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RESEARCH PAPER

Some integral inequalities via new family of preinvex functions

Muhammad Tariq[1,*,‡], Soubhagya Kumar Sahoo[2,‡], Hijaz Ahmad[3,‡], Asif Ali Shaikh[1,‡], Bibhakar Kodamasingh[2,‡] and Dawood Khan[4,‡]

[1]Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro 76062, Pakistan, [2]Department of Mathematics, Institute of Technical Education and Research, Siksha 'O' Anusandhan University, Bhubaneswar 751030, Odisha, India, [3]Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy, [4]Department of Mathematics, University of Balochistan, 87300 Pakistan

[*]Corresponding Author
[†]captaintariq2187@gmail.com (Muhammad Tariq); soubhagyalulu@gmail.com (Soubhagya Kumar Sahoo); hijaz555@gmail.com (Hijaz Ahmad); asif.shaikh@faculty.muet.edu.pk (Asif Ali Shaikh); bibhakarkodamasingh@soa.ac.in (Bibhakar Kodamasingh); dawooddawood601@gmail.com (Dawood Khan)

Abstract

The main objective of this work is to introduce and define the concept of s−type m−preinvex function and derive the new sort of Hermite−Hadamard inequality via the newly discussed idea. Furthermore, to enhance the quality of paper, we prove two new lemmas and we attain some extensions of Hermite−Hadamard-type inequality in the manner of newly explored definition for these lemmas. The concepts and tools of this paper may invigorate and revitalize for additional research in this mesmerizing and absorbing field of mathematics.

Key words: Preinvex function; s−type preinvexity; s−type m−preinvexity; Hermite−Hadamard inequality
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1 Introduction

The theory of convexity has assumed a key part and has gotten exceptional consideration by numerous scientists in the improvement of different fields of pure and applied sciences. It all started with the book by Hardy, Littlewood and Pólya [1], where the term convexity was used. This theory presents us with a characteristic and general system to examine a wide class of irrelevant issues. Because of its importance, the ideas of convex sets and convex functions have been generalized in various ways utilizing novel and creative ideas. The convex function is a class of significant functions popularly accepted in mathematical analysis. This class represents prominent parts of the theory of inequality. Moreover, convex functions have been widely utilized in many research fields such as optimization, engineering, physics, financial activities, etc. In optimization, the concept of generalized convexity along with inequality theory is often used. The Hermite-Hadamard integral inequalities containing convex functions are an intense research topic for many mathematicians because of their relevance and efficiency in use. Convex functions have a very strong association with integral inequalities. As of late, several mathematicians have explored the close relationship and correlated work on symmetry and convexity. It is also explained that while working on any one of the concepts, it tends to be applied to the other one too. Many familiar and relevant inequalities are modifications of convex functions. In literature, there are some well-known inequalities such as Hermite-Hadamard inequality and Jensen inequality that interpret the geometrical meaning of convex functions (see [2, 3, 4, 5, 6, 7, 8, 9, 10]) and the references cited therein.

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In [11], G. Toader introduced the class of $m-$convex functions. Soon after this many mathematicians like Latif [12] and Kalsoom [13] worked on the investigation of $m-$preinvexity.
Hanson [14] presented another new class of convex functions called invex functions, with the plan to generalize the legitimacy of the sufficiency of the Kuhn-Tucker conditions in nonlinear programming. Weir and Mond [15] introduced the preinvex function, which is an important extension of the convex function and it helped in handling numerous critical problems. It is realized that every convex function is a preinvex function but the converse is not true.

2 Preliminaries

Here, we remember several known definitions.

Definition 1 (see [16]) *Let $\Psi : \mathbb{A} \times \mathbb{A} \neq \emptyset \to \mathbb{R}$, then $\mathbb{A}$ is an invex w.r.t $\zeta(.,.)$ if $\nu + \delta\zeta(\mu, \nu) \in \mathbb{A}$, for every $\mu, \nu \in \mathbb{A}$ and $\delta \in [0, 1]$.*

Note that, $\mathbb{A}$ is also called $\zeta-$connected set.
The above definition collapses to classical convexity if $\zeta(\mu, \nu) = \mu - \nu$. Therefore, every convex set is an invex but the converse is not true in general, (see [16] and [17]).

Definition 2 (see [15]) *The function $\Psi : \mathbb{A} \neq \emptyset \to \mathbb{R}$ on an invex set is called preinvex w.r.t ζ if*

$$\Psi(\nu + \delta\zeta(\mu, \nu)) \leq \delta\Psi(\mu) + (1 - \delta)\Psi(\nu), \quad \forall \mu, \nu \in \mathbb{A}, \ \delta \in [0, 1].$$

Definition 3 (see [18]) *A set $\mathbb{A} \subseteq \mathbb{R}^n$ is said to be $m-$invex w.r.t $\zeta : \mathbb{A} \times \mathbb{A} \times (0, 1] \to \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if*

$$m\nu + \delta\zeta(\mu, \nu, m) \in \mathbb{A},$$

holds for every $\mu, \nu \in \mathbb{A}$, $m \in (0, 1]$ and $\delta \in [0, 1]$.

Definition 4 *[13] A $\Psi : \mathbb{A} \to \mathbb{R}$ is called generalized $m-$preinvex w.r.t $\zeta : \mathbb{A} \times \mathbb{A} \times (0, 1] \to \mathbb{R}^n$ for fixed $m \in (0, 1]$, if*

$$\Psi(m\nu + \delta\zeta(\mu, \nu, m)) \leq \delta\Psi(\mu) + m(1 - \delta)\Psi(\nu), \tag{1}$$

holds for every $\mu, \nu \in \mathbb{A}$, $\delta \in [0, 1]$.

Definition 5 (see [19]) *A nonnegative function $\Psi : \mathbb{A} \to \mathbb{R}$ is called $s-$type convex function if $\mu, \nu \in \mathbb{A}$, $s \in [0, 1]$ and $\delta \in [0, 1]$, if*

$$\Psi(\delta\mu + (1 - \delta)\nu) \leq \left[1 - (s(1 - \delta))\right]\Psi(\mu) + [1 - s\delta]\Psi(\nu). \tag{2}$$

We also want the following hypothesis regarding the function ζ which is due to Mohan and Neogy [20].
Condition-C: Let $\mathbb{A} \subset \mathbb{R}^n$ be an open invex subset w.r.t $\zeta : \mathbb{A} \times \mathbb{A} \to \mathbb{R}$. For any $\mu, \nu \in \mathbb{A}$ and $\delta \in [0, 1]$

$$\begin{aligned} \zeta(\nu, \nu + \delta\zeta(\mu, \nu)) &= -\delta\zeta(\mu, \nu) \\ \zeta(\mu, \nu + \delta\zeta(\mu, \nu)) &= (1 - \delta)\zeta(\mu, \nu). \end{aligned} \tag{3}$$

For any $\mu, \nu \in \mathbb{A}$ and $\delta_1, \delta_2 \in [0, 1]$ from condition C, we have

$$\zeta(\nu + \delta_2\zeta(\mu, \nu), \nu + \delta_1\zeta(\mu, \nu)) = (\delta_2 - \delta_1)\zeta(\mu, \nu).$$

Extended condition-C ([21]): For any $\mu, \nu \in \mathbb{A}$, $\delta \in [0, 1]$ and $\mathbb{A} \subset \mathbb{R}^n$ be an open $m-$invex subset with respect to $\zeta : \mathbb{A} \times \mathbb{X} \times (0, 1] \to \mathbb{R}$. Then we have

$$\begin{aligned} \zeta(\nu, m\nu + \delta\zeta(\mu, \nu, m), m) &= -\delta\zeta(\mu, \nu, m) \\ \zeta(\mu, m\nu + \delta\zeta(\mu, \nu, m), m) &= (1 - \delta)\zeta(\mu, \nu, m) \\ \zeta(\mu, \nu, m) &= -\zeta(\nu, \mu, m). \end{aligned}$$

3 Generalized preinvex function

In this part, we are to define and explore a new class of preinvex functions namely $s-$type $m-$preinvex function.

Definition 6 *Let $\mathbb{A} \subset \mathbb{R}$ be a nonempty $m-$invex set w.r.t $\zeta : \mathbb{A} \times \mathbb{A} \times (0, 1] \to \mathbb{R}$. Then the function $\Psi : \mathbb{A} \to \mathbb{R}$ is called $s-$type $m-$preinvex, if*

$$\Psi(m\nu + \delta\zeta(\mu, \nu, m)) \leq \left[1 - (s(1 - \delta))\right]\Psi(\mu) + m[1 - s\delta]\Psi(\nu), \tag{4}$$

holds $\forall \mu, \nu \in \mathbb{A}$, $s \in [0, 1]$, $m \in (0, 1]$ and $\delta \in [0, 1]$.

Remark 1 *(i) If $s = m = 1$, then the above definition collapses to preinvex function [15].*
(ii) If $m = 1$ and $\zeta(\mu, \nu, m) = \mu - m\nu$, then the above definition collapses to $s-$type convex function [19].
(iii) If $s = m = 1$ and $\zeta(\mu, \nu, m) = \mu - m\nu$, then the above definition collapses to convex function [3].

4 Hermite–Hadamard type inequality via generalized preinvex function

Here, we are to explore the new sort of H-H inequality via s–type m-preinvex function.

Theorem 1 *Let $\mathbb{A}^\circ \subseteq \mathbb{R}$ be an open invex subset w.r.t $\zeta : \mathbb{A}^\circ \times \mathbb{A}^\circ \to \mathbb{R}$ and $\mu, \nu \in \mathbb{A}^\circ$ with $m\nu + \zeta(\mu, \nu, m) \leq \nu$. Suppose $\Psi : [m\nu + \zeta(\mu, \nu, m), \nu] \to (0, \infty)$ is s–type m-preinvex function, $\Psi \in L[m\nu + \zeta(\mu, \nu, m), \nu]$ for all $m \in (0, 1]$ and satisfies Condition–C then*

$$\frac{2}{2-s}\Psi(m\nu + \frac{1}{2}\zeta(\mu, \nu, m)) \leq \frac{1}{\zeta(\mu, \nu, m)}\left[\int_{m\nu}^{m\nu+\zeta(\mu,\nu,m)} \Psi(x)dx + m\int_{\frac{m\nu+\zeta(\mu,\nu,m)}{m}}^{\nu} \Psi(x)dx\right] \leq (2-s)[\Psi(\mu) + m\Psi(\nu)].$$

Proof Since $\mu, \nu \in \mathbb{A}^\circ$ and $\mathbb{A}^\circ$ is an invex set with respect to ζ, for every $m \in (0, 1]$ and $\delta \in [0, 1]$, we have $m\nu + \delta\zeta(\mu, \nu, m) \in \mathbb{A}^\circ$. For the left side, using the Definition 6, put $\delta = \frac{1}{2}$,

$$\Psi(my + \delta\zeta(x, y, m)) \leq [1 - (s(1-\delta))]\Psi(x) + m[1 - (s\delta)]\Psi(y)$$
$$\Psi(my + \frac{1}{2}\zeta(x, y, m)) \leq \left[1 - (\frac{s}{2})\right]\left[\Psi(x) + m\Psi(y)\right],$$

put $x = m\nu + \delta\zeta(\mu, \nu, m)$ and $my = m\nu + (1-\delta)\zeta(\mu, \nu, m)$ in the above inequality So we obtain

$$\Psi(my + \frac{1}{2}\zeta(x, y, m)) = \Psi(m\nu + (1-\delta)\zeta(\mu, \nu, m) + \frac{1}{2}\zeta(m\nu + \delta\zeta(\mu, \nu, m), m\nu + (1-\delta)\zeta(\mu, \nu, m), m)). \quad (5)$$

Now by using Condition C, we have

$$\zeta(m\nu + \delta\zeta(\mu, \nu, m), m\nu + (1-\delta)\zeta(\mu, \nu, m)) = (\delta - 1 + \delta)\zeta(\mu, \nu, m)$$

$$\zeta(m\nu + \delta\zeta(\mu, \nu, m), m\nu + (1-\delta)\zeta(\mu, \nu, m)) = (2\delta - 1)\zeta(\mu, \nu, m).$$

Now we put the value of ζ in (5), then as a result, we get

$$\Psi(my + \frac{1}{2}\zeta(x, y, m)) = \Psi(m\nu + (1-\delta)\zeta(\mu, \nu, m) + \frac{1}{2}(2\delta - 1)\zeta(\mu, \nu, m))$$

$$\Psi(my + \frac{1}{2}\zeta(x, y, m)) = \Psi(m\nu + (1 - \delta + \delta - \frac{1}{2})\zeta(\mu, \nu, m))$$

$$\Psi(my + \frac{1}{2}\zeta(x, y, m)) = \Psi(m\nu + \frac{1}{2}\zeta(\mu, \nu, m)).$$

Thus,

$$\begin{aligned}&\Psi(m\nu + \frac{1}{2}\zeta(\mu, \nu, m))\\ &\leq \left[1 - (\frac{s}{2})\right]\left[\int_0^1 \Psi(m\nu + \delta\zeta(\mu, \nu, m))d\delta + m\int_0^1 \Psi(\nu + \frac{(1-\delta)}{m}\zeta(\mu, \nu, m))d\delta\right]\\ &\leq \left[1 - (\frac{s}{2})\right]\frac{1}{\zeta(\mu, \nu, m)}\left[\int_{m\nu}^{m\nu+\zeta(\mu,\nu,m)} \Psi(x)dx + m\int_{\frac{m\nu+\zeta(\mu,\nu,m)}{m}}^{\nu} \Psi(x)dx\right].\end{aligned}$$

For the right side of the inequality and from the property of s–type m–preinvexity, we have

$$\begin{aligned}&\frac{1}{\zeta(\mu, \nu, m)}\left[\int_{m\nu}^{m\nu+\zeta(\mu,\nu,m)} \Psi(x)dx + m\int_{\frac{m\nu+\zeta(\mu,\nu,m)}{m}}^{\nu} \Psi(x)dx\right]\\ &\leq \left[\int_0^1 \Psi(m\nu + \delta\zeta(\mu, \nu, m))d\delta + m\int_0^1 \Psi(\nu + \frac{(1-\delta)}{m}\zeta(\mu, \nu, m))d\delta\right]\\ &\leq \int_0^1 [1 - (s(1-\delta))]\Psi(\mu)d\delta + m\int_0^1 [1 - (s\delta)]\Psi(\nu)d\delta\\ &+ \int_0^1 [1 - s\delta]\Psi(\mu)d\delta + m\int_0^1 [1 - (s(1-\delta))]\Psi(\nu)d\delta\\ &\leq \frac{(2-s)}{2}[\Psi(\mu) + \Psi(\mu) + m(\Psi(\nu) + \Psi(\nu))]\\ &\leq (2-s)[\Psi(\mu) + m(\Psi(\nu))].\end{aligned}$$

This is the required proof. ■

Corollary 1 *If $s = m = 1$ and $\zeta(\mu, \nu, m) = \mu - m\nu$, then we get Hermite-Hadamard inequality in [22].*

Remark 2 *If $m = 1$, then we attain the inequality*

$$\frac{2}{2-s}\Psi(\nu+\frac{1}{2}\zeta(\mu,\nu)) \le \frac{1}{\zeta(\mu,\nu)}\left[\int_{\nu}^{\nu+\zeta(\mu,\nu)}\Psi(x)dx + \int_{\nu+\zeta(\mu,\nu)}^{\nu}\Psi(x)dx\right] \le (2-s)\left[\Psi(\mu)+\Psi(\nu)\right].$$

Remark 3 *If $s = 1$, then we get the inequality*

$$2\Psi(m\nu+\frac{1}{2}\zeta(\mu,\nu,m)) \le \frac{1}{\zeta(\mu,\nu,m)}\left[\int_{m\nu}^{m\nu+\zeta(\mu,\nu,m)}\Psi(x)dx + m\int_{\frac{m\nu+\zeta(\mu,\nu,m)}{m}}^{\nu}\Psi(x)dx\right] \le \left[\Psi(\mu)+m\Psi(\nu)\right].$$

Remark 4 *If $s = m = 1$, then we get the inequality*

$$2\Psi(\nu+\frac{1}{2}\zeta(\mu,\nu)) \le \frac{1}{\zeta(\mu,\nu)}\left[\int_{\nu}^{\nu+\zeta(\mu,\nu)}\Psi(x)dx + \int_{\nu+\zeta(\mu,\nu)}^{\nu}\Psi(x)dx\right] \le \left[\Psi(\mu)+\Psi(\nu)\right].$$

5 Refinements of Hermite–Hadamard type inequality

The main aim of this section is to examine the refinements of Hermite–Hadamard inequality via s–type preinvex functions.

Lemma 1 *Let $\Psi : \left[\mu, m\mu+\zeta(\frac{\nu}{c},\mu,m)\right] \to \mathbb{R}$ be a differentiable mapping on $(\mu, m\mu+\zeta(\frac{\nu}{c},\mu,m))$ with $0 < c \le 1$ and $m\mu+\zeta(\nu,\mu,m) > \mu > 0$. If $\Psi' \in \mathcal{L}\,(\mu, m\mu+\zeta(\frac{\nu}{c},\mu,m))$ and for all $m \in (0,1]$, then*

$$\frac{\Psi(\mu)+\Psi(m\mu+\zeta(\frac{\nu}{c},\mu,m))}{2} - \frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)}\Psi(x)dx = \frac{\zeta(\nu,c\mu,m)}{2c}\int_0^1(1-2\delta)\Psi'\left(m\frac{\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)d\delta. \quad (6)$$

Proof

$$\frac{\zeta(\nu,c\mu,m)}{2c}\int_0^1(1-2\delta)\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)d\delta = \frac{\zeta(\nu,c\mu,m)}{2c}\left[\frac{(1-2\delta)\Psi(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c}))}{\zeta(\mu,\frac{\nu}{c},m)}\Bigg|_0^1 + 2\int_0^1\frac{\Psi(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m))}{\zeta(\mu,\frac{\nu}{c},m)}d\delta\right]$$

$$= \frac{\zeta(\nu,c\mu,m)}{2c}\left[\frac{c(\Psi(\mu)+\Psi(m\mu+\zeta(\frac{\nu}{c},\mu,m))}{\zeta(\nu,c\mu)} - \frac{2c}{\zeta(\nu,c\mu)}\int_0^1\Psi(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m))d\delta\right]$$

$$= \frac{\Psi(\mu)+\Psi(m\mu+\zeta(\frac{\nu}{c},\mu,m))}{2} - \frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)}\Psi(x)dx,$$

which gives the proof. ■

Lemma 2 *Let $\Psi : \left[\mu, m\mu+\zeta(\frac{\nu}{c},\mu,m)\right] \to \mathbb{R}$ be a differentiable mapping on $(\mu, m\mu+\zeta(\frac{\nu}{c},\mu,m))$ with $0 < c \le 1$ and $m\mu+\zeta(\nu,\mu,m) > \mu > 0$. If $\Psi' \in \mathcal{L}\,(\mu, m\mu+\zeta(\frac{\nu}{c},\mu,m))$ and for all $m \in (0,1]$, then*

$$\begin{aligned}&\frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)}\Psi(x)dx - \Psi\left(\frac{2m\mu+\zeta(\nu,\mu,m)}{2c}\right)\\ &= \frac{\zeta(\nu,c\mu,m)}{c}\left\{\int_0^1\delta\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)d\delta - \int_{1/2}^1\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)d\delta\right\}.\end{aligned} \quad (7)$$

Proof

$$\begin{aligned}&\frac{\zeta(\nu,c\mu,m)}{c}\left\{\int_0^1\delta\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)d\delta - \int_{1/2}^1\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)d\delta\right\}\\ &= \frac{\zeta(\nu,c\mu,m)}{c}\times\left[\frac{\delta\Psi(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m))}{\zeta(\mu,\frac{\nu}{c},m)}\Bigg|_0^1 - \int_0^1\frac{\Psi(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m))}{\zeta(\mu,\frac{\nu}{c},m)}d\delta - \frac{\Psi(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m))}{\zeta(\mu,\frac{\nu}{c},m)}d\delta\Bigg|_{\frac{1}{2}}^1\right]\\ &= \frac{\zeta(\nu,c\mu,m)}{c}\left[\frac{c\Psi(\mu)}{\zeta(c\mu,\nu,m)} - \frac{c}{\zeta(c\mu,\nu,m)}\int_0^1\Psi(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m))d\delta - \frac{c}{\zeta(c\mu,\nu,m)}\left(\Psi(\mu)-\Psi\left(\frac{(2m\mu+\zeta(\nu,\mu,m))}{2c}\right)\right)\right]\\ &= \frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{m\mu+\zeta(\frac{m\nu}{c},\mu,m)}\Psi(x)dx - \Psi\left(\frac{2m\mu+\zeta(\nu,\mu,m)}{2c}\right),\end{aligned}$$

which gives the proof. ■

Theorem 2 *Let $\mathbb{X}^{\circ} \subseteq \mathbb{R}$ be an open invex subset w.r.t $\zeta : \mathbb{X}^{\circ}\times\mathbb{X}^{\circ} \to \mathbb{R}$ and $\mu,\nu \in \mathbb{X}^{\circ}$ with $m\nu+\zeta(\mu,\nu,m) \le \nu$. Suppose $\Psi : [m\nu+\zeta(\mu,\nu,m),\nu]$ be a differentiable function on $\mathbb{X}^{\circ}$. If $|\Psi'|$ is s–type m–preinvex function on $(\mu, m\mu+\zeta(\nu,\mu,m))$ for $m \in (0,1]$ and $s \in [0,1]$, then*

$$\left|\frac{\Psi(\mu)+\Psi(m\mu+\zeta(\frac{\nu}{c},\mu,m))}{2} - \frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)}\Psi(x)dx\right| \le \frac{\zeta(\nu,c\mu,m)}{2c}\left\{\frac{2-s}{4}\left[|\Psi'(\mu)| + m|\Psi'\left(\frac{\nu}{c}\right)|\right]\right\}. \quad (8)$$

Proof According to Lemma 1, one has

$$\left|\frac{\Psi(\mu)+\Psi(m\mu+\zeta(\frac{\nu}{c},\mu,m))}{2}-\frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)}\Psi(x)dx\right|=\frac{\zeta(\nu,c\mu,m)}{2c}\int_0^1|1-2\delta||\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)|d\delta.$$

Since $|\Psi'|$ is $s-$type $m-$preinvex on $(\mu,\mu+\zeta(\nu,\mu))$, we have

$$\left|\frac{\Psi(\mu)+\Psi(m\mu+\zeta(\frac{\nu}{c},\mu,m))}{2}-\frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)}\Psi(x)dx\right|\leq\frac{\zeta(\nu,c\mu,m)}{2c}\int_0^1|1-2\delta|\left[(1-s(1-\delta))\Psi'(\mu)|+m(1-s\delta)|\Psi'\left(\frac{\nu}{c}\right)|\right]d\delta$$

$$\leq\frac{\zeta(\nu,c\mu,m)}{2c}\left\{|\Psi'(\mu)|\int_0^1|1-2\delta|1-s(1-\delta)d\delta+m|\Psi'\left(\frac{\nu}{c}\right)|\int_0^1|1-2\delta|(1-s\delta)d\delta\right\}. \tag{9}$$

Since,

$$\int_0^1(1-s(1-\delta))\,|1-2\delta|d\delta=\int_0^1(1-s\delta)\,|1-2\delta|d\delta=-\frac{s-2}{4}.$$

Putting the value of the above computation in (9), then we obtain the required proof. ■

Theorem 3 *Let $\mathbb{X}^\circ\subseteq\mathbb{R}$ be an open invex subset w.r.t $\zeta:\mathbb{X}^\circ\times\mathbb{X}^\circ\to\mathbb{R}$ and $\mu,\nu\in\mathbb{X}^\circ$ with $m\nu+\zeta(\mu,\nu,m)\leq\nu$. Suppose $\Psi:[m\nu+\zeta(\mu,\nu,m),\nu]$ be a differentiable mapping on $\mathbb{X}^\circ$. If $|\Psi'|^q$ is $s-$type $m-$preinvex on $(\mu,m\mu+\zeta(\nu,\mu,m))$ for $p,q>1$, $\frac{1}{q}+\frac{1}{p}=1$, $m\in(0,1]$ and $s\in[0,1]$, then*

$$\left|\frac{\Psi(\mu)+\Psi(m\mu+\zeta(\frac{\nu}{c},\mu,m))}{2}-\frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{\mu+\zeta(\frac{\nu}{c},\mu)}\Psi(x)dx\right|\leq\frac{\zeta(\nu,c\mu,m)}{2c}\left[\frac{1}{p+1}\right]^{1/p}\left\{\frac{2-s}{2}\left[|\Psi'(\mu)|^q+m|\Psi'\left(\frac{\nu}{c}\right)|^q\right]\right\}^{1/q}. \tag{10}$$

Proof According to Lemma 1 and applying Hölder's inequality, one has

$$\left|\frac{\Psi(\mu)+\Psi(m\mu+\zeta(\frac{\nu}{c},\mu,m))}{2}-\frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)}\Psi(x)dx\right|=\frac{\zeta(\nu,c\mu,m)}{2c}\int_0^1|1-2\delta||\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)|d\delta$$

$$\leq\frac{\zeta(\nu,c\mu,m)}{2c}\left(\int_0^1|1-2\delta|^pd\delta\right)^{1/p}\left(\int_0^1|\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)|^qd\delta\right)^{1/q}. \tag{11}$$

Since $|\Psi'|^q$ is s-type $m-$preinvex on $(\mu,m\mu+\zeta(\nu,\mu,m))$, we have

$$\int_0^1|\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)|^qd\delta=|\Psi'(\mu)|^q\int_0^1(1-s(1-\delta))d\delta+m|\Psi'\left(\frac{\nu}{c}\right)|^q\int_0^1(1-s\delta)d\delta.$$

Now, equation (11) becomes

$$\left|\frac{\Psi(\mu)+\Psi(m\mu+\zeta(\frac{\nu}{c},\mu,m))}{2}-\frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)}\Psi(x)dx\right|$$

$$\leq\frac{\zeta(\nu,c\mu,m)}{2c}\left[\frac{1}{p+1}\right]^{1/p}\left(|\Psi'(\mu)|^q\int_0^1(1-s(1-\delta))d\delta+m|\Psi'\left(\frac{\nu}{c}\right)|^q\int_0^1(1-s\delta)d\delta\right)^{1/q}. \tag{12}$$

Since,

$$\int_0^1(1-s\,(1-\delta))\,d\delta=\int_0^1(1-s\delta)\,d\delta=-\frac{s-2}{2}\int_0^1|1-2\delta|^pd\delta=\left[\frac{1}{p+1}\right].$$

Putting the value of the above computation in (12), then we obtain the required proof. ■

Theorem 4 *Let $\mathbb{X}^\circ\subseteq\mathbb{R}$ be an open invex subset w.r.t $\zeta:\mathbb{X}^\circ\times\mathbb{X}^\circ\to\mathbb{R}$ and $\mu,\nu\in\mathbb{X}^\circ$ with $m\nu+\zeta(\mu,\nu,m)\leq\nu$. Suppose $\Psi:[m\nu+\zeta(\mu,\nu,m),\nu]$ be a differentiable mapping on $\mathbb{X}^\circ$. If $|\Psi'|^q$ is $s-$type $m-$preinvex on $(\mu,m\mu+\zeta(\nu,\mu,m))$ for $p,q>1$, $\frac{1}{q}+\frac{1}{p}=1$, $m\in(0,1]$ and $s\in[0,1]$, then*

$$\left|\frac{\Psi(\mu)+\Psi(m\mu+\zeta(\frac{\nu}{c},\mu,m))}{2}-\frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)}\Psi(x)dx\right|\leq\frac{\zeta(\nu,c\mu,m)}{c}\left[\frac{1}{2^{\left(\frac{p+1}{p}\right)}}\right]\left\{\frac{2-s}{4}\left[|\Psi'(\mu)|^q+m|\Psi'\left(\frac{\nu}{c}\right)|^q\right]\right\}^{1/q}. \tag{13}$$

Proof According to Lemma 1 and applying Hölder's inequality, one has

$$\left|\frac{\Psi(\mu)+\Psi(m\mu+\zeta(\frac{\nu}{c},\mu,m))}{2}-\frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{\mu+\zeta(\frac{\nu}{c},\mu)}\Psi(x)dx\right|\leq\frac{\zeta(\nu,c\mu,m)}{2c}\int_0^1|1-2\delta||\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)|d\delta$$

$$=\frac{\zeta(\nu,c\mu,m)}{2c}\int_0^1|1-2\delta|^{1/p}|1-2\delta|^{1/q}|\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)|d\delta$$

$$\leq\frac{\zeta(\nu,c\mu,m)}{2c}\left(\int_0^1|1-2\delta|d\delta\right)^{1/p}\left(\int_0^1|1-2\delta||\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)|^qd\delta\right)^{1/q}. \tag{14}$$

Since $|\Psi'|^q$ is s-type $m-$preinvex on $(\mu, m\mu + \zeta(\nu, \mu, m))$, we have

$$\int_0^1 |\Psi'\left(\frac{m\nu}{c} + \delta\zeta(\mu, \frac{\nu}{c}, m)\right)|^q d\delta = |\Psi'(\mu)|^q \int_0^1 (1 - s(1-\delta))d\delta + m|\Psi'\left(\frac{\nu}{c}\right)|^q \int_0^1 (1 - s\delta)d\delta.$$

Now, equation (14) becomes

$$\left|\frac{\Psi(\mu) + \Psi(m\mu + \zeta(\frac{\nu}{c}, \mu, m))}{2} - \frac{c}{\zeta(\nu, c\mu, m)} \int_\mu^{m\mu+\zeta(\frac{\nu}{c},\mu,m)} \Psi(x)dx\right|$$
$$\leq \frac{\zeta(\nu, c\mu, m)}{2c} \left(\int_0^1 |1-2\delta|d\delta\right)^{1/p} \left(|\Psi'(\mu)|^q \int_0^1 |1-2\delta|(1-s(1-\delta))d\delta \;+m|\Psi'\left(\frac{\nu}{c}\right)|^q \int_0^1 |1-2\delta|(1-sk)d\delta\right)^{1/q}. \tag{15}$$

Since,

$$\int_0^1 |1-2\delta|\,(1-s\,(1-\delta))\,d\delta = \int_0^1 |1-2\delta|\,(1-s\delta)\,d\delta = -\frac{s-2}{4}$$
$$\int_0^1 |1-2\delta|d\delta = \frac{1}{2}.$$

Putting the values of the above computations in (15), then we obtain the required proof. ■

Theorem 5 *Let $\mathbb{X}^\circ \subseteq \mathbb{R}$ be an open invex subset w.r.t $\zeta : \mathbb{X}^\circ \times \mathbb{X}^\circ \to \mathbb{R}$ and $\mu, \nu \in \mathbb{X}^\circ$ with $m\nu + \zeta(\mu, \nu, m) \leq \nu$. Suppose $\Psi : [m\nu + \zeta(\mu, \nu, m), \nu]$ be a differentiable mapping on $\mathbb{X}^\circ$. If $|\Psi'|^q$ is $s-$type $m-$preinvex on $(\mu, m\mu + \zeta(\nu, \mu, m))$ for $p, q > 1, \frac{1}{q} + \frac{1}{p} = 1, m \in (0, 1]$ and $s \in [0, 1]$, then*

$$\left|\frac{\Psi(\mu) + \Psi(m\mu + \zeta(\frac{\nu}{c}, \mu, m))}{2} - \frac{c}{\zeta(\nu, c\mu, m)} \int_\mu^{m\mu+\zeta(\frac{\nu}{c},\mu,m)} \Psi(x)dx\right|$$
$$\leq \frac{\zeta(\nu, c\mu, m)}{2c}\left[\frac{1}{2(p+1)}\right]^{1/p}\left[\left\{\frac{3-2s}{6}|\Psi'(\mu)|^q + \frac{3-s}{6}m|\Psi'\left(\frac{\nu}{c}\right)|^q\right\}^{1/q} + \left\{\frac{3-s}{6}|\Psi'(\mu)|^q + \frac{3-2s}{6}m|\Psi'\left(\frac{\nu}{c}\right)|^q\right\}^{1/q}\right]. \tag{16}$$

Proof According to Lemma 1 and applying Hölder-Íscan inequality, one has

$$\left|\frac{\Psi(\mu) + \Psi(m\mu + \zeta(\frac{\nu}{c}, \mu, m))}{2} - \frac{c}{\zeta(\nu, c\mu, m)} \int_\mu^{m\mu+\zeta(\frac{\nu}{c},\mu,m)} \Psi(x)dx\right| \leq \frac{\zeta(\nu, c\mu, m)}{2c}$$
$$\times\left[\left(\int_0^1 (1-\delta)|1-2\delta|^p d\delta\right)^{1/p}\left(\int_0^1 (1-\delta)|\Psi'\left(\frac{m\nu}{c} + \delta\zeta(\mu, \frac{\nu}{c}, m)\right)|^q d\delta\right)^{1/q} + \left(\int_0^1 \delta|1-2\delta|^p d\delta\right)^{1/p}\left(\int_0^1 \delta|\Psi'\left(\frac{m\nu}{c} + \delta\zeta(\mu, \frac{\nu}{c}, m)\right)|^q d\delta\right)^{1/q}\right]. \tag{17}$$

Since $|\Psi'|^q$ is s-type $m-$preinvex on $(\mu, m\mu + \zeta(\nu, \mu, m))$, we have

$$\int_0^1 |\Psi'\left(\frac{m\nu}{c} + \delta\zeta(\mu, \frac{\nu}{c}, m)\right)|^q d\delta = |\Psi'(\mu)|^q \int_0^1 (1 - s(1-\delta))d\delta + m|\Psi'\left(\frac{\nu}{c}\right)|^q \int_0^1 (1 - s\delta)d\delta.$$

Now, equation (17) becomes

$$\left|\frac{\Psi(\mu) + \Psi(m\mu + \zeta(\frac{\nu}{c}, \mu, m))}{2} - \frac{c}{\zeta(\nu, c\mu, m)} \int_\mu^{m\mu+\zeta(\frac{\nu}{c},\mu,m)} \Psi(x)dx\right|$$
$$\leq \frac{\zeta(\nu, c\mu, m)}{2c}\left[\left(\int_0^1 (1-\delta)|1-2\delta|^p d\delta\right)^{1/p}\left(|\Psi'(\mu)|^q\int_0^1 (1-\delta)(1-s(1-\delta))d\delta \;+m|\Psi'\left(\frac{\nu}{c}\right)|^q\int_0^1 (1-\delta)(1-s\delta)d\delta\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1 \delta|1-2\delta|^p d\delta\right)^{1/p}\left(|\Psi'(\mu)|^q\int_0^1 \delta(1-s(1-\delta))d\delta + m|\Psi'\left(\frac{\nu}{c}\right)|^q\int_0^1 \delta(1-s\delta)d\delta\right)^{1/q}\right]. \tag{18}$$

Since,

$$\int_0^1 (1-\delta)\,(1-s\,(1-\delta))\,d\delta = \int_0^1 \delta\,(1-s\delta)\,d\delta = -\frac{2s-3}{6}$$
$$\int_0^1 \delta\,(1-s\,(1-\delta))\,d\delta = \int_0^1 (1-\delta)\,(1-s\delta)\,d\delta = -\frac{s-3}{6}$$
$$\int_0^1 \delta|1-2\delta|^p d\delta = \int_0^1 (1-\delta)|1-2\delta|^p d\delta = \left[\frac{1}{2(p+1)}\right].$$

Putting the values of the above computations in (18), then we obtain the required result. ■

Theorem 6 *Let $\mathbb{X}^\circ \subseteq \mathbb{R}$ be an open invex subset w.r.t $\zeta : \mathbb{X}^\circ \times \mathbb{X}^\circ \to \mathbb{R}$ and $\mu, \nu \in \mathbb{X}^\circ$ with $m\nu + \zeta(\mu, \nu, m) \leq \nu$. Suppose $\Psi : [m\nu + \zeta(\mu, \nu, m), \nu]$ be a differentiable mapping on $\mathbb{X}^\circ$. If $|\Psi'|^q$ is $s-$type $m-$preinvex on $(\mu, m\mu + \zeta(\nu, \mu, m))$ for $q \geq 1$, $m \in (0, 1]$ and $s \in [0, 1]$, then*

$$\left|\frac{\Psi(\mu) + \Psi(m\mu + \zeta(\frac{\nu}{c}, \mu, m))}{2} - \frac{c}{\zeta(\nu, c\mu, m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)} \Psi(x)dx\right| \leq \frac{\zeta(\nu, c\mu, m)}{2c}\left[\frac{1}{4}\right]^{1-1/q}$$
$$\times \left[\left\{\frac{4-3s}{16}|\Psi'(\mu)|^q + \frac{4-s}{16}m|\Psi'\left(\frac{\nu}{c}\right)|^q\right\}^{1/q} + \left\{\frac{4-s}{16}|\Psi'(\mu)|^q + \frac{4-3s}{16}m|\Psi'\left(\frac{\nu}{c}\right)|^q\right\}^{1/q}\right]. \tag{19}$$

Proof According to Lemma 1 and applying Improved power-mean inequality, one has

$$\left|\frac{\Psi(\mu) + \Psi(m\mu + \zeta(\frac{\nu}{c}, \mu, m))}{2} - \frac{c}{\zeta(\nu, c\mu, m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)} \Psi(x)dx\right|$$
$$\leq \frac{\zeta(\nu, c\mu, m)}{2c}\left[\left(\int_0^1 (1-\delta)|1-2\delta|d\delta\right)^{1-1/q}\left(\int_0^1 (1-\delta)|1-2\delta||\Psi'\left(\frac{m\nu}{c} + \delta\zeta(\mu, \frac{\nu}{c}, m)\right)|^q d\delta\right)^{1/q}\right.$$
$$\left. + \left(\int_0^1 \delta|1-2\delta|d\delta\right)^{1-1/q}\left(\int_0^1 \delta|1-2\delta||\Psi'\left(\frac{m\nu}{c} + \delta\zeta(\mu, \frac{\nu}{c}, m)\right)|^q d\delta\right)^{1/q}\right]. \tag{20}$$

Since $|\Psi'|^q$ is s-type $m-$preinvex on $(\mu, m\mu + \zeta(\nu, \mu, m))$, we have

$$\int_0^1 |\Psi'\left(\frac{m\nu}{c} + \delta\zeta(\mu, \frac{\nu}{c}, m)\right)|^q d\delta = |\Psi'(\mu)|^q \int_0^1 (1 - s(1-\delta))d\delta + m|\Psi'\left(\frac{\nu}{c}\right)|^q \int_0^1 (1 - s\delta)d\delta.$$

Now, equation (20) becomes

$$\left|\frac{\Psi(\mu) + \Psi(m\mu + \zeta(\frac{\nu}{c}, \mu, m))}{2} - \frac{c}{\zeta(\nu, c\mu, m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)} \Psi(x)dx\right|$$
$$\leq \frac{\zeta(\nu, c\mu, m)}{2c}\left[\left(\int_0^1 (1-\delta)|1-2\delta|d\delta\right)^{1-1/q}\left(|\Psi'(\mu)|^q \int_0^1 (1-\delta)|1-2\delta|(1-s(1-\delta))d\delta\right.\right.$$
$$\left. + m|\Psi'\left(\frac{\nu}{c}\right)|^q \int_0^1 (1-\delta)|1-2\delta|(1-s\delta)d\delta\right)^{1/q}$$
$$\left. + \left(\int_0^1 \delta|1-2\delta|d\delta\right)^{1-1/q}\left(|\Psi'(\mu)|^q \int_0^1 \delta|1-2\delta|(1-s(1-\delta))d\delta + m|\Psi'\left(\frac{\nu}{c}\right)|^q \int_0^1 \delta|1-2\delta|(1-s\delta)d\delta\right)^{1/q}\right]. \tag{21}$$

Since,

$$\int_0^1 (1-\delta)|1-2\delta|\,(1-s(1-\delta))\,d\delta = \int_0^1 \delta|1-2\delta|\,(1-s\delta)\,d\delta = -\frac{3s-4}{16}$$
$$\int_0^1 \delta|1-2\delta|\,(1-s(1-\delta))\,d\delta = \int_0^1 (1-\delta)|1-2\delta|\,(1-s\delta)\,d\delta = -\frac{s-4}{16}$$
$$\int_0^1 \delta|1-2\delta|d\delta = \int_0^1 (1-\delta)|1-2\delta|d\delta = \left[\frac{1}{4}\right]$$

Putting the values of the above computations in (21), then we obtain the required proof. ∎

Theorem 7 *Let $\mathbb{X}^\circ \subseteq \mathbb{R}$ be an open invex subset w.r.t $\zeta : \mathbb{X}^\circ \times \mathbb{A}^\circ \to \mathbb{R}$ and $\mu, \nu \in \mathbb{X}^\circ$ with $m\nu + \zeta(\mu, \nu, m) \leq \nu$. Suppose $\Psi : [m\nu + \zeta(\mu, \nu, m), \nu]$ be a differentiable mapping on $\mathbb{X}^\circ$. If $|\Psi'|^q$ is $s-$type $m-$preinvex on $(\mu, m\mu + \zeta(\nu, \mu, m))$ for $q \geq 1$, $m \in (0, 1]$ and $s \in [0, 1]$, then*

$$\left|\frac{\Psi(\mu) + \Psi(m\mu + \zeta(\frac{\nu}{c}, \mu, m))}{2} - \frac{c}{\zeta(\nu, c\mu, m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)} \Psi(x)dx\right| \leq \frac{\zeta(\nu, c\mu, m)}{2c}\left[\frac{1}{2}\right]^{1-\frac{1}{q}}\left\{\frac{2-s}{4}\left[|\Psi'(\mu)|^q + m|\Psi'\left(\frac{\nu}{c}\right)|^q\right]\right\}^{1/q}. \tag{22}$$

Proof According to Lemma 1 and applying Power-mean inequality, one has

$$\left|\frac{\Psi(\mu) + \Psi(m\mu + \zeta(\frac{\nu}{c}, \mu, m))}{2} - \frac{c}{\zeta(\nu, c\mu, m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)} \Psi(x)dx\right| \leq \frac{\zeta(\nu, c\mu, m)}{2c}\int_0^1 |1-2\delta||\Psi'\left(\frac{m\nu}{c} + \delta\zeta(\mu, \frac{\nu}{c}, m)\right)|d\delta$$
$$\leq \frac{\zeta(\nu, c\mu, m)}{2c}\left(\int_0^1 |1-2\delta|d\delta\right)^{1-1/q}\left(\int_0^1 |1-2\delta||\Psi'\left(\frac{m\nu}{c} + \delta\zeta(\mu, \frac{\nu}{c}, m)\right)|^q d\delta\right)^{1/q}. \tag{23}$$

Since $|\Psi'|^q$ is s-type $m-$preinvex on $(\mu, m\mu + \zeta(\nu, \mu, m))$, we have

$$\int_0^1 |\Psi'\left(\frac{m\nu}{c} + \delta\zeta(\mu, \frac{\nu}{c}, m)\right)|^q d\delta = |\Psi'(\mu)|^q \int_0^1 (1 - s(1-\delta))d\delta + m|\Psi'\left(\frac{\nu}{c}\right)|^q \int_0^1 (1 - s\delta)d\delta.$$

Now, equation (23) becomes

$$\left|\frac{\Psi(\mu)+\Psi(m\mu+\zeta(\frac{\nu}{c},\mu,m))}{2}-\frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{\mu+\zeta(\frac{m\nu}{c},\mu,m)}\Psi(x)dx\right|$$
$$\leq\frac{\zeta(\nu,c\mu,m)}{2c}\left(\int_0^1|1-2\delta|d\delta\right)^{1-1/q}\left(|\Psi'(\mu)|^q\int_0^1|1-2\delta|(1-s(1-\delta))d\delta\;+m|\Psi'\left(\frac{\nu}{c}\right)|^q\int_0^1|1-2\delta|(1-s\delta)d\delta\right)^{1/q}. \tag{24}$$

Since,

$$\int_0^1|1-2\delta|\,(1-s\,(1-\delta))\,d\delta=\int_0^1|1-2\delta|\,(1-s\delta)\,d\delta=-\frac{s-2}{4}$$
$$\int_0^1|1-2\delta|d\delta=\frac{1}{2}.$$

Putting the values of the above computations in (24), then we obtain the required proof. ∎

Theorem 8 *Let $\mathbb{X}^\circ\subseteq\mathbb{R}$ be an open invex subset w.r.t $\zeta:\mathbb{X}^\circ\times\mathbb{X}^\circ\to\mathbb{R}$ and $\mu,\nu\in\mathbb{X}^\circ$ with $m\nu+\zeta(\mu,\nu,m)\leq\nu$. Suppose $\Psi:[m\nu+\zeta(\mu,\nu,m),\nu]$ be a differentiable mapping on $\mathbb{X}^\circ$. If $|\Psi'|^q$ is s–type m–preinvex on $(\mu,m\mu+\zeta(\nu,\mu,m))$ for $p,q>1$, $\frac{1}{q}+\frac{1}{p}=1$, $m\in(0,1]$ and $s\in[0,1]$, then*

$$\left|\frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)}\Psi(x)dx-\Psi\left(\frac{2m\mu+\zeta(\nu,\mu,m)}{2c}\right)\right|$$
$$\leq\frac{\zeta(\nu,c\mu,m)}{c}\left[\left(\frac{1}{p+1}\right)^{1/p}\left\{\frac{2-s}{2}[|\Psi'(\mu)|^q+m|\Psi'(\tfrac{\nu}{c})|^q]\right\}^{1/q}+\left\{\frac{4-3s}{8}[|\Psi'(\mu)|^q+m|\Psi'(\tfrac{\nu}{c})|^q]\right\}^{1/q}\right]. \tag{25}$$

Proof According to Lemma 2 and applying Hölder's inequality, one has

$$\left|\frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)}\Psi(x)dx-\Psi\left(\frac{2m\mu+\zeta(\nu,\mu,m)}{2c}\right)\right|$$
$$=\frac{\zeta(\nu,c\mu,m)}{c}\left\{\int_0^1\delta\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)d\delta-\int_{1/2}^1\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)d\delta\right\}$$
$$\leq\frac{\zeta(\nu,c\mu,m)}{c}\left[\left(\int_0^1\delta^pd\delta\right)^{1/p}\left\{\int_0^1|\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)|^qd\delta\right\}^{1/q}+\left\{\int_{1/2}^1|\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)|^qd\delta\right\}^{1/q}\right]$$
$$\leq\frac{\zeta(\nu,c\mu,m)}{c}\left[\left(\frac{1}{p+1}\right)^{1/p}\left\{\int_0^1[1-s(1-\delta)]|\Psi'(\mu)|^qd\delta+m\int_0^1[1-s\delta]|\Psi'\left(\frac{\nu}{c}\right)|^qd\delta\right\}^{1/q}\right.$$
$$\left.+\left\{\int_{1/2}^1[1-s(1-\delta)]|\Psi'(\mu)|^qd\delta+m\int_{1/2}^1[1-s\delta]|\Psi'\left(\frac{\nu}{c}\right)|^qd\delta\right\}^{1/q}\right]$$
$$=\frac{\zeta(\nu,c\mu,m)}{c}\left[\left(\frac{1}{p+1}\right)^{1/p}\left\{\frac{2-s}{2}[|\Psi'(\mu)|^q+m|\Psi'(\tfrac{\nu}{c})|^q]\right\}^{1/q}+\left\{\frac{4-3s}{8}[|\Psi'(\mu)|^q+m|\Psi'(\tfrac{\nu}{c})|^q]\right\}^{1/q}\right],$$

which gives the proof. ∎

Theorem 9 *Let $\mathbb{X}^\circ\subseteq\mathbb{R}$ be an open invex subset w.r.t $\zeta:\mathbb{X}^\circ\times\mathbb{A}^\circ\to\mathbb{R}$ and $\mu,\nu\in\mathbb{X}^\circ$ with $m\nu+\zeta(\mu,\nu,m)\leq\nu$. Suppose $\Psi:[m\nu+\zeta(\mu,\nu,m),\nu]$ be a differentiable mapping on $\mathbb{X}^\circ$. If $|\Psi'|^q$ is s–type m–preinvex on $(\mu,m\mu+\zeta(\nu,\mu,m))$ for $q\geq1$, $m\in(0,1]$ and $s\in[0,1]$, then*

$$\left|\frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)}\Psi(x)dx-\Psi\left(\frac{2m\mu+\zeta(\nu,\mu,m)}{2c}\right)\right|$$
$$\leq\frac{\zeta(\nu,c\mu,m)}{c}\left[\left(\frac{1}{2}\right)^{1-1/q}\left\{\frac{3-s}{6}|\Psi'(\mu)|^q+\frac{3-2s}{6}m|\Psi'(\tfrac{\nu}{c})|^q\right\}^{1/q}+\left\{\frac{4-3s}{8}[|\Psi'(\mu)|^q+m|\Psi'(\tfrac{\nu}{c})|^q]\right\}^{1/q}\right]. \tag{26}$$

Proof From Lemma 2 and applying power-mean inequality, one has

$$\left|\frac{c}{\zeta(\nu,c\mu,m)}\int_{\mu}^{m\mu+\zeta(\frac{\nu}{c},\mu,m)}\Psi(x)dx-\Psi\left(\frac{2m\mu+\zeta(\nu,\mu,m)}{2c}\right)\right|$$
$$=\frac{\zeta(\nu,c\mu,m)}{c}\left\{\int_0^1\delta\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)d\delta-\int_{1/2}^1\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)d\delta\right\}$$
$$\leq\frac{\zeta(\nu,c\mu,m)}{c}\left[\left(\int_0^1\delta d\delta\right)^{1-1/q}\left\{\int_0^1\delta|\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)|^qd\delta\right\}^{1/q}+\left\{\int_{1/2}^1|\Psi'\left(\frac{m\nu}{c}+\delta\zeta(\mu,\frac{\nu}{c},m)\right)|^qd\delta\right\}^{1/q}\right]$$

$$\leq \frac{\zeta(\nu, c\mu, m)}{c} \times \left[\left(\frac{1}{2}\right)^{1-1/q} \left\{\int_0^1 \delta[1-s(1-\delta)]|\Psi'(\mu)|^q d\delta + m\int_0^1 \delta[1-s\delta]|\Psi'\left(\frac{\nu}{c}\right)|^q d\delta\right\}^{1/q}\right]$$
$$+\frac{\zeta(\nu, c\mu, m)}{c} \times \left\{\int_{1/2}^1 [1-s(1-\delta)]|\Psi'(\mu)|^q d\delta + m\int_{1/2}^1 [1-s\delta]|\Psi'\left(\frac{\nu}{c}\right)|^q d\delta\right\}^{1/q}$$
$$=\frac{\zeta(\nu, c\mu, m)}{c} \left[\left(\frac{1}{2}\right)^{1-1/q} \left\{\frac{3-s}{6}|\Psi'(\mu)|^q + \frac{3-2s}{6} m|\Psi'(\frac{\nu}{c})|^q\right\}^{1/q} + \left\{\frac{4-3s}{8}[|\Psi'(\mu)|^q + m|\Psi'(\frac{\nu}{c})|^q]\right\}^{1/q}\right],$$

which gives the required proof.

■

6 Conclusion

In this work, we showed and investigated a novel idea of preinvex function namely s–type m–preinvex function and the new sort of Hermite–Hadamard type inequality via newly introduced definition are examined. Further, our attaining results in the order of lemma can be considered as refinements and remarkable extensions to the new family of preinvex functions. In the future, we hope the results of this paper and the new idea can be extended in different directions like fractional calculus, quantum calculus, time scale calculus, etc. We hope the consequences and techniques of this article will energize and inspire the researcher to explore a more interesting sequel in this area.

Declarations

Consent for publication

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Conflicts of interest

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Author's contributions

M.T.: Conceptualization, Methodology, Software, Writing- Reviewing and Editing. H.A.: Supervision, Investigation, Data Curation, Writing-Original draft preparation. S.K.S.: Visualization, Investigation, Methodology, Software, Writing- Reviewing and Editing. A.A.S.: Conceptualization, Methodology, Software, Supervision, Investigation, Data Curation. B.K.: Methodology, Software, Supervision, Investigation. D.K.: Conceptualization, Methodology, Investigation, Data Curation. All authors discussed the results, approved and contributed to the final manuscript.

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