_{[X,Y]}^t N + X(g^\gamma(\nabla_Y^t N, \xi))N \\
&- Y(g^\gamma(\nabla_X^t N, \xi))N - g^\gamma(\nabla_{[X,Y]}^t N, \xi)N \\
=&-\nabla_X^t g^\gamma(\nabla_Y^t N, \xi)N + \nabla_Y^t g^\gamma(\nabla_X^t N, \xi)N + g^\gamma(\nabla_{[X,Y]}^t N, \xi)N \\
&+ X(g^\gamma(\nabla_Y^t N, \xi))N - Y(g^\gamma(\nabla_X^t N, \xi))N - g^\gamma(\nabla_{[X,Y]}^t N, \xi)N \\
=&-X(g^\gamma(\nabla_Y^t N, \xi))N - g^\gamma(\nabla_Y^t N, \xi)\nabla_X^t N \\
&+ Y(g^\gamma(\nabla_X^t N, \xi)) + g^\gamma(\nabla_X^t N, \xi)\nabla_Y^t N \\
&+ X(g^\gamma(\nabla_Y^t N, \xi))N - Y(g^\gamma(\nabla_X^t N, \xi))N \\
=&-g^\gamma(\nabla_Y^t N, \xi)\nabla_X^t N + g^\gamma(\nabla_X^t N, \xi)\nabla_Y^t N \\
=&-g^\gamma(\nabla_Y^t N, \xi)g^\gamma(\nabla_X^t N, \xi) + g^\gamma(\nabla_X^t N, \xi)g^\gamma(\nabla_Y^t N, \xi) \\
=&0.
\end{aligned}$$

(vi)

$$\begin{aligned}
0 =&\{D_2(Y, A_N X) - D_2(X, A_N Y) + 2d\rho_2(X, Y) \\
&+ \rho_1(Y)\rho_2(X) - \rho_1(X)\rho_2(Y)\}u.
\end{aligned}$$

Let C_6 be the right hand side of the previous equation. If we write $X = pX + a\xi$ and $Y = PY + b\xi$, the fact that ∇ is a torsion-free connection, decompositions (5.26) and (5.27); and equations (5.34), (5.32) yield

$$\begin{aligned}
C_6 =&\{\frac{\sigma}{r}g^\gamma(Y, A_N^\gamma X) - \frac{\sigma}{r}g^\gamma(X, A_N^\gamma Y) + [X(\rho_2(Y)) - Y(\rho_2(X)) - \rho_2([X,Y])] \\
&+ \frac{\sigma}{r}g^\gamma(\nabla_Y^t N, \xi)g^\gamma(X, N) - \frac{\sigma}{r}g^\gamma(\nabla_X^t N, \xi)g^\gamma(Y, N)\}u \\
=&\frac{\sigma}{r}\{g^\gamma(Y, A_N^\gamma X) - g^\gamma(X, A_N^\gamma Y) + X(g^\gamma(Y, N)) - Y(g^\gamma(X, N)) \\
&- g^\gamma([X,Y], N) + g^\gamma(\nabla_Y^t N, \xi)g^\gamma(X, N) - g^\gamma(\nabla_X^t N, \xi)g^\gamma(Y, N)\}u \\
=&\frac{\sigma}{r}\{g^\gamma(PY, A_N^\gamma X) - g^\gamma(PX, A_N^\gamma Y) + X(g^\gamma(b\xi, N)) - Y(g^\gamma(a\xi, N)) \\
&- g^\gamma(\nabla_X Y - \nabla_Y X, N) + g^\gamma(\nabla_Y^t N, \xi)g^\gamma(a\xi, N) - g^\gamma(\nabla_X^t N, \xi)g^\gamma(b\xi, N)\}u \\
=&\frac{\sigma}{r}\{g^\gamma(PY, A_N^\gamma X) - g^\gamma(PX, A_N^\gamma Y) + X(b) - Y(a) \\
&- g^\gamma(\nabla_X PY, N) - g^\gamma(\nabla_X b\xi, N) + g^\gamma(\nabla_Y PX, N) + g^\gamma(\nabla_Y a\xi, N) \\
&+ ag^\gamma(\nabla_Y^t N, \xi) - bg^\gamma(\nabla_X^t N, \xi)\}u \\
=&\frac{\sigma}{r}\{g^\gamma(PY, A_N^\gamma X) - g^\gamma(PX, A_N^\gamma Y) + X(b) - Y(a) \\
&- g^\gamma(h^*(X, PY), N) - g^\gamma(X(b)\xi + b\nabla_X \xi, N) + g^\gamma(h^*(Y, PX), N) \\
&+ g^\gamma(Y(a)\xi + a\nabla_Y \xi, N) + ag^\gamma(\nabla_Y^t N, \xi) - bg^\gamma(\nabla_X^t N, \xi)\}u \\
=&\frac{\sigma}{r}\{X(b) - Y(a) - X(b) + Y(a) - bg^\gamma(\nabla_X \xi, N) \\
&+ ag^\gamma(\nabla_Y \xi, N) + ag^\gamma(\nabla_Y^t N, \xi) - bg^\gamma(\nabla_X^t N, \xi)\}u \\
=&\frac{\sigma}{r}\{-bg^\gamma(\nabla_X^\perp \xi, N) - ag^\gamma(\nabla_Y^\perp \xi, N) + ag^\gamma(\nabla_Y^t N, \xi) - bg^\gamma(\nabla_X^t N, \xi)\}u \\
=&0.
\end{aligned}$$

(vii)

$$-\nabla_X(A_u Y) + \nabla_Y(A_u X) + A_u[X, Y] + \epsilon_1(X)A_N Y - \epsilon_1(Y)A_N X = 0.$$

Denote by C_7 the left hand side of the previous equation. Because ∇ is a torsion-free connection, we have

$$\begin{aligned}C_7 =& - \nabla_X \left(\frac{1}{r}Y\right) + \nabla_Y \left(\frac{1}{r}X\right) + \frac{1}{r}[X,Y]\\ =&\frac{1}{R}(-\nabla_X Y + \nabla_Y X + [X,Y])\\ =&0.\end{aligned}$$

(viii)

$$\{D_1(Y, A_u X) - D_1(X, A_u Y) + 2d\epsilon_1(X,Y) + \rho_1(X)\epsilon_1(Y) - \rho_1(Y)\epsilon_1(X)\}N = 0.$$

Let C_8 be the left hand side of the previous equation. Because B is smooth, symmetric and bilinear, we obtain

$$\begin{aligned}C_8 &= \left\{B\left(Y, \frac{1}{r}X\right) - B\left(X, \frac{1}{r}Y\right)\right\}N\\ &= \frac{1}{r}\left\{B(Y,X) - B(X,Y)\right\}N\\ &= 0.\end{aligned}$$

(ix)

$$\{D_2(Y, A_u X) - D_2(X, A_u Y) + \epsilon_1(Y)\rho_2(X) - \epsilon_1(X)\rho_2(Y)\}u = 0.$$

Taking C_9 as the left hand side of the previous equation, and applying bilinearity of g, give us

$$\begin{aligned}C_9 =&\{D_2(Y, A_u X) - D_2(X, A_u Y)\}u\\ =&\left\{\frac{\sigma}{r}g\left(Y, \frac{1}{r}X\right) - \frac{\sigma}{r}g\left(X, \frac{1}{r}Y\right)\right\}u\\ =&\frac{\sigma}{r^2}\left\{g(Y,X) - g(X,Y)\right\}u\\ =&0.\end{aligned}$$

□

Finally, we prove the Fundamental Theorem for isometric immersions of null hypersurfaces in semi-Riemannian space forms with constant sectional curvature different from zero.

Theorem 5.2. *Let $(M, g, S(TM))$ a null simply connected manifold of dimension $n+1$, with a screen distribution of dimension n and index $q-1$. Let ϵ be a vector bundle over M of dimension 1 and let $\overline{g}^\gamma$ be the metric over $\gamma = TM \oplus \epsilon$ given by (5.30). Further, let ∇^γ be a connection over γ which satisfies (5.31) - (5.37). Moreover, suppose that the Gauss-Codazzi-Ricci equations for $c = \sigma/r^2$ given by (5.38)-(5.41) hold for γ, where $\sigma = sign(c)$.*

Then there exists an isometric immersion $f : M \to \mathbb{Q}^{n+2}_{c,q}$, such that $\overline{f} = i \circ f$. Furthermore, there exist an isometry of vector bundles $\phi : \delta \oplus \epsilon \to \mathrm{tr}^f(TM)$, such that

$$\begin{aligned}h^f &= \phi h,\\ \nabla^{t,f}\phi &= \phi\nabla^t.\end{aligned}$$

Moreover, let $f, g : M \to \mathbb{Q}^{n+2}_{c,q}$ be two such isometric immersions of a null manifold and suppose there exists an isometry of vector bundles $\overline{\psi} : f^*T\mathbb{Q}^{n+2}_q \to f^*T\mathbb{Q}^{n+2}_q$ such that

$$\overline{\psi}(f_*) = g_*$$

and $\overline{\psi}|_{\mathrm{tr}^f(TM)} = \psi$ satisfies

$$\begin{aligned}\psi(\phi^f(N)) &= \phi^g(N),\\ \psi(\phi^f(u)) &= \phi^g(u).\end{aligned}$$

Then there exists an isometry $\tau : \mathbb{Q}^{n+2}_q \to \mathbb{Q}^{n+2}_q$, such that

$$\tau f = g \quad \text{and} \quad \tau_*|_{\mathrm{tr}^f(TM)} = \psi$$

Proof. Let δ be a semi-Riemannian bundle of dimension 1 over M, with index d and let u a unit element respect to δ.

We define the metric $\overline{g}^{\overline{\epsilon}}$ on the bundle $\overline{\epsilon} = \gamma \oplus \delta$, as in Proposition 5.6 and the connection $\overline{\nabla}^{\overline{\epsilon}}$ over $\overline{\epsilon}$ as in (5.42), (5.43) and (5.44). In this way, by Proposition 5.7 the properties (5.7)-(5.14) hold.

Now, since we are assuming (5.38), (5.40) and (5.41), Proposition 5.8 implies that (5.15) - (5.17) are satisfied.

Therefore, by Theorem 5.1, there exists an isometric immersion $\overline{f} : M \to \mathbb{R}^{n+3}_{q+d}$ and a vector bundle isometry $\overline{\phi} : \epsilon \oplus \delta \to tr(TM)$ such that $D_1 = D_1^0$, $D_2 = D_2^0$, $\rho_1 = \rho_1^0$, $\rho_2 = \rho_2^0$ and $\epsilon_1 = \epsilon_1^0$.

From the election of D_1 and ρ_1, and properties (5.31), (5.32) it follows that

$$\begin{aligned} h^0 &= \overline{\phi} \circ h, \\ \nabla^{t,0}\overline{\phi} &= \overline{\phi}\nabla^t. \end{aligned}$$

On the other hand, we consider the canonical isometry $i : \mathbb{Q}^{n+2}_{c,q} \to \mathbb{R}^{n+3}_{c,q+\lambda(c)}$, whose image is the hyperquadric

$$\mathbb{Q}^{n+2}_{c,q} = \{X \in \mathbb{R}^{n+3}_{q+\lambda(c)} : g^0(X, X) = 1/c\},$$

where $\lambda(c) = 1$ if $c < 0$ and $\lambda(c) = 0$ if $c > 0$. Observe that $\lambda(c) = d$.

We affirm that $\overline{f}(M) \subset \mathbb{Q}^{n+2}_{c,q}$, modulo a translation on $\mathbb{R}^{n+3}_{q+d}$ if necessary. For that, we notice

$$\overline{\phi}(\overline{\nabla}^{\overline{\epsilon}}_X u) = \overline{\phi}(-A_u X) = \overline{\phi}\left(-\frac{1}{r}X\right) = -\frac{1}{r}\overline{\phi}(X) = -\frac{1}{r}f_* X,$$

from which it follows that

$$\overline{f} + r\overline{\phi}(u) = K,$$

where K is some constant vector in $\mathbb{R}^{n+3}_{q+d}$. Then

$$g^0(\overline{f} - K, \overline{f} - K) = g^0(-r\overline{\phi}(u), -r\overline{\phi}(u)) = r^2 g^0(\overline{\phi}(u), \overline{\phi}(u)) = \sigma r^2 = \frac{1}{c}$$

In this way, there exists an isometric immersion $f : M \to \mathbb{Q}^{n+2}_{c,q}$, such that $\overline{f} = i \circ f$. Moreover, there exists a vector bundle isometry $\phi : \delta \oplus \epsilon \to \mathrm{tr}^f(TM)$, such that

$$\begin{aligned} h^f &= \phi h, \\ \nabla^{t,f}\phi &= \phi\nabla^t. \end{aligned}$$

Now we tackle the uniqueness part of the proof. For that, let f, g be two such isometric immersions and let $i : \mathbb{Q}^{n+2}_{c,q} \to \mathbb{R}^{n+3}_{q+d}$ the canonical inclusion. Consider the functions $\overline{f} = i \circ f$, $\overline{g} = i \circ g$, which are isometric inmersions of M in $\mathbb{R}^{n+3}_{q+d}$.

Because $\phi^{\overline{f}}(\delta)$ and $\phi^{\overline{g}}(\delta)$ are orthogonal to $\overline{\phi}^{\overline{f}}(\gamma)$ and $\overline{\phi}^{\overline{g}}(\gamma)$, respectively. Thus $\overline{\psi}$ define the bundle isometry $\overline{\zeta} : \overline{f}^* T\mathbb{R}^{n+3}_{q+d} \to \overline{g}^* T\mathbb{R}^{n+3}_{q+d}$, such that

$$\overline{\zeta f}_* = \overline{g}_*,$$

and $\overline{\zeta}|_{tr^{\overline{f}}(TM)} = \zeta$, satisfies

$$\begin{aligned} \zeta(\phi^{\overline{f}}(N)) &= \phi^{\overline{g}}(N), \\ \overline{\zeta}(\phi^{\overline{f}}(u)) &= \phi^{\overline{g}}(u). \end{aligned}$$

Then, by Theorem 5.1, there exists a bundle isometry $\overline{\tau} : \mathbb{R}^{n+3}_{q+d} \to \mathbb{R}^{n+3}_{q+d}$, such that

$$\overline{\tau}\overline{f} = \overline{g} \quad \text{and} \quad \overline{\tau}_*|_{tr^{\overline{f}}(TM)} = \zeta,$$

which in turn induces the bundle isometry $\tau : \mathbb{Q}^{n+2}_{c,q} \to \mathbb{Q}^{n+2}_{c,q}$ that satisfies

$$\tau f = g \quad \text{and} \quad \tau_*|_{\mathrm{tr}^f(TM)} = \psi.$$

□

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Author's contributions

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