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Four New Sequence Spaces Obtained from the Domain of Quadruple Band Matrix Operator

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Abstract. In this work, we construct the new sequence spaces $c_0(Q)$, $c(Q)$, $\ell_\infty(Q)$ and $\ell_p(Q)$ derived by the domain of quadruple band matrix, which generalizes the matrices Δ^3, $B(r,s,t)$, Δ^2, $B(r,s)$, Δ, where Δ^3, $B(r,s,t)$, Δ^2, $B(r,s)$ and Δ are called third order difference, triple band, second order difference, double band and difference matrix, in turn. Also, we investigate some topological properties and some inclusion relations related to those spaces. Furthermore, we give the Schauder basis of the spaces $c_0(Q)$, $c(Q)$ and $\ell_p(Q)$, and determine $\alpha-\beta-$ and $\gamma-$duals of those spaces. Lastly, we characterize some matrix classes related to some of those spaces.

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1. Introduction

The set of all complex valued sequences is a vector space under point-wise addition and scalar multiplication. This space is denoted by w. Each vector subspace of w is called a sequence space. By ℓ_∞, c_0, c and ℓ_p, we denote the spaces of all bounded, null, convergent and absolutetely p-summable sequences, respectively, where $1 \leq p < \infty$.

Let X be a sequence space such that it is a complete metric linear space with continuous coordinate projections. Then, we say that X is an FK-space. If the metric of an FK-space X is given by a complete norm, then we say that X is a BK-space [11].

We know that the spaces ℓ_∞, c_0 and c are all BK-spaces with the norm $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$. Also, ℓ_p is a BK-space with the norm

$$\|x\|_p = \Big(\sum_{k=0}^{\infty} |x_k|^p\Big)^{\frac{1}{p}},$$

where $1 \leq p < \infty$.

Let $x = (x_k) \in w$ and $A = (a_{nk})$ be an infinite matrix with $a_{nk} \in \mathbb{C}$ for all $n, k \in \mathbb{N}$. Then, the A-transform of x is defined by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k$$

and is assumed to be convergent for all $n \in \mathbb{N}$ [18]. Here, our convention is that the summation without limits runs from 0 to ∞.

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Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers. Then, the matrix domain of A in X is defined by

$$X_A = \Big\{x = (x_k) \in w : Ax \in X\Big\}. \tag{1.1}$$

Also, the class of all matrices $A = (a_{nk})$ such that $X \subset Y_A$ is denoted by $(X : Y)$. An infinite matrix $A = (a_{nk})$ is called a triangle provided $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n, k \in \mathbb{N}$. The inverse of a triangle exists, is unique and a triangle [18].

The spaces of all bounded and convergent series are defined by $bs = (\ell_\infty)_S$ and $cs = c_S$, respectively, where $S = (s_{nk})$ is called summation matrix defined by

$$s_{nk} = \begin{cases} 1 & , \quad 0 \leq k \leq n \\ 0 & , \quad k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$.

A Banach Limit is a functional $L : \ell_\infty \longrightarrow \mathbb{R}$ provided
(i) $L(ax_n + by_n) = aL(x_n) + bL(y_n)$ $a, b \in \mathbb{R}$
(ii) $L(x_n) \geq 0$ if $x_n \geq 0$, $\forall n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$
(iii) $L(p^j(x_n)) = L(x_n)$, $p^j(x_n) = x_{n+j}$, $j = 1, 2, 3, ...$
(iv) $L(e) = 1$ where $e = (1, 1, ...)$

Let $x = (x_n)$ be a bounded sequence. Then, $x = (x_n)$ is called almost convergent to the generalized limit ξ if $L(x_n) = \xi$ holds for all Banach Limit L and denoted by $f - \lim x_n = \xi$ [10].

Lorenz [10] proved that $f - \lim x_n = \xi$ if and only if $\lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = \xi$ uniformly in n.

So the spaces f and fs are defined by

$$f = \left\{x = (x_k) \in w : \exists \xi \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = \xi \text{ uniformly in n}\right\},$$

and

$$fs = \left\{x = (x_k) \in w : \exists \xi \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{k=0}^{m} \sum_{j=0}^{n+k} \frac{x_j}{m+1} = \xi \text{ uniformly in n}\right\},$$

and called the space of all almost convergent sequences and almost convergent series, respectively.

The theory of matrix transformation is of great importance in the summability theory which was obtained by Cesàro, Norlund, Borel, Riesz.... So, many authors have constructed new sequence spaces by using matrix domain of infinite matrices. The usage of matrix domain of difference matrices was first motivated by Kızmaz. He defined the spaces $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$ in [8], and many authors followed him by defining the spaces $\Delta c_0(p)$, $\Delta c(p)$ and $\Delta\ell_\infty(p)$ in [1], $c_0(u, \Delta, p)$, $c(u, \Delta, p)$ and $\ell_\infty(u, \Delta, p)$ in [2], $c_0(\Delta^2)$, $c(\Delta^2)$ and $\ell_\infty(\Delta^2)$ in [6], $c_0(u, \Delta^2)$, $c(u, \Delta^2)$ and $\ell_\infty(u, \Delta^2)$ in [13], $c_0(u, \Delta^2, p)$, $c(u, \Delta^2, p)$ and $\ell_\infty(u, \Delta^2, p)$ in [5], $c_0(\Delta^m)$, $c(\Delta^m)$ and $\ell_\infty(\Delta^m)$ in [7], $\hat{\ell_\infty}$, $\hat{c_0}$, $\hat{c}$ and $\hat{\ell_p}$ in [9], $c_0(B)$, $c(B)$, $\ell_\infty(B)$ and $\ell_p(B)$ in [14], $\ell_1(\Delta_i^3)$ and $bv(\Delta_i^3)$ in [17], $h(\Delta_i^3)$ in [12, 16].

2. Four New Spaces

In this section, we construct four new sequence spaces $c_0(Q)$, $c(Q)$, $\ell_\infty(Q)$ and $\ell_p(Q)$ by using the domain of the quadruple band matrix, where $1 \leq p < \infty$. Furthermore, we show that the spaces $c_0(Q)$, $c(Q)$, $\ell_\infty(Q)$ and $\ell_p(Q)$ are linearly isomorphic to the spaces c_0, c, ℓ_∞ and ℓ_p, respectively. Moreover, we prove some strict inclusion relations.

Let $r, s, t, u \in \mathbb{R} \setminus \{0\}$. Then, the quadruple band matrix $Q = Q(r, s, t, u) = (q_{nk}(r, s, t, u))$ is defined by

$$q_{nk}(r, s, t, u) = \begin{cases} r & , \quad k = n \\ s & , \quad k = n-1 \\ t & , \quad k = n-2 \\ u & , \quad k = n-3 \\ 0 & , \quad otherwise, \end{cases}$$

for all $n, k \in \mathbb{N}$. Here, we would like to bring attention that $Q(1, -3, 3, -1) = \Delta^3$, $Q(1, \frac{-3}{2}, 1, \frac{-1}{4}) = \Delta_i^3$, $Q(r, s, t, 0) = B(r, s, t)$, $Q(1, -2, 1, 0) = \Delta^2$, $Q(r, s, 0, 0) = B(r, s)$ and $Q(1, -1) = \Delta$, where Δ^3, $B(r, s, t)$, Δ^2, $B(r, s)$ and Δ are called third order difference, triple band, second order difference, double band(generalized difference) and difference matrix, respectively. Therefore, our results derived from the matrix domain of the quadruple band matrix are more general and more comprehensive than the results on the matrix domain of the others mentioned above.

Now, we define the spaces $c_0(Q)$, $c(Q)$, $\ell_\infty(Q)$ and $\ell_p(Q)$ by means of the domain of quadruple band matrix as follows:

$$c_0(Q) = \Big\{x = (x_k) \in w : \lim_{k\to\infty}(rx_k + sx_{k-1} + tx_{k-2} + ux_{k-3}) = 0\Big\},$$

$$c(Q) = \Big\{x = (x_k) \in w : \lim_{k\to\infty}(rx_k + sx_{k-1} + tx_{k-2} + ux_{k-3}) \text{ exists}\Big\},$$

$$\ell_\infty(Q) = \Big\{x = (x_k) \in w : \sup_{k\in\mathbb{N}}|rx_k + sx_{k-1} + tx_{k-2} + ux_{k-3}| < \infty\Big\},$$

and

$$\ell_p(Q) = \Big\{x = (x_k) \in w : \sum_k |rx_k + sx_{k-1} + tx_{k-2} + ux_{k-3}|^p < \infty\Big\},$$

where $1 \leq p < \infty$. Unless stated otherwise, any term with negative subscript is assumed to be zero. By considering the notation of (1.1), we write

$$c_0(Q) = (c_0)_Q,\ c(Q) = c_Q,\ \ell_\infty(Q) = (\ell_\infty)_Q \text{ and } \ell_p(Q) = (\ell_p)_Q. \tag{2.1}$$

Let $x = (x_k) \in w$. Then the Q-transform of x is defined by

$$(Qx)_k = y_k = rx_k + sx_{k-1} + tx_{k-2} + ux_{k-3},$$

for all $k \in \mathbb{N}$.

Theorem 2.1. *The followings hold.*

(a) *The sequence spaces $c_0(Q)$, $c(Q)$ and $\ell_\infty(Q)$ are BK-spaces with the norm defined by*

$$\|x\|_{c_0(Q)} = \|x\|_{c(Q)} = \|x\|_{\ell_\infty(Q)} = \|Qx\|_\infty = \sup_{k\in\mathbb{N}}|rx_k + sx_{k-1} + tx_{k-2} + ux_{k-3}|.$$

(b) *The sequence space $\ell_p(Q)$ is a BK-space with the norm defined by*

$$\|x\|_{\ell_p(Q)} = \|Qx\|_p = \Big(\sum_{k=0}^{\infty}|rx_k + sx_{k-1} + tx_{k-2} + ux_{k-3}|^p\Big)^{\frac{1}{p}},$$

where $1 \leq p < \infty$.

Proof. (a) It is clear that c_0, c and ℓ_∞ are BK-spaces according to their sup-norm and $Q = Q(r,s,t,u)$ is a triangle matrix. Moreover, the relations (2.1) hold. By combining these three conditions and Theorem 4.3.12 of Wilansky [18], we make inferences that $c_0(Q)$, $c(Q)$ and $\ell_\infty(Q)$ are BK-spaces.

(b) This part can be proved by using a similar way. So, we omit the details. This completes the proof. □

Now, let us consider the equation

$$rz^3 + sz^2 + tz + u = 0,$$

where $r,s,t,u \in \mathbb{R} \setminus \{0\}$. It is well known that, this equation has three roots such that $z_1 = \frac{1}{3r}[a - b - s]$, $z_2 = -\frac{1}{6r}[(1 - i\sqrt{3})a - (1 + i\sqrt{3})b + 2s]$ and $z_3 = -\frac{1}{6r}[(1 + i\sqrt{3})a - (1 - i\sqrt{3})b + 2s]$, where

$$a = \sqrt[3]{\frac{\sqrt{(-27r^2u + 9rst - 2s^3)^2 + 4(3rt - s^2)^3} - 27r^2u + 9rst - 2s^3}{2}}$$

and

$$b = \sqrt[3]{\frac{\sqrt{(-27r^2u + 9rst - 2s^3)^2 + 4(3rt - s^2)^3} + 27r^2u - 9rst + 2s^3}{2}}.$$

Here and in the following, unless stated otherwise, we assume that μ_1, μ_2 and μ_3 are random three roots of the equation $rz^3 + sz^2 + tz + u = 0$.

Also, by using a simple calculation, we have

$$\mu_1 + \mu_2 + \mu_3 = -\frac{s}{r},\ \ \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 = \frac{t}{r}\ \text{ and }\ \mu_1\mu_2\mu_3 = -\frac{u}{r},$$

$$\mu_1^3 + \frac{s}{r}\mu_1^2 + \frac{t}{r}\mu_1 + \frac{u}{r} = 0, \tag{2.2}$$

$$\mu_1^2 + \mu_2^2 + \frac{s}{r}(\mu_1 + \mu_2) + \mu_1\mu_2 + \frac{t}{r} = 0, \tag{2.3}$$

$$\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 + \frac{s}{r}(\mu_1 + \mu_2 + \mu_3) + \frac{t}{r} = 0, \tag{2.4}$$

$$\mu_1 + \mu_2 + \mu_3 + \frac{s}{r} = 0. \tag{2.5}$$

Theorem 2.2. *The sequence speces $c_0(Q)$, $c(Q)$, $\ell_\infty(Q)$ and $\ell_p(Q)$ are linearly isomorphic to the sequence spaces c_0, c, ℓ_∞ and ℓ_p, respectively, where $1 \le p < \infty$.*

Proof. In order not to reduce repetition, we give the proof of theorem for only the sequence space $c(Q)$.

For the proof, to display of existence of a linear bijection between the spaces $c(Q)$ and c is needed. Consider the transformation L defined by $L : c(Q) \longrightarrow c$, $L(x) = Qx$. The linearity of L is clear. Also, it is obvious that $x = \theta$ whenever $Qx = \theta$. So, L is injective.

Let $y = (y_k) \in c$ and define a sequence $x = (x_k)$ by

$$x_k = \frac{1}{r}\sum_{j=0}^{k}\sum_{i=0}^{k-j}\sum_{v=0}^{k-j-i}\mu_1^{k-j-i-v}\mu_2^v\mu_3^i y_j$$

for all $k \in \mathbb{N}$. Then, by considering (2.2)-(2.5), we obtain

$$\begin{aligned}
(Qx)_k &= rx_k + sx_{k-1} + tx_{k-2} + ux_{k-3} \\
&= \sum_{j=0}^{k}\sum_{i=0}^{k-j}\sum_{v=0}^{k-j-i}\mu_1^{k-j-i-v}\mu_2^v\mu_3^i y_j + \frac{s}{r}\sum_{j=0}^{k-1}\sum_{i=0}^{k-j-1}\sum_{v=0}^{k-j-i-1}\mu_1^{k-j-i-v-1}\mu_2^v\mu_3^i y_j \\
&+ \frac{t}{r}\sum_{j=0}^{k-2}\sum_{i=0}^{k-j-2}\sum_{v=0}^{k-j-i-2}\mu_1^{k-j-i-v-2}\mu_2^v\mu_3^i y_j + \frac{u}{r}\sum_{j=0}^{k-3}\sum_{i=0}^{k-j-3}\sum_{v=0}^{k-j-i-3}\mu_1^{k-j-i-v-3}\mu_2^v\mu_3^i y_j \\
&= \sum_{j=0}^{k-3}\Bigg[\sum_{i=0}^{k-j-3}\Bigg[\sum_{v=0}^{k-j-i-3}\mu_3^i\mu_2^v\mu_1^{k-j-i-v-3}\left(\mu_1^3 + \frac{s}{r}\mu_1^2 + \frac{t}{r}\mu_1 + \frac{u}{r}\right) \\
&+ \mu_3^i\mu_2^{k-j-i-2}\left(\mu_1^2 + \mu_2^2 + \frac{s}{r}(\mu_1 + \mu_2) + \mu_1\mu_2 + \frac{t}{r}\right) \\
&+ \mu_3^{k-j-2}\left(\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 + \frac{s}{r}(\mu_1 + \mu_2 + \mu_3) + \frac{t}{r}\right)\Bigg]\Bigg]y_j \\
&+ \Bigg[y_{k-2}\left(\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 + \frac{s}{r}(\mu_1 + \mu_2 + \mu_3) + \frac{t}{r}\right) \\
&+ y_{k-1}\left(\mu_1 + \mu_2 + \mu_3 + \frac{s}{r}\right) + y_k\Bigg] \\
&= y_k
\end{aligned}$$

for all $k \in \mathbb{N}$. Therefore, $Qx = y$ and since $y \in c$, we deduce that $Qx \in c$, namely $x = (x_k) \in c(Q)$ and $L(x) = y$. Thus, L is surjective. Also, we have from the Theorem 2.1 that

$$\|L(x)\|_\infty = \|Qx\|_\infty = \|x\|_{c(Q)}$$

for all $x = (x_k) \in c(Q)$. Since L is norm preserving. Hence, L is a linear bijection between the spaces $c(Q)$ and c, as desired. So the proof is complete. □

Now, we list the following statements which are needed in the next lemma.

$$\sup_{n\in\mathbb{N}}\sum_k |a_{nk}|^q < \infty, \tag{2.6}$$

$$\sup_{n,k\in\mathbb{N}} |a_{nk}| < \infty, \tag{2.7}$$

$$\sup_{k\in\mathbb{N}}\sum_n |a_{nk}|^p < \infty, \tag{2.8}$$

$$\sup_{K\in\mathcal{F}}\sum_k \Big|\sum_{n\in K} a_{nk}\Big|^q < \infty, \tag{2.9}$$

$$\sup_{K\in\mathcal{F}}\sum_n \Big|\sum_{k\in K} a_{nk}\Big|^p < \infty, \tag{2.10}$$

$$\lim_{n\to\infty} a_{nk} = \xi_k \text{ for all } k\in\mathbb{N}, \tag{2.11}$$

$$\lim_{n\to\infty}\sum_k a_{nk} = \xi, \tag{2.12}$$

$$\lim_{n\to\infty}\sum_k |a_{nk}| = \sum_k \Big|\lim_{n\to\infty} a_{nk}\Big|, \tag{2.13}$$

where $\mathcal{F}$ is the collection of all finite subsets of $\mathbb{N}$, $\frac{1}{p}+\frac{1}{q}=1$ and $1\le p\le\infty$.

Lemma 2.3 ([15]). *For an infinite matrix $A=(a_{nk})$, the following statements hold:*

(i) $A=(a_{nk})\in(\ell_\infty:\ell_\infty)=(c:\ell_\infty)=(c_0:\ell_\infty)\Leftrightarrow$ (2.6) *holds with* $q=1$.
(ii) $A=(a_{nk})\in(\ell_p:\ell_\infty)\Leftrightarrow$ (2.6) *holds with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $1<p<\infty$.
(iii) $A=(a_{nk})\in(\ell_1:\ell_\infty)\Leftrightarrow$ (2.7) *holds.*
(iv) $A=(a_{nk})\in(\ell_\infty:c)\Leftrightarrow$ (2.11) *and* (2.13) *hold.*
(v) $A=(a_{nk})\in(c:c)\Leftrightarrow$ (2.6), (2.11) *and* (2.12) *hold with* $q=1$.
(vi) $A=(a_{nk})\in(c_0:c)\Leftrightarrow$ (2.6) *and* (2.11) *hold with* $q=1$.
(vii) $A=(a_{nk})\in(\ell_p:c)\Leftrightarrow$ (2.6) *and* (2.11) *hold with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $1<p<\infty$.
(viii) $A=(a_{nk})\in(\ell_1:c)\Leftrightarrow$ (2.7) *and* (2.11) *hold.*
(ix) $A=(a_{nk})\in(c_0:c_0)\Leftrightarrow$ (2.6) *and* (2.11) *hold with* $q=1$ *and* $\xi_k=0$ *for all* $k\in\mathbb{N}$.
(x) $A=(a_{nk})\in(\ell_1:\ell_1)\Leftrightarrow$ (2.8) *holds with* $p=1$.
(xi) $A=(a_{nk})\in(\ell_p:\ell_1)\Leftrightarrow$ (2.9) *holds with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $1<p\le\infty$.
(xii) $A=(a_{nk})\in(c_0:\ell_1)=(c:\ell_1)\Leftrightarrow$ (2.10) *holds with* $p=1$.
(xiii) $A=(a_{nk})\in(\ell_p:\ell_p)\Leftrightarrow A=(a_{nk})\in(\ell_\infty:\ell_\infty)\cap(\ell_1:\ell_1),\ 1<p<\infty$.
(xiv) $A=(a_{nk})\in(c:\ell_p)\Leftrightarrow$ (2.10) *holds with* $1\le p<\infty$.
(xv) $A=(a_{nk})\in(\ell_1:\ell_p)\Leftrightarrow$ (2.8) *holds with* $1\le p<\infty$.

Theorem 2.4. *Let $X\in\{c_0,c,\ell_\infty,\ell_p\}$ and $Q=Q(r,s,t,u)$. Then,*

(a) $X=X_Q$, *if* $|\mu_\sigma|<1$, $\forall\sigma\in\{1,2,3\}$
(b) $X\subset X_Q$ *is strict, if* $|\mu_\sigma|\ge 1$, $\exists\sigma\in\{1,2,3\}$

Proof. Let us take $X\in\{c_0,c,\ell_\infty,\ell_p\}$ and $Q=Q(r,s,t,u)$. By considering the Lemma 2.3 (i), (v), (ix), (x) and (xiii), we have

$$\sup_{n\in\mathbb{N}}\sum_k |q_{nk}(r,s,t,u)| = |u|+|t|+|s|+|r|,$$
$$\lim_{n\to\infty} q_{nk}(r,s,t,u) = 0 \text{ for all } k\in\mathbb{N},$$
$$\lim_{n\to\infty}\sum_k q_{nk}(r,s,t,u) = r+s+t+u,$$

and

$$\sup_{k\in\mathbb{N}}\sum_n |q_{nk}(r,s,t,u)| = |r|+|s|+|t|+|u|.$$

Therefore, we obtain $Q\in(X:X)$, namely $X\subset X_Q$.

(a) Let $|\mu_\sigma|<1$, $\forall\sigma\in\{1,2,3\}$ and $H=(h_{nk})$ be the inverse of $Q=Q(r,s,t,u)$ defined by

$$h_{nk}=\begin{cases}\frac{1}{r}\sum\limits_{i=0}^{n-k}\sum\limits_{v=0}^{n-k-i}\mu_1^{n-k-i-v}\mu_2^{v}\mu_3^{i} & , \quad 0\le k\le n\\ 0 & , \quad k>n\end{cases}$$

for all $n, k \in \mathbb{N}$. Then, we have:

<u>In case of:</u> $\mu_1 \neq \mu_2 \neq \mu_3$

$$\begin{aligned}
\sup_{n\in\mathbb{N}}\sum_k |h_{nk}| &\leq \frac{1}{|r|}\sup_{n\in\mathbb{N}}\Bigg[\frac{|\mu_1|^2-|\mu_1|^{n+3}}{(1-|\mu_1|)|(\mu_1-\mu_2)(\mu_1-\mu_3)|}+\frac{|\mu_2|^2-|\mu_2|^{n+3}}{(1-|\mu_2|)|(\mu_1-\mu_2)(\mu_3-\mu_2)|}\\
&+ \frac{|\mu_3|^2-|\mu_3|^{n+3}}{(1-|\mu_3|)|(\mu_1-\mu_3)(\mu_2-\mu_3)|}\Bigg]\\
&\leq \frac{1}{|r|}\Bigg[\frac{1}{(1-|\mu_1|)|(\mu_1-\mu_2)(\mu_1-\mu_3)|}+\frac{1}{(1-|\mu_2|)|(\mu_1-\mu_2)(\mu_3-\mu_2)|}\\
&+ \frac{1}{(1-|\mu_3|)|(\mu_1-\mu_3)(\mu_2-\mu_3)|}\Bigg]\\
&< \infty,
\end{aligned}$$

$$\lim_{n\to\infty} h_{nk} = \lim_{n\to\infty}\frac{(\mu_2-\mu_3)\mu_1^{n-k+2}+(\mu_3-\mu_1)\mu_2^{n-k+2}+(\mu_1-\mu_2)\mu_3^{n-k+2}}{r(\mu_1-\mu_2)(\mu_1-\mu_3)(\mu_2-\mu_3)} = 0$$

for all $k \in \mathbb{N}$,

$$\begin{aligned}
\lim_{n\to\infty}\sum_k h_{nk} &= \frac{1}{r}\lim_{n\to\infty}\Bigg[\frac{\mu_1^2-\mu_1^{n+3}}{(1-\mu_1)(\mu_1-\mu_2)(\mu_1-\mu_3)}+\frac{\mu_2^2-\mu_2^{n+3}}{(1-\mu_2)(\mu_1-\mu_2)(\mu_3-\mu_2)}\\
&+ \frac{\mu_3^2-\mu_3^{n+3}}{(1-\mu_3)(\mu_1-\mu_3)(\mu_2-\mu_3)}\Bigg]\\
&= \frac{1}{r}\Bigg[\frac{\mu_1^2}{(1-\mu_1)(\mu_1-\mu_2)(\mu_1-\mu_3)}+\frac{\mu_2^2}{(1-\mu_2)(\mu_1-\mu_2)(\mu_3-\mu_2)}\\
&+ \frac{\mu_3^2}{(1-\mu_3)(\mu_1-\mu_3)(\mu_2-\mu_3)}\Bigg],
\end{aligned}$$

$$\begin{aligned}
\sup_{k\in\mathbb{N}}\sum_n |h_{nk}| &\leq \frac{1}{|r|}\sup_{k\in\mathbb{N}}\Bigg[\frac{|\mu_1|^2}{(1-|\mu_1|)|(\mu_1-\mu_2)(\mu_1-\mu_3)|}+\frac{|\mu_2|^2}{(1-|\mu_2|)|(\mu_1-\mu_2)(\mu_3-\mu_2)|}\\
&+ \frac{|\mu_3|^2}{(1-|\mu_3|)|(\mu_1-\mu_3)(\mu_2-\mu_3)|}\Bigg]\\
&\leq \frac{1}{|r|}\Bigg[\frac{1}{(1-|\mu_1|)|(\mu_1-\mu_2)(\mu_1-\mu_3)|}+\frac{1}{(1-|\mu_2|)|(\mu_1-\mu_2)(\mu_3-\mu_2)|}\\
&+ \frac{1}{(1-|\mu_3|)|(\mu_1-\mu_3)(\mu_2-\mu_3)|}\Bigg]\\
&< \infty.
\end{aligned}$$

<u>In case of:</u> $\mu = \mu_1 = \mu_2 = \mu_3$

$$\begin{aligned}
\sup_{n\in\mathbb{N}}\sum_k |h_{nk}| &= \sup_{n\in\mathbb{N}}\frac{(n^2+5n+6)|\mu|^{n+1}-2(n^2+4n+3)|\mu|^{n+2}+(n^2+3n+2)|\mu|^{n+3}-2}{2|r|(|\mu|-1)^3}\\
&\leq \frac{1}{|r|(1-|\mu|)^3}\\
&< \infty,
\end{aligned}$$

$$\lim_{n\to\infty} h_{nk} = \lim_{n\to\infty}\frac{\mu^{n-k}(n-k+2)(n-k+1)}{2r} = 0$$

for all $k \in \mathbb{N}$,

$$\begin{aligned}\lim_{n\to\infty}\sum_k h_{nk} &= \lim_{n\to\infty}\frac{(n^2+5n+6)\mu^{n+1}-2(n^2+4n+3)\mu^{n+2}+(n^2+3n+2)\mu^{n+3}-2}{2r(\mu-1)^3}\\ &= \frac{1}{r(1-\mu)^3}\,,\end{aligned}$$

$$\sup_{k\in\mathbb{N}}\sum_n |h_{nk}| = \frac{1}{|r|(1-|\mu|)^3} < \infty.$$

In case of: $\mu=\mu_i=\mu_j\neq\mu_\lambda$, $i,j,\lambda\in\{1,2,3\}$

$$\begin{aligned}\sup_{n\in\mathbb{N}}\sum_k |h_{nk}| &\leq \frac{1}{|r|}\sup_{n\in\mathbb{N}}\left[\frac{|\mu|}{|\mu-\mu_\lambda|(|\mu|-1)^2}+\frac{|\mu_\lambda||\mu|}{|\mu-\mu_\lambda|^2(1-|\mu|)}\right.\\ &+ \left.\frac{|\mu_\lambda|^2}{|\mu-\mu_\lambda|^2(1-|\mu_\lambda|)}\right]\\ &\leq \frac{1}{|r|}\left[\frac{1}{|\mu-\mu_\lambda|(|\mu|-1)^2}+\frac{1}{|\mu-\mu_\lambda|^2(1-|\mu|)}\right.\\ &+ \left.\frac{1}{|\mu-\mu_\lambda|^2(1-|\mu_\lambda|)}\right]\\ &< \infty,\end{aligned}$$

$$\lim_{n\to\infty} h_{nk} = \lim_{n\to\infty}\frac{\mu^{n-k+1}\Big[(\mu-\mu_\lambda)(n-k+1)-\mu_\lambda\Big]+\mu_\lambda^{n-k+2}}{r(\mu-\mu_\lambda)^2} = 0$$

for all $k \in \mathbb{N}$,

$$\begin{aligned}\lim_{n\to\infty}\sum_k h_{nk} &= \frac{1}{r}\lim_{n\to\infty}\left[\frac{-(n+2)\mu^{n+2}+(n+1)\mu^{n+3}+\mu}{(\mu-\mu_\lambda)(\mu-1)^2}\right.\\ &- \left.\frac{\mu_\lambda(\mu^{n+2}-\mu)}{(\mu-\mu_\lambda)^2(\mu-1)}+\frac{\mu_\lambda^{n+3}-\mu_\lambda^2}{(\mu-\mu_\lambda)^2(\mu_\lambda-1)}\right]\\ &= \frac{1}{r}\left[\frac{\mu}{(\mu-\mu_\lambda)(\mu-1)^2}-\frac{\mu_\lambda\mu}{(\mu-\mu_\lambda)^2(1-\mu)}\right.\\ &+ \left.\frac{\mu_\lambda^2}{(\mu-\mu_\lambda)^2(1-\mu_\lambda)}\right],\end{aligned}$$

$$\begin{aligned}\sup_{k\in\mathbb{N}}\sum_n |h_{nk}| &\leq \frac{1}{|r|}\sup_{k\in\mathbb{N}}\left[\frac{|\mu|}{|\mu-\mu_\lambda|(|\mu|-1)^2}+\frac{|\mu_\lambda||\mu|}{|\mu-\mu_\lambda|^2(1-|\mu|)}\right.\\ &+ \left.\frac{|\mu_\lambda|^2}{|\mu-\mu_\lambda|^2(1-|\mu_\lambda|)}\right]\\ &\leq \frac{1}{|r|}\left[\frac{1}{|\mu-\mu_\lambda|(|\mu|-1)^2}+\frac{1}{|\mu-\mu_\lambda|^2(1-|\mu|)}\right.\\ &+ \left.\frac{1}{|\mu-\mu_\lambda|^2(1-|\mu_\lambda|)}\right]\\ &< \infty.\end{aligned}$$

Therefore, we obtain $H \in (X : X)$, namely $X_Q \subset X$. Hence, $X = X_Q$.

(b) We know that the inclusion $X \subset X_Q$ holds. Let $|\mu_\sigma| \geq 1$, $\exists\sigma \in \{1,2,3\}$. Now, we define four sequences such that

$$\varphi_1 = \left\{\frac{1}{r}\sum_{i=0}^{n-2}\sum_{v=0}^{n-i-2}\mu_1^{n-i-2-v}\mu_2^v\mu_3^i\right\}_{n\in\mathbb{N}},$$

$$\varphi_2 = \Big\{n\Big\}_{n\in\mathbb{N}}, \quad \varphi_3 = \Big\{(-1)^n(n+3)\Big\}_{n\in\mathbb{N}} \text{ and } \varphi_4 = \Big\{(-1)^n\Big\}_{n\in\mathbb{N}}.$$

If $|\mu_\sigma| > 1$, $\exists\sigma \in \{1,2,3\}$, then

In case of: $\mu_1 \neq \mu_2 \neq \mu_3$

$$\varphi_1 = \left\{\frac{(\mu_2-\mu_3)\mu_1^n + (\mu_3-\mu_1)\mu_2^n + (\mu_1-\mu_2)\mu_3^n}{r(\mu_1-\mu_2)(\mu_1-\mu_3)(\mu_2-\mu_3)}\right\}_{n\in\mathbb{N}} \notin X.$$

In case of: $\mu = \mu_1 = \mu_2 = \mu_3$

$$\varphi_1 = \left\{\frac{\mu^{n-2}n(n-1)}{2r}\right\}_{n\in\mathbb{N}} \notin X.$$

In case of: $\mu = \mu_i = \mu_j \neq \mu_\lambda$, $i, j, \lambda \in \{1,2,3\}$

$$\varphi_1 = \left\{\frac{\mu^{n-1}[(\mu-\mu_\lambda)(n-1)-\mu_\lambda] + \mu_\lambda^n}{r(\mu-\mu_\lambda)^2}\right\}_{n\in\mathbb{N}} \notin X.$$

But $Q\varphi_1 = e^{(2)} = (0,0,1,0,...) \in X$, that is $\varphi_1 \in X_Q$. So, $\varphi_1 \in X_Q \setminus X$.

If $|\mu_\sigma| = 1$, $\exists\sigma \in \{1,2,3\}$, then $\varphi_1 \in X_Q \setminus X$ whenever $X \in \{c_0, \ell_p\}$.

Let $X \in \{c, \ell_\infty\}$. Then, there are two cases such that

(I) in case of $\mu_\sigma = 1$, namely $r+s+t+u = 0$, we obtain $Q\varphi_2 = (r-t-2u, r-t-2u, ...) \in X$. So, $\varphi_2 \in X_Q \setminus X$

(II) in case of $\mu_\sigma = -1$, that is $r-s+t-u = 0$, we obtain $Q\varphi_3 = \Big\{(-1)^n(2r-s+u)\Big\}_{n\in\mathbb{N}} \in \ell_\infty$, $Q\varphi_4 = (0,0,0,...) \in c$. So, $\varphi_3 \in (\ell_\infty)_Q \setminus \ell_\infty$ and $\varphi_4 \in c_Q \setminus c$. Considering the result of all these, we obtain that the inclusion $X \subset X_Q$ is strict. This completes the proof. □

Theorem 2.5. *If $r+s+t+u = 0$, the inclusion $c \subset c_0(Q)$ is strict.*

Proof. Suppose that $r+s+t+u = 0$ and $x = (x_k) \in c$, that is $\lim_{k\to\infty} x_k = l$. Then, we have $\lim_{k\to\infty}(Qx)_k = (r+s+t+u)l = 0$. So, $Qx = ((Qx)_k) \in c_0$, that is $x = (x_k) \in c_0(Q)$. This gives us that the inclusion $c \subset c_0(Q)$ holds. Now, we define a sequence $z = (z_k)$ such that $z_k = ln(k+4)$ for all $k \in \mathbb{N}$. Then, it is clear that $z = (z_k) \notin c$, but $Qx = ((Qx)_k) = (sln(\frac{k+3}{k+4}) + tln(\frac{k+2}{k+4}) + uln(\frac{k+1}{k+4})) \in c_0$, namely $z = (z_k) \in c_0(Q)$. As a consequence, the inclusion $c \subset c_0(Q)$ strictly holds. The proof is complete. □

Theorem 2.6. *The inclusions $\ell_p(Q) \subset c_0(Q) \subset c(Q) \subset \ell_\infty(Q)$ strictly hold.*

Proof. It is known that the inclusions $\ell_p \subset c_0 \subset c \subset \ell_\infty$ hold. Then, it is clear that the inclusions $\ell_p(Q) \subset c_0(Q) \subset c(Q) \subset \ell_\infty(Q)$ hold. Let us now consider three sequences $x = (x_k)$, $y = (y_k)$ and $z = (z_k)$ defined by

$$x_k = \frac{1}{r}\sum_{j=0}^{k}\sum_{i=0}^{k-j}\sum_{v=0}^{k-j-i}\mu_1^{k-j-i-v}\mu_2^v\mu_3^i(j+1)^{-\frac{1}{p}},$$

$$y_k = \frac{1}{r}\sum_{j=0}^{k}\sum_{i=0}^{k-j}\sum_{v=0}^{k-j-i}\mu_1^{k-j-i-v}\mu_2^v\mu_3^i,$$

and

$$z_k = \frac{1}{r}\sum_{j=0}^{k}\sum_{i=0}^{k-j}\sum_{v=0}^{k-j-i}\mu_1^{k-j-i-v}\mu_2^v\mu_3^i(-1)^j$$

for all $k \in \mathbb{N}$. Then, we have $Qx = \left(\frac{1}{(k+1)^{\frac{1}{p}}}\right) \in c_0 \setminus \ell_p$, $Qy = (1,1,1,...) \in c \setminus c_0$ and $Qz = ((-1)^k) \in \ell_\infty \setminus c$, namely $x = (x_k) \in c_0(Q) \setminus \ell_p(Q)$, $y = (y_k) \in c(Q) \setminus c_0(Q)$ and $z = (z_k) \in \ell_\infty(Q) \setminus c(Q)$. As a consequence the inclusions $\ell_p(Q) \subset c_0(Q) \subset c(Q) \subset \ell_\infty(Q)$ are strict. This completes the proof. □

Lemma 2.7 ([15]). *$A = (a_{nk}) \in (\ell_\infty : c_0)$ if and only if $\lim_{n\to\infty}\sum_k |a_{nk}| = 0$.*

Theorem 2.8. *The sequence spaces ℓ_∞ and $c_0(Q)$ overlap but do not include each other.*

Proof. Let us consider the quadruple band matrix $Q = (q_{nk}(r, s, t, u))$ and Lemma 2.7. Then, we obtain

$$\lim_{n\to\infty} \sum_k |q_{nk}(r, s, t, u)| = |r| + |s| + |t| + |u| \neq 0$$

for all $r, s, t, u \in \mathbb{R} \setminus \{0\}$. This means that, $Q \notin (\ell_\infty : c_0)$, namely for at least $a = (a_k) \in \ell_\infty$, $Qa \notin c_0$. So, $\ell_\infty \not\subset c_0(Q)$. Now, we define two sequences $b = (b_k)$ and $d = (d_k)$ such that $b_k = 4^{-k}$ and $d_k = 5^{k+3}$ for all $k \in \mathbb{N}$. Then, it is clear that $b = (b_k) \in \ell_\infty \cap c_0(Q)$. By assuming $r = t = 1$ and $s = u = -5$, we obtain that $Qd = (0, 0, 0, ...) \in c_0$, namely $d = (d_k) \in c_0(Q)$ but $d = (d_k) \notin \ell_\infty$. So, $\ell_\infty \cap c_0(Q) \neq \emptyset$ and $c_0(Q) \not\subset \ell_\infty$. So the proof is complete. □

Theorem 2.9. *The inclusions $\ell_p(B) \subset \ell_p(Q)$, $c_0(B) \subset c_0(Q)$, $c(B) \subset c(Q)$ and $\ell_\infty(B) \subset \ell_\infty(Q)$ strictly hold, where $B = B(b_1, b_2, b_3) = (b_{nk})$ is triple band matrix defined by*

$$b_{nk} = \begin{cases} b_1 & , \quad k = n \\ b_2 & , \quad k = n-1 \\ b_3 & , \quad k = n-2 \\ 0 & , \quad otherwise \end{cases}$$

for all $n, k \in \mathbb{N}$ and $b_1, b_2, b_3 \in \mathbb{R} \setminus \{0\}$.

Proof. Consider the quadruple and matrix $Q = Q(r, s, t, u)$. In case of $u = 0$, we obtain $Q(r, s, t, 0) = B(b_1, b_2, b_3)$. So, the inclusions $\ell_p(B) \subset \ell_p(Q)$, $c_0(B) \subset c_0(Q)$, $c(B) \subset c(Q)$ and $\ell_\infty(B) \subset \ell_\infty(Q)$ hold. Assume that $r, s, t, u \in \mathbb{R} \setminus \{0\}$. Let us define two sequences $x = (x_k)$ and $y = (y_k)$ such that $x_k = k^2 + 3k + 2$ and $y_k = (k + 2)^3$ for all $k \in \mathbb{N}$. Then, we obtain

$$(Bx)_k = (b_1 + b_2 + b_3)k^2 + (3b_1 + b_2 - b_3)k + 2b_1,$$

$$(Qx)_k = (r + s + t + u)k^2 + (3r + s - t - 3u)k + (2r + 2u),$$

$$(By)_k = (b_1 + b_2 + b_3)k^3 + (6b_1 + 3b_2)k^2 + (12b_1 + 3b_2)k + (8b_1 + b_2),$$

and

$$(Qy)_k = (r + s + t + u)k^3 + (6r + 3s - 3u)k^2 + (12r + 3s + 3u)k + (8r + s - u)$$

for all $k \in \mathbb{N}$. So, we can see that

(I)In case of $b_1, b_2, b_3 \in \mathbb{R} \setminus \{0\}$, $Bx \notin c_0, \ell_p$ and $By \notin c, \ell_\infty$, namely $x \notin c_0(B), \ell_p(B)$ and $y \notin c(B), \ell_\infty(B)$,

(II)In case of $s = -3r$, $t = 3r$, $u = -r$ and $r \in \mathbb{R} \setminus \{0\}$, $Qx \in c_0, \ell_p$ and $Qy \in c, \ell_\infty$, namely $x \in c_0(Q), \ell_p(Q)$ and $y \in c(Q), \ell_\infty(Q)$,

As a results of these the inclusions $\ell_p(B) \subset \ell_p(Q)$, $c_0(B) \subset c_0(Q)$, $c(B) \subset c(Q)$ and $\ell_\infty(B) \subset \ell_\infty(Q)$ strictly hold. This completes the proof. □

3. The Schauder Basis and $\alpha-, \beta-, \gamma-$ Duals

In this section, we give the Schauder basis of the spaces $c_0(Q)$, $c(Q)$ and $\ell_p(Q)$ and determine α-, β- and γ-duals of the spaces $c_0(Q)$, $c(Q)$, $\ell_\infty(Q)$ and $\ell_p(Q)$, where $1 \leq p < \infty$.

Let $(X, \|.\|_X)$ be a Banach space and (b_n) be a sequence in X. Then, (b_n) is called a Schauder basis if for every $x \in X$ there is a unique sequence (λ_n) of scalars such that

$$\lim_{n\to\infty} \left\| x - \sum_{k=0}^{n} \lambda_k b_k \right\|_X = 0.$$

Let X be a sequence space. Then, α-, β- and γ-duals of X are defined by

$$X^\alpha = \Big\{ a = (a_k) \in w : \forall x \in X\ ,\ ax = (a_k x_k) \in \ell_1 \Big\},$$

$$X^\beta = \Big\{ a = (a_k) \in w : \forall x \in X\ ,\ ax = (a_k x_k) \in cs \Big\},$$

and

$$X^\gamma = \Big\{ a = (a_k) \in w : \forall x \in X\ ,\ ax = (a_k x_k) \in bs \Big\},$$

respectively.

Theorem 3.1. *Let $\chi_k = (Qx)_k$ for all $k \in \mathbb{N}$, and let $l = \lim_{k\to\infty} \chi_k$. Let $d^{(k)}(r,s,t,u) = \left\{d_n^{(k)}(r,s,t,u)\right\}_{n\in\mathbb{N}}$ for evey fixed $k \in \mathbb{N}$ and $g = (g_n)$ be two sequences such that*

$$d_n^{(k)}(r,s,t,u) = \begin{cases} 0 & , \quad 0 \le n < k \\ \frac{1}{r}\sum\limits_{i=0}^{n-k}\sum\limits_{v=0}^{n-k-i}\mu_1^{n-k-i-v}\mu_2^v\mu_3^i & , \quad n \ge k \end{cases}$$

and

$$g_n = \frac{1}{r}\sum_{j=0}^{n}\sum_{i=0}^{n-j}\sum_{v=0}^{n-j-i}\mu_1^{n-j-i-v}\mu_2^v\mu_3^i$$

for all $n \in \mathbb{N}$. Then,

(a) *Schauder basis of the sequence spaces $c_0(Q)$ and $\ell_p(Q)$ is the sequence $\left\{d^{(k)}(r,s,t,u)\right\}_{k\in\mathbb{N}}$ and every $x \in c_0(Q)$ or $\ell_p(Q)$ can be uniquely represented by the form*

$$x = \sum_k \chi_k d^{(k)}(r,s,t,u).$$

(b) *Schauder basis of the sequence space $c(Q)$ is the sequence $\left\{g, d^{(0)}(r,s,t,u), d^{(1)}(r,s,t,u), ...\right\}$ and every $x \in c(Q)$ can be uniquely represented by the form*

$$x = lg + \sum_k [\chi_k - l] d^{(k)}(r,s,t,u).$$

Proof. (a) One can easily seen that

$$Qd^{(k)}(r,s,t,u) = e^{(k)} \in c_0, \ell_p\ , \ 1 \le p < \infty\ , \ k \in \mathbb{N},$$

where $e^{(k)}$ is called k-th unit vector so that $e^{(k)} = (0,0,...,1.0,0,...)$ where only the k^{th} place is 1 and zero otherwise. Therefore, $\left\{d^{(k)}(r,s,t,u)\right\}_{k\in\mathbb{N}} \subset c_0(Q)$ and $\left\{d^{(k)}(r,s,t,u)\right\}_{k\in\mathbb{N}} \subset \ell_p(Q)$ hold.

Let us take $x \in c_0(Q)$ or $\ell_p(Q)$, $1 \le p < \infty$ and define

$$x^{[m]} = \sum_{k=0}^{m} \chi_k d^{(k)}(r,s,t,u)$$

for all $m \in \mathbb{N}$. Then, by applying the quadruple band matrix $Q = Q(r,s,t,u)$ to $x^{[m]}$, we get

$$Qx^{[m]} = \sum_{k=0}^{m} \chi_k Qd^{(k)}(r,s,t,u) = \sum_{k=0}^{m} (Qx)_k e^{(k)}$$

and

$$\left\{Q(x - x^{[m]})\right\}_n = \begin{cases} 0 & , \quad 0 \le n \le m \\ (Qx)_n & , \quad n > m \end{cases}$$

for all $n, m \in \mathbb{N}$.

For arbitrarily given $\epsilon > 0$, there exists a $m_0 \in \mathbb{N}$ such that

$$|(Qx)_m| < \frac{\epsilon}{2}$$

and

$$\sum_{n=m_0+1}^{\infty} |(Qx)_n|^p \le \left(\frac{\epsilon}{2}\right)^p$$

for all $m \ge m_0$ and $1 \le p < \infty$. Then, we have

$$\|x - x^{[m]}\|_{c_0(Q)} = \sup_{m\le n} |(Qx)_n| \le \sup_{m_0 \le n} |(Qx)_n| \le \frac{\epsilon}{2} < \epsilon$$

and

$$\|x - x^{[m]}\|_{\ell_p(Q)} = \Big(\sum_{n=m+1}^{\infty} |(Qx)_n|^p\Big)^{\frac{1}{p}} \le \Big(\sum_{n=m_0+1}^{\infty} |(Qx)_n|^p\Big)^{\frac{1}{p}} \le \frac{\epsilon}{2} < \epsilon$$

for all $m \geq m_0$. This gives us that,

$$x = \sum_k \chi_k d^{(k)}(r,s,t,u).$$

Now, we suppose that $x = (x_k)$ has another representation such that

$$x = \sum_k \lambda_k d^{(k)}(r,s,t,u).$$

Because of the transformation L defined by $L : c_0(Q) \longrightarrow c_0$ or $L : \ell_p(Q) \longrightarrow \ell_p$, $L(x) = Qx$ is continuous, we obtain that

$$(Qx)_n = \sum_k \lambda_k \left\{Qd^{(k)}(r,s,t,u)\right\}_n = \sum_k \lambda_k e_n^{(k)} = \lambda_n$$

for all $n \in \mathbb{N}$. This result contradicts the fact that, $(Qx)_n = \chi_n$ for all $n \in \mathbb{N}$. As a consequence, the representation of the sequence $x \in c_0(Q)$, $\ell_p(Q)$ is unique.

(b) It is obvious that, $\left\{d^{(k)}(r,s,t,u)\right\}_{k\in\mathbb{N}} \subset c_0(Q)$ and $Qg = e \in c$. This implies that, the inclusion $\left\{g, d^{(k)}(r,s,t,u)\right\}_{k\in\mathbb{N}} \subset c(Q)$ holds.

Let $y = x - lg$ for arbitrarily taken $x = (x_k) \in c(Q)$, where $l = \lim_{k\to\infty} \chi_k$. Then, it is clear that $y = (y_k) \in c_0(Q)$. It follows that $y = (y_k)$ has a unique representation according to part (a). So, every $x = (x_k) \in c(Q)$ can be uniquely written on the form

$$x = lg + \sum_k [\chi_k - l] d^{(k)}(r,s,t,u).$$

So the proof is complete. □

By using the results of Theorem 2.1 and Theorem 3.1, the next result can be given.

Corollary 3.2. *The sequence spaces $c_0(Q)$, $c(Q)$ and $\ell_p(Q)$ are separable, where $1 \leq p < \infty$.*

Theorem 3.3. *Define the sets τ_1, τ_2 and τ_3 by*

$$\tau_1 = \left\{a = (a_k) \in w : \sup_{K\in\mathcal{F}} \sum_n \left| \frac{1}{r} \sum_{k\in K} \sum_{i=0}^{n-k} \sum_{v=0}^{n-k-i} \mu_1^{n-k-i-v} \mu_2^v \mu_3^i a_n \right| < \infty \right\},$$

$$\tau_2 = \left\{a = (a_k) \in w : \sup_{k\in\mathbb{N}} \sum_n \left| \frac{1}{r} \sum_{i=0}^{n-k} \sum_{v=0}^{n-k-i} \mu_1^{n-k-i-v} \mu_2^v \mu_3^i a_n \right| < \infty \right\},$$

and

$$\tau_3 = \left\{a = (a_k) \in w : \sup_{K\in\mathcal{F}} \sum_k \left| \frac{1}{r} \sum_{n\in K} \sum_{i=0}^{n-k} \sum_{v=0}^{n-k-i} \mu_1^{n-k-i-v} \mu_2^v \mu_3^i a_n \right|^q < \infty \right\},$$

where $\mathcal{F}$ is the collection of all finite subsets of $\mathbb{N}$, $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p \leq \infty$.

Then, the followings hold.

(i) $\{c_0(Q)\}^\alpha = \{c(Q)\}^\alpha = \tau_1$.
(ii) $\{\ell_1(Q)\}^\alpha = \tau_2$.
(iii) $\left\{\ell_p(Q)\right\}^\alpha = \tau_3$, *where $1 < p \leq \infty$.*

Proof. We only prove the part (i) as a sample. The rest of the theorem can be proved by using similar methods and Lemma 2.3.

(i) By considering the Theorem 2.2, take a sequence $x = (x_n)$ with

$$x_n = \frac{1}{r} \sum_{k=0}^{n} \sum_{i=0}^{n-k} \sum_{v=0}^{n-k-i} \mu_1^{n-k-i-v} \mu_2^v \mu_3^i y_k \tag{3.1}$$

for all $n \in \mathbb{N}$. Then, for each $a = (a_n) \in w$, we have

$$a_n x_n = \frac{1}{r} \sum_{k=0}^{n} \sum_{i=0}^{n-k} \sum_{v=0}^{n-k-i} \mu_1^{n-k-i-v} \mu_2^v \mu_3^i a_n y_k = (V^{r,s,t,u} y)_n$$

for all $n \in \mathbb{N}$, where $V^{r,s,t,u} = (v_{nk}^{r,s,t,u})$ is defined by

$$v_{nk}^{r,s,t,u} = \begin{cases} \frac{1}{r}\sum\limits_{i=0}^{n-k}\sum\limits_{v=0}^{n-k-i}\mu_1^{n-k-i-v}\mu_2^v\mu_3^i a_n & , \quad 0 \leq k \leq n \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}$.

The above result implies that $ax = (a_k x_k) \in \ell_1$ whenever $x = (x_k) \in c_0(Q)$ or $c(Q)$ if and only if $V^{r,s,t,u}y \in \ell_1$ whenever $y = (y_k) \in c_0$ or c, namely $a = (a_n)\,\{c_0(Q)\}^\alpha = \{c(Q)\}^\alpha$ if and only if $V^{r,s,t,u} \in (c_0 : \ell_1) = (c : \ell_1)$. It follows from this result and Lemma 2.3 (xii) that

$$a = (a_n)\,\{c_0(Q)\}^\alpha = \{c(Q)\}^\alpha \iff \sup_{K\in\mathcal{F}}\sum_n\left|\frac{1}{r}\sum_{k\in K}\sum_{i=0}^{n-k}\sum_{v=0}^{n-k-i}\mu_1^{n-k-i-v}\mu_2^v\mu_3^i a_n\right| < \infty.$$

Consequently, $\{c_0(Q)\}^\alpha = \{c(Q)\}^\alpha = \tau_1$. This completes the proof. □

Theorem 3.4. *Define the sets τ_4, τ_5, τ_6, τ_7 and τ_8 by*

$$\tau_4 = \left\{a = (a_k) \in w : \sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|\frac{1}{r}\sum_{j=k}^{n}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^v\mu_3^i a_j\right|^q < \infty\right\},\ 1 \leq q < \infty,$$

$$\tau_5 = \left\{a = (a_k) \in w : \frac{1}{r}\sum_{j=k}^{\infty}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^v\mu_3^i a_j \text{ exists for each } k \in \mathbb{N}\right\},$$

$$\tau_6 = \left\{a = (a_k) \in w : \frac{1}{r}\lim_{n\to\infty}\sum_{k=0}^{n}\sum_{j=k}^{n}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^v\mu_3^i a_j \text{ exists}\right\},$$

$$\tau_7 = \left\{a = (a_k) \in w : \sup_{n,k\in\mathbb{N}}\left|\frac{1}{r}\sum_{j=k}^{n}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^v\mu_3^i a_j\right| < \infty\right\},$$

and

$$\tau_8 = \left\{a = (a_k) \in w : \lim_{n\to\infty}\sum_k\left|\frac{1}{r}\sum_{j=k}^{n}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^v\mu_3^i a_j\right| = \sum_k\left|\frac{1}{r}\sum_{j=k}^{\infty}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^v\mu_3^i a_j\right|\right\}.$$

Then, the followings hold.

(i) $\{c_0(Q)\}^\beta = \tau_4 \cap \tau_5$, *where* $q = 1$.
(ii) $\{c(Q)\}^\beta = \tau_4 \cap \tau_5 \cap \tau_6$, *where* $q = 1$.
(iii) $\{\ell_1(Q)\}^\beta = \tau_5 \cap \tau_7$.
(iv) $\left\{\ell_p(Q)\right\}^\beta = \tau_5 \cap \tau_4$, *where* $1 < p < \infty$ *and* $1 < q < \infty$.
(v) $\{\ell_\infty(Q)\}^\beta = \tau_5 \cap \tau_8$.
(vi) $\{c_0(Q)\}^\gamma = \{c(Q)\}^\gamma = \tau_4$, *where* $q = 1$.
(vii) $\{\ell_1(Q)\}^\gamma = \tau_7$.
(viii) $\left\{\ell_p(Q)\right\}^\gamma = \tau_4$, *where* $1 < p < \infty$ *and* $1 < q < \infty$.
(ix) $\{\ell_\infty(Q)\}^\gamma = \tau_4$, *where* $q = 1$.

Proof. We only prove the part (i) as a sample. The rest of the theorem may be proved by using similar methods and Lemma 2.3.

(i) Let us take an arbitrary sequence $a = (a_k) \in w$ and consider the sequence $x = (x_k)$ defined by the relation (3.1). Then, we write

$$\begin{aligned}\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n}\left[\frac{1}{r}\sum_{j=0}^{k}\sum_{i=0}^{k-j}\sum_{v=0}^{k-j-i}\mu_1^{k-j-i-v}\mu_2^{v}\mu_3^{i}y_j\right]a_k\\ &= \sum_{k=0}^{n}\left[\frac{1}{r}\sum_{j=k}^{n}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^{v}\mu_3^{i}a_j\right]y_k\\ &= \left(E^{r,s,t,u}y\right)_n\end{aligned}$$

for all $n \in \mathbb{N}$, where $E^{r,s,t,u} = (e_{nk}^{r,s,t,u})$ is defined by

$$e_{nk}^{r,s,t,u} = \begin{cases} \frac{1}{r}\sum\limits_{j=k}^{n}\sum\limits_{i=0}^{j-k}\sum\limits_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^{v}\mu_3^{i}a_j & , \quad 0 \le k \le n\\ 0 & , \quad k > n\end{cases}$$

for all $n, k \in \mathbb{N}$.

The above calculation implies that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in c_0(Q)$ if and only if $E^{r,s,t,u}y \in c$ whenever $y = (y_k) \in c_0$, namely $a = (a_n) \in \{c_0(Q)\}^{\beta}$ if and only if $E^{r,s,t,u} \in (c_0 : c)$. By considering Lemma 2.3 (vi), we see that $a = (a_n) \in \{c_0(Q)\}^{\beta}$ if and only if

$$\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|\frac{1}{r}\sum_{j=k}^{n}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^{v}\mu_3^{i}a_j\right| < \infty$$

and

$$\frac{1}{r}\sum_{j=k}^{\infty}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^{v}\mu_3^{i}a_j \text{ exists for each } k \in \mathbb{N}.$$

As a consequence of this, we deduce that $\{c_0(Q)\}^{\beta} = \tau_4 \cap \tau_5$, where $q = 1$. This completes the proof. □

4. Some Matrix Transformation Related to the Four New Sequence Spaces

In this section, we characterize some matrix classes related to the spaces $c_0(Q)$, $c(Q)$, $\ell_\infty(Q)$ and $\ell_p(Q)$, where $1 \le p < \infty$.

In order to simplify notation, we shall use the equality defined by

$$g_{nk}^{r,s,t,u} = \frac{1}{r}\sum_{j=k}^{\infty}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^{v}\mu_3^{i}a_{nj}$$

for all $n, k \in \mathbb{N}$, throughout the section 4.

Now, we give following two lemmas to use it in the section 4.

Lemma 4.1 ([18]). *The matrix mappings between BK-spaces are continuous.*

Lemma 4.2 ([3]). *Let X, Y be arbitrary two sequence spaces. Then, $A \in (X : Y_T) \iff TA \in (X : Y)$, where A is an infinite matrix and T is a triangle matrix.*

Theorem 4.3. *For an arbitrarily given infinite matrix $A = (a_{nk})$ of complex numbers, the followings hold.*

(i) *$A \in (c(Q) : \ell_p)$ if and only if*

$$\sup_{K\in\mathcal{F}}\sum_{n}\left|\sum_{k\in K} g_{nk}^{r,s,t,u}\right|^{p} < \infty, \tag{4.1}$$

$$g_{nk}^{r,s,t,u} \text{ exists for all } k, n \in \mathbb{N}, \tag{4.2}$$

$$\sum_{k} g_{nk}^{r,s,t,u} \text{ converges for all } n \in \mathbb{N}, \tag{4.3}$$

$$\sup_{m\in\mathbb{N}}\sum_{k=0}^{m}\left|\frac{1}{r}\sum_{j=k}^{m}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^{v}\mu_3^{i}a_{nj}\right|<\infty\ ,\ n\in\mathbb{N}, \tag{4.4}$$

where $1\le p<\infty$.

(ii) $A\in(c(Q):\ell_\infty)$ *if and only if* (4.2) *and* (4.4) *hold, and*

$$\sup_{n\in\mathbb{N}}\sum_{k}|g_{nk}^{r,s,t,u}|<\infty. \tag{4.5}$$

Proof. Because of the part (ii) can be proved by using a similar way, by taking Lemma 2.3 (i) instead of Lemma 2.3 (xiv), we shall give the proof of theorem 4.3(i).

(i) Let us take $x=(x_k)\in c(Q)$ and assume that the conditions (4.1)-(4.4) hold. Then, by using Theorem 3.4 (ii), we have $\{a_{nk}\}_{k\in\mathbb{N}}\in\{c(Q)\}^\beta$ for all $n\in\mathbb{N}$. Hence, the A-transform of x exists. Let us define a matrix $H^{r,s,t,u}=(h_{nk}^{r,s,t,u})$ such that $h_{nk}^{r,s,t,u}=g_{nk}^{r,s,t,u}$ for all $n,k\in\mathbb{N}$. Accordig to condition (4.1), $H^{r,s,t,u}=(h_{nk}^{r,s,t,u})$ satisfies Lemma 2.3 (xiv). So, $H^{r,s,t,u}\in(c:\ell_p)$.

Now, we write the following:

$$\sum_{k=0}^{m}a_{nk}x_k=\frac{1}{r}\sum_{k=0}^{m}\sum_{j=k}^{m}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^{v}\mu_3^{i}a_{nj}y_k$$

for all $n,m\in\mathbb{N}$.

So, letting $m\to\infty$ side by side, we see that

$$\sum_{k}a_{nk}x_k=\sum_{k}g_{nk}^{r,s,t,u}y_k \tag{4.6}$$

whence, taking ℓ_p-norm, we obtain

$$\|Ax\|_{\ell_p}=\|H^{r,s,t,u}y\|_{\ell_p}<\infty.$$

This gives us that $Ax\in\ell_p$, namely $A\in(c(Q):\ell_p)$.

On the contrary, let us assume that $A\in(c(Q):\ell_p)$. By combining Lemma 4.1 and the fact that the spaces $c(Q)$ and ℓ_p are BK-spaces, we can say that

$$\|Ax\|_{\ell_p}\le M\|x\|_{c(Q)}$$

for every $x=(x_k)\in c(Q)$.

Consider the sequence $x=(x_k)$ defined by $x_k=\sum_{k\in K}d^{(k)}(r,s,t,u)$ for every fixed $k\in\mathbb{N}$, where $d^{(k)}(r,s,t,u)=\left\{d_n^{(k)}(r,s,t,u)\right\}_{n\in\mathbb{N}}$ and $K\in\mathcal{F}$. Because of the inequality (4) holds for all $x=(x_k)\in c(Q)$, we can write

$$\|Ax\|_{\ell_p}=\Big(\sum_{n}\Big|\sum_{k\in K}g_{nk}^{r,s,t,u}\Big|^p\Big)^{\frac{1}{p}}\le M\|x\|_{c_{(Q)}}=M.$$

Hence (4.1) holds.

By considering the assumption, we figure out that the matrix $A=(a_{nk})$ can be applied to the space $c(Q)$. As a consequence of this the conditions (4.2)-(4.4) obviously hold. This completes the proof. □

Theorem 4.4. *For an arbitrarily given infinite matrix* $A=(a_{nk})$ *of complex numbers,* $A\in(c(Q):c)$ *if and only if* (4.2), (4.4) *and* (4.5) *hold, and*

$$\lim_{n\to\infty}g_{nk}^{r,s,t,u}=\alpha_k \text{ for all } k\in\mathbb{N}, \tag{4.7}$$

$$\lim_{n\to\infty}\sum_{k}g_{nk}^{r,s,t,u}=\alpha. \tag{4.8}$$

Proof. By taking arbitrary sequence $x=(x_k)\in c(Q)$, we assume that the conditions (4.2), (4.4), (4.5), (4.7) and (4.8) hold for an infinite matrix $A=(a_{nk})$. If we consider the Theorem 3.4 and the supposed conditions, we conclude that $\{a_{nk}\}_{k\in\mathbb{N}}\in\{c(Q)\}^\beta$ for all $n\in\mathbb{N}$, and therefore Ax exists. According to conditions (4.5) and (4.7), we can write

$$\sum_{j=0}^{k}|\alpha_j|=\lim_{n\to\infty}\sum_{j=0}^{k}|g_{nj}^{r,s,t,u}|\le\sup_{n}\sum_{j}|g_{nj}^{r,s,t,u}|<\infty$$

for all $k \in \mathbb{N}$, which yields that $(\alpha_k) \in \ell_1$, and so the series $\sum_k \alpha_k(y_k - l)$ converges, where $\lim_{k\to\infty} y_k = l$, namely $y \in c$. If we connect Lemma 2.3 (v) with conditions (4.5), (4.7) and (4.8), we obtain that $G^{r,s,t,u} \in (c : c)$. Moreover, according to condition (4.6), we get

$$\sum_k a_{nk}x_k = \sum_k g_{nk}^{r,s,t,u}(y_k - l) + l\sum_k g_{nk}^{r,s,t,u} \tag{4.9}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ side by side in (4.9), we obtain that

$$\lim_{n\to\infty}(Ax)_n = \sum_k \alpha_k(y_k - l) + l\alpha,$$

which yields that $A \in (c(Q) : c)$.

On the contrary, we suppose that $A \in (c(Q) : c)$. If we consider the inclusion relation $c \subset \ell_\infty$, we deduce that $A \in (c(Q) : \ell_\infty)$. By connecting this result and Theorem 4.3 (ii), we obtain that the conditions (4.2), (4.4) and (4.5) hold.

Now, let us consider the sequences $d^{(k)}(r,s,t,u) = \left\{d_n^{(k)}(r,s,t,u)\right\}_{n\in\mathbb{N}}$ and $x = \sum_k d^{(k)}(r,s,t,u)$. Then, it is clear that $Ad^{(k)}(r,s,t,u) = \left\{g_{nk}^{r,s,t,u}\right\}_{n\in\mathbb{N}} \in c$ and $Ax = \left\{\sum_k g_{nk}^{r,s,t,u}\right\}_{n\in\mathbb{N}} \in c$, for all $k \in \mathbb{N}$. This gives us that the necessity of (4.7) and (4.8) hold. This completes the proof. □

Theorem 4.5. *For an arbitrarily given infinite matrix $A = (a_{nk})$ of complex numbers, the followings hold:*

(i) $A \in (\ell_1(Q) : \ell_\infty)$ *if and only if*

$$\sup_{k,n\in\mathbb{N}} |g_{nk}^{r,s,t,u}| < \infty. \tag{4.10}$$

(ii) $A \in (\ell_p(Q) : \ell_\infty)$ *if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k \left|g_{nk}^{r,s,t,u}\right|^q < \infty, \tag{4.11}$$

$$\{a_{nk}\}_{k\in\mathbb{N}} \in \tau_4, \tag{4.12}$$

where $1 < p < \infty$.

(iii) $A \in (\ell_\infty(Q) : \ell_\infty)$ *if and only if* (4.5) *holds, and*

$$\lim_{m\to\infty} \sum_k \left|\sum_{j=k}^{m}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i} \mu_1^{j-k-i-v}\mu_2^{v}\mu_3^{i}a_{nj}\right| = \sum_k \left|g_{nk}^{r,s,t,u}\right| \ (n \in \mathbb{N}). \tag{4.13}$$

Proof. Because of the parts (i) and (iii) can be proved by using a similar way, we shall give the proof of theorem for only part (ii).

(ii) By taking arbitrary sequence $x = (x_k) \in \ell_p(Q)$, we assume that the conditions (4.11) and (4.12) hold for an infinite matrix $A = (a_{nk})$. Then according to Theorem 3.4 (iv), it is clear that $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{\ell_p(Q)\right\}^\beta$ for all $n \in \mathbb{N}$. Therefore, Ax exists.

If we take ℓ_∞-norm (4.6) side by side and apply Hölder's inequality, respectively, we have

$$\begin{aligned} \|Ax\|_\infty &= \sup_{n\in\mathbb{N}}\left|\sum_k a_{nk}x_k\right| = \sup_{n\in\mathbb{N}}\left|\sum_k g_{nk}^{r,s,t,u}y_k\right| \\ &\le \sup_{n\in\mathbb{N}}\left(\sum_k \left|g_{nk}^{r,s,t,u}\right|^q\right)^{\frac{1}{q}}\left(\sum_k |y_k|^p\right)^{\frac{1}{p}} < \infty, \end{aligned}$$

which yields that $Ax \in \ell_\infty$, namely $A \in (\ell_p(Q) : \ell_\infty)$.

On the contrary, let us assume that $A \in (\ell_p(Q) : \ell_\infty)$. Then, because of $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{\ell_p(Q)\right\}^\beta$ for all $n \in \mathbb{N}$ by the hypothesis, the condition (4.12) holds and $\{g_{nk}^{r,s,t,u}\}_{n,k\in\mathbb{N}}$ exists. Moreover, according to same reason, the condition (4.6) holds and the sequences $a_n = \{a_{nk}\}_{k\in\mathbb{N}}$ define the continuous linear functionals f_n on $\ell_p(Q)$ by

$$f_n(x) = \sum_k a_{nk}x_k$$

for all $n \in \mathbb{N}$. Also, from Theorem 2.2, we know that $\ell_p(Q)$ and ℓ_p are norm isomorphic. By connecting this result and condition (4.6), we obtain

$$\|f_n\| = \|\{g_{nk}^{r,s,t,u}\}_{k\in\mathbb{N}}\|_q,$$

which yields us that the functionals are pointwise bounded. According to Banach-Steinhaus theorem, we deduce that the functionals are uniformly bounded, namely there exists a $M > 0$ such that

$$\Big(\sum_k \left|g_{nk}^{r,s,t,u}\right|^q\Big)^{\frac{1}{q}} = \|f_n\| \le M$$

holds for all $n \in \mathbb{N}$. This gives us that the condition (4.11) holds. This completes the proof. □

Theorem 4.6. *For an arbitrarily given infinite matrix $A = (a_{nk})$ of complex numbers, $A \in (\ell_1(Q) : \ell_p)$ if and only if*

$$\sup_{k\in\mathbb{N}} \sum_n \left|g_{nk}^{r,s,t,u}\right|^p < \infty, \tag{4.14}$$

where $1 \le p < \infty$.

Proof. By taking arbitrary sequence $x = (x_k) \in \ell_1(Q)$, we suppose that the condition (4.14) holds. Then, it is obvious that $y \in \ell_1$ and according to Theorem 3.4 (iii) $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\ell_1(Q)\}^\beta$ for all $n \in \mathbb{N}$, namely Ax exists. Hence, $\sum_k g_{nk}^{r,s,t,u} y_k$ is absolutely convergent for every fixed $n \in \mathbb{N}$ and each $y \in \ell_1$.

Now, by applying Minkowski's inequality to (4.6), we get

$$\Big(\sum_n \left|(Ax)_n\right|^p\Big)^{\frac{1}{p}} \le \sum_n |y_n| \Big(\sum_n \left|g_{nk}^{r,s,t,u}\right|^p\Big)^{\frac{1}{p}} < \infty,$$

which yields us that $Ax \in \ell_p$, namely $A \in (\ell_1(Q) : \ell_p)$.

On the contrary, let us suppose that $A \in (\ell_1(Q) : \ell_p)$, where $1 \le p < \infty$. Then, it is clear that Ax exists and belongs to ℓ_p for all $x = (x_k) \in \ell_1(Q)$. So, $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\ell_1(Q)\}^\beta$ for all $n \in \mathbb{N}$, which gives us the relation (4.6) holds.

Now, we define a matrix $D = (d_{nk})$ such that $d_{nk} = g_{nk}^{r,s,t,u}$ for all $n, k \in \mathbb{N}$. Therefore, it is obvious that $D \in (\ell_1 : \ell_p)$. By considering the Lemma 2.3 (xv), we deduce that the condition (4.14) holds. This completes the proof. □

Theorem 4.7. *For an arbitrarily given infinite matrix $A = (a_{nk})$ of complex numbers, the followings hold.*

(i) *$A \in (\ell_1(Q) : f)$ if and only if (4.10) holds, and*

$$f - \lim g_{nk}^{r,s,t,u} = \lambda_k\ ,\ \forall k \in \mathbb{N}. \tag{4.15}$$

(ii) *$A \in (\ell_p(Q) : f)$ if and only if (4.11), (4.12) and (4.15) hold, where $1 < p < \infty$.*

(iii) *$A \in (\ell_\infty(Q) : f)$ if and only if (4.5), (4.13) and (4.15) hold, and*

$$\lim_{m\to\infty} \sum_k \left| \frac{1}{m+1} \sum_{\sigma=0}^m g_{n+\sigma,k}^{r,s,t,u} - \lambda_k \right| = 0 \ uniformly\ in\ n.$$

Proof. Since the parts (i) and (iii) can be proved similarly, we shall give the proof of theorem for only part (ii).

(ii) By taking arbitrary sequence $x = (x_k) \in \ell_p(Q)$, we suppose that the conditions (4.11), (4.12) and (4.15) hold. Then, Ax obviously exists. If we consider the condition (4.15), we can write

$$\left| \frac{1}{m+1} \sum_{\sigma=0}^m g_{n+\sigma,k}^{r,s,t,u} \right|^q \longrightarrow |\lambda_k|^q \ (m \to \infty)$$

unifromly in n, for all $k \in \mathbb{N}$. This leads us with (4.11) to the inequality

$$\begin{aligned} \sum_{\sigma=0}^k |\lambda_\sigma|^q &= \lim_{m\to\infty} \sum_{\sigma=0}^k \left| \frac{1}{m+1} \sum_{\varphi=0}^m g_{n+\varphi,\sigma}^{r,s,t,u} \right|^q \ (uniformly\ in\ n) \\ &\le \sup_{n,m\in\mathbb{N}} \sum_\sigma \left| \frac{1}{m+1} \sum_{\varphi=0}^m g_{n+\varphi,\sigma}^{r,s,t,u} \right|^q = M < \infty \end{aligned}$$

holds for each $k \in \mathbb{N}$. Hence, $(\lambda_k) \in \ell_1$.

By the hypothesis, we know that $y = (y_k) \in \ell_p$ whenever $x = (x_k) \in \ell_p(Q)$. Then, by using Hölder's inequality, we can write

$$\sum_k |\lambda_k y_k| \leq \Big(\sum_k |\lambda_k|^q\Big)^{\frac{1}{q}} \Big(\sum_k |y_k|^p\Big)^{\frac{1}{p}} < \infty,$$

which yields $(\lambda_k y_k) \in \ell_1$. Also, since $y = (y_k) \in \ell_p$, for all $\epsilon > 0$ there exists a fixed $k_0 \in \mathbb{N}$ such that

$$\Big(\sum_{k=k_0+1}^{\infty} |y_k|^p\Big)^{\frac{1}{p}} < \frac{\epsilon}{4M^{\frac{1}{q}}}.$$

Then, there exists some $m_0 \in \mathbb{N}$ by (4.15) such that

$$\left|\sum_{k=0}^{k_0}\Big[\frac{1}{m+1}\sum_{\sigma=0}^{m} g_{n+\sigma,k}^{r,s,t,u} - \lambda_k\Big] y_k\right| < \frac{\epsilon}{2}$$

for all $m \geq m_0$, uniformly in n. By considering these two results and by applying Hölder's inequality, we can write

$$\begin{aligned}
\left|\frac{1}{m+1}\sum_{\varphi=0}^{m}(Ax)_{n+\varphi} - \sum_k \lambda_k y_k\right| &= \left|\sum_k\Big[\frac{1}{m+1}\sum_{\sigma=0}^{m} g_{n+\sigma,k}^{r,s,t,u} - \lambda_k\Big] y_k\right| \\
&\leq \left|\sum_{k=0}^{k_0}\Big[\frac{1}{m+1}\sum_{\sigma=0}^{m} g_{n+\sigma,k}^{r,s,t,u} - \lambda_k\Big] y_k\right| \\
&+ \left|\sum_{k=k_0+1}^{\infty}\Big[\frac{1}{m+1}\sum_{\sigma=0}^{m} g_{n+\sigma,k}^{r,s,t,u} - \lambda_k\Big] y_k\right| \\
&< \frac{\epsilon}{2} + \left(\sum_{k=k_0+1}^{\infty}\left[\left|\frac{1}{m+1}\sum_{\sigma=0}^{m} g_{n+\sigma,k}^{r,s,t,u}\right| + |\lambda_k|\right]^q\right)^{\frac{1}{q}} \\
&\cdot \left(\sum_{k=k_0+1}^{\infty} |y_k|^p\right)^{\frac{1}{p}} \\
&< \frac{\epsilon}{2} + 2M^{\frac{1}{q}}\frac{\epsilon}{4M^{\frac{1}{q}}} = \epsilon
\end{aligned}$$

for all $m \geq m_0$, uniformly in n, which yields us that $Ax \in f$, namely $A \in (\ell_p(Q) : f)$.

On the contrary, we assume that $A \in (\ell_p(Q) : f)$. Then, if we consider the inclusion $f \subset \ell_\infty$, we deduce that $A \in (\ell_p(Q) : \ell_\infty)$, which yields us that the conditions (4.11) and (4.12) hold.

Now, we consider the sequence $d^{(k)}(r,s,t,u) = \left\{d_n^{(k)}(r,s,t,u)\right\}_{n\in\mathbb{N}} \in \ell_p(Q)$ defined in the Theorem 3.1. Because of the assumption $A \in (\ell_p(Q) : f)$, we can write $Ad^{(k)}(r,s,t,u) = \left\{g_{nk}^{r,s,t,u}\right\}_{n\in\mathbb{N}} \in f$ for all $k \in \mathbb{N}$. This gives us the condition (4.15) holds. This completes the proof. □

In Theorem 4.7, if we use ordinary limit instead of f-limit, we can give the next corollary.

Corollary 4.8. *For an arbitrarily given infinite matrix $A = (a_{nk})$ of complex numbers, the followings hold.*

(i) $A \in (\ell_1(Q) : c)$ *if and only if* (4.7) *and* (4.10) *hold,*
(ii) $A \in (\ell_p(Q) : c)$ *if and only if* (4.7), (4.11) *and* (4.12) *hold, where* $1 < p < \infty$,
(iii) $A \in (\ell_\infty(Q) : c)$ *if and only if* (4.5), (4.7) *and* (4.13) *hold, and*

$$\lim_{n\to\infty}\sum_k \left|g_{nk}^{r,s,t,u} - \lambda_k\right| = 0.$$

Finally, by the aid of the Lemma 4.2, Theorems 4.3, 4.4, 4.5, 4.6, 4.7 and Corollary 4.8, the next result can be given.

Corollary 4.9. *Assume that the matrix $C = (c_{nk})$ is defined by*

$$c_{nk} = ra_{nk} + sa_{n-1,k} + ta_{n-2,k} + ua_{n-3,k}$$

for all $n, k \in \mathbb{N}$, where $A = (a_{nk})$ is an infinite matrix with complex entries. Then, the necessary and sufficient conditions in order to A belongs any of the classes $(c(Q) : \ell_p(Q))$, $(c(Q) : \ell_\infty(Q))$, $(c(Q) : c(Q))$, $(\ell_1(Q) : \ell_\infty(Q))$, $(\ell_p(Q) : \ell_\infty(Q))$, $(\ell_\infty(Q) : \ell_\infty(Q))$, $(\ell_1(Q) : \ell_p(Q))$, $(\ell_1(Q) : c(Q))$, $(\ell_p(Q) : c(Q))$ and $(\ell_\infty(Q) : c(Q))$ are determined, if the matrix $A = (a_{nk})$ is replaced by the matrix $C = (c_{nk})$ in the required ones in Theorems 4.3, 4.4, 4.5, 4.6, 4.7 and Corollary 4.8.

5. Conclusion

By remembering the definition of Quadruple band matrix, we arrive at a decision that $Q(1, -3, 3, -1) = \Delta^3$, $Q(1, \frac{-3}{2}, 1, \frac{-1}{4}) = \Delta_i^3$, $Q(r, s, t, 0) = B(r, s, t)$, $Q(1, -2, 1, 0) = \Delta^2$, $Q(r, s, 0, 0) = B(r, s)$ and $Q(1, -1) = \Delta$, where Δ^3, $B(r, s, t)$, Δ^2, $B(r, s)$ and Δ are called third order difference, triple band, second order difference, double band(generalized difference) and difference matrix, in turn. Also, Quadruple band matrix is not a special case of m-th order generalized difference matrix B^m defined in [4] and is not a special case of the weighed mean matrices. Therefore, this work fills up a gap in the known literature.

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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