Mathematical Modelling and Numerical Simulation with Applications, 2022, 2(3), 127–146

https://www.mmnsa.org
ISSN Online: 2791-8564 / Open Access
https://doi.org/10.53391/mmnsa.2022.011

RESEARCH PAPER

An efficient application of scrambled response approach to estimate the population mean of the sensitive variables

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Abstract

In the presence of one auxiliary variable and two auxiliary variables, we analyze various exponential estimators. The ranks of the auxiliary variables are also connected with the study variables, and there is a linkage between the study variables and the auxiliary variables. These ranks can be used to improve an estimator's accuracy. The Optional Randomized Response Technique (ORRT) and the Quantitative Randomized Response Technique are two techniques we utilize to estimate the sensitive variables from the population mean (QRRT). We used the scrambled response technique and checked the proposed estimators up to the first-order of approximation. The mean square error (MSE) equations are obtained for all the proposed ratio exponential estimators and show that our proposed exponential type estimator is more efficient than ratio estimators. The expression of mean square error is obtained up to the first degree of approximation. The empirical and theoretical comparison of the proposed estimators with existing estimators is also be carried out. We have shown that the proposed optional randomized response technique and quantitative randomized response model are always better than existing estimators. The simulation study is also carried out to determine the performance of the estimators. Few real-life data sets are also be applied in support of proposed estimators. It is observed that our suggested estimator is more efficient as compared to an existing estimator.

Key words: Randomized response technique; simple random sampling; scrambling response; sensitive and non-sensitive variables; exponential-type estimators
AMS 2020 Classification: 62D05; 62D10; 62E10; 62E17

1 Introduction

In the optional randomized response technique, while collecting data, sometimes the interviewer faces the problem of non-response. The interviewee hesitates to respond to sensitive questions regarding their private life, abortion, drug addiction, HIV infection status, duration of suffering from AIDS, sexual behavior, the incidence of domestic violence, and tax evasion, for example,

➤ **Received:** 25.05.2022 ➤ **Revised:** 18.07.2022 ➤ **Accepted:** 27.07.2022 ➤ **Published:** 10.08.2022

any sensitive query. In general, people do not feel comfortable when asked about their past medication use status relative to medical problems, sexual activity, premature births, etc. [1] proposed an optional randomized response model to manage those cases. This model elaborates that one query can be delicate and may not be sensitive to another. In this technique, the interviewee can choose whether to provide the answer or scramble the answer. Under this model, the mean and sensitivity level of parameters are estimated. We cannot trace the reaction provided by the interviewee through investigation. The story of sensitivity of the question is the proportion of individuals who give the scrambled answers. Ratio-type estimators are developed when a variable of concern is sensitive and auxiliary variables are non-sensitive information. Additional information is used to increase the estimator's precision at designing and at the estimation stages. We use ratio, product, and regression estimators more frequently when there is a connection between the dependent and independent variables. In a similar expression, the variables of independent ranks are also correlated with the relating upsides of the study variables [2]. Along these lines, the ranked auxiliary variable (that contains the ranks of the auxiliary variable) can be treated as another additional variable. This data may help us expand the proficiency of an estimator. In ORRT, most of the respondents assumed that the aspects of inquiry are sensitive, but some are more willing to answer directly. In ORT, respondents are given an option either to supply RR using a specified randomized device or to respond directly according to the extent to which the respondent feels that the question is sensitive or not. Most of the methods developed for ORT are limited to SRSWR sampling only. A few of the ORT techniques are available for complex surveys.

We are investigating a multiplicative randomized response strategy for providing quantitative randomized responses (QRRT) to sensitive queries. The respondent multiplies his sensitive response by a random number from a known distribution and gives the product to the interviewer, who has no idea what the random number is valuable and consequently receives a scrambled response. The respondent generates S using some specified method and multiplies his sensitive answer Y by S.

The interviewer receives the scrambled answer Z = YS. The particular values S are unknown to the interviewer, but its distribution is known, Let Scrambling variable is denoted by S with $E(S) = \theta_1$ and their variance is $V(S) = \sigma_1^2$. Some specific distributions for random scrambling numbers are proposed and investigated, as well as methods for creating scrambling numbers.

The application of the proposed Quantitative randomized response technique (QRRT) has also been discussed. A new composite class of estimators is defined using scrambled response to estimate the population means of a sensitive variable. Methods for studying sensitive behavior include randomized response techniques, which provide anonymity to interviewees who answer sensitive questions. The quantitative randomized response technique (QRRT) variation on this approach allows researchers to estimate the frequency or quantity of sensitive behaviors. The application of the proposed optional Randomized Response Technique has been discussed. The randomized response technique in a survey reduces potential bias due to nonresponse and social desirability when asking questions about sensitive behaviors and beliefs [3]. Use of randomization device (outcome unobserved by the interviewer) conceals individual responses and protects respondent privacy Auxiliary variables are first used in a ratio-type estimator by [4]. The use of more than one auxiliary characteristic improves the Estimation. To know about the variability present in a finite population variance may be required, which is also essential for future predictions and studies. Therefore, we review different estimators in the literature and propose a new class of estimators. We have some auxiliary information that is used in variance estimation. We are interested in comparing the different variance estimators. We sought to recommend a variance estimator for use in the analysis of the content evaluation survey. There are various ways and examples of the use of assisting (auxiliary) variables like

- A hospital survey may identify insufficient quantity in a specific hospital.
- In Socioeconomic surveys, in advance, may well know the availability of food, educational status, and medical facilities of a region.
- The entirely cultivated area in the agriculture production survey may well be known in advance.

In this paper, generalized two-stage optional randomized response technique (ORRT) and Quantitative randomized response technique are derived for a finite population mean of a delicate variable based on Randomized Response technique using non-sensitivity additional information.

2 Few existing estimators in simple random sampling

Now we discuss MSE estimators which exist in the literature. Firstly, note the simple sample MSE estimator.

i) The unbiased usual estimator of the population mean of Z is given

$$MSE(\hat{\mu}_{YS}) = \left(\frac{1-f}{n}\right)\bar{Z}^2[\sigma_y^2 + \sigma_s^2]. \tag{1}$$

ii) [5] proposed a ratio estimator, which is given as

$$MSE(\hat{\mu}_{RS}) = \left(\frac{1-f}{n}\right)\bar{Z}^2[C_y^2 + \frac{\sigma_s^2}{\mu_y^2} + C_x^2 - 2C_xC_y\rho_{yx}]. \tag{2}$$

iii) [6] suggested a ratio estimator, which is given as

$$MSE(\hat{\mu}_{RG}) = \bar{Z}^2\left(\frac{1-f}{n}\right)[C_y^2 + \frac{W\sigma_s^2}{\mu_y^2} + C_x^2 - 2C_xC_y\rho_{yx}]. \tag{3}$$

iv) [7] suggested a ratio estimator.

$$MSE(\hat{\mu}_{RN}) = \bar{Z}^2\left(\frac{1-f}{n}\right)[C_y^2 + \frac{WK^2\sigma_s^2}{\mu_y^2} + C_x^2 - 2C_xC_y\rho_{yx}]. \tag{4}$$

v) [8] proposed the exponential-type estimator created on generalized two-stage optional-scrambled reply method which is given as

$$MSE(\hat{t}_{GRR}) = \bar{Z}^2\left(\frac{1-f}{n}\right)[C_y^2 + \frac{K^2W(1-T)^2\sigma_s^2}{\mu_y^2} - C_y^2\rho^2{}_{yx}]. \tag{5}$$

3 Proposed model I

Mean estimator for generalized two-stage optional scramble response

Let the set proportion of the population is denoted by $T(0 \leq T \leq 1)$, for which we can assume to give the true responses along with $W(0 \leq W \leq 1)$ to be the Sensitivity level related to that sensitive question. Here we have a scrambling variable S with zero (0) mean and variance σ_S^2 and let $K(-1 \leq K \leq 1)$ is suitably chosen scalar. [8] proposed the ORRT for the estimation of population mean in the case of sensitive study variables. Their suggested scenario states: Let the sensitive study variable be denoted by Y having the population mean μ_y and unknown population variance σ_y^2. The ORRT can be written as

$$Z = \left\{ \begin{array}{ll} Y & \text{with probability T + (1 - W)(1 - T)} \\ Y + KS & \text{with probability } W(1-T) \end{array} \right\}, \tag{6}$$

$$\begin{aligned} E(Z) &= [T + (1-W)(1-T)E(Y) + [W(1-T)E(Y+KS)], \\ E(Z) &= \mu_{YM}. \end{aligned}$$

Generalized exponential-ratio-type estimator using one-auxiliary variable for generalized two stages optional scramble response

Let a simple random sample without replacement of size n be drawn from the population consisting of N units. Let Z, Y and X be the optional randomized response variable, the study variable, and the auxiliary variable respectively. Let the population (sample) means of Z, Y and X are μ_Z, μ_Y and μ_X ($\bar{z}$, $\bar{y}$ and $\bar{x}$) symbols, respectively, and notations to be used are:

$$\begin{aligned} E(Z) &= \mu_Z, \mu_Z, \mu_Y \text{ and } \mu_X, E(U) = \mu_U, E(X) = \mu_X, E(S) = \mu_S = 0, \\ E(e_0) &= E(e_1) = E(e_2) = E(e_3) = 0, \bar{r}_x = \bar{R}_x(1+e_2), \bar{x} = \bar{X}(1+e_1), \ \bar{y} = \bar{Y}(1+e_0), \\ E(e_3^2) &= \lambda C_z^2, E(e_0e_1) = \lambda\rho_{yx}C_yC_x, \ E(e_0e_2) = \lambda\rho_{yr_x}C_yC_r, \ E(e_0e_3) = \lambda\rho_{yz}C_yC_z, \\ E(e_0e_3) &= \lambda\rho_{yz}C_yC_z, \ E(e_1e_2) = \lambda\rho_{xr_x}C_xC_r, \ ]E(e_1e_3) = \lambda\rho_{xz}C_xC_z, \ E(e_3e_2) = \lambda\rho_{rx_z}C_xC_r, \\ C_z^2 &= C_y^2 + \frac{K^2W(1-T)\sigma_S^2}{\mu_y^2}, \ \rho_{zx} = \frac{\rho_{yx}}{\sqrt{1+\frac{K^2W(1-T)\sigma_S^2}{\sigma_y^2}}}, \ \rho_{zr_x} = \frac{cov(x,y)}{C_{rx}C_y\sqrt{1+\frac{K^2W(1-T)\sigma_S^2}{\mu_y^2\sigma_y^2}}}, \ \theta = \frac{a\bar{X}}{a\bar{X}+b}. \end{aligned}$$

Table 1. Special case for the proposed estimator

Estimator	a	b
$\hat{\hat{Z}}_{pr}^{(1)}$	1	C_x
$\hat{\hat{Z}}_{pr}^{(2)}$	1	$\beta_2(x)$
$\hat{\hat{Z}}_{pr}^{(3)}$	$\beta_2(x)$	C_x
$\hat{\hat{Z}}_{pr}^{(4)}$	C_x	$\beta_2(x)$
$\hat{\hat{Z}}_{pr}^{(5)}$	1	ρ_{xy}
$\hat{\hat{Z}}_{pr}^{(6)}$	C_x	ρ_{xy}
$\hat{\hat{Z}}_{pr}^{(7)}$	ρ_{xy}	C_x
$\hat{\hat{Z}}_{pr}^{(8)}$	$\beta_2(x)$	ρ_{xy}
$\hat{\hat{Z}}_{pr}^{(9)}$	ρ_{xy}	$\beta_2(x)$

Motivated by [9], [10], and [11] proposed a difference- ratio-type exponential an estimator $\hat{\hat{Z}}_{pr}$ that is given by

$$\hat{\hat{Z}}_{pr} = \{\omega_1\bar{z} + \omega_2(\bar{X} - \bar{x}) + \omega_3(\bar{R}_x - \bar{r}_x)\}\exp\left(\frac{a(\bar{X}-\bar{x})}{a(\bar{X}-\bar{x}) + 2b)}\right). \tag{7}$$

By taking expectation, we can find the bias $\hat{\hat{Z}}_{pr}$ under first approximation, which is given as

$$Bias(\hat{\hat{Z}}_{pr}) = \frac{1}{8}\Big[-8\bar{Z} + 4\lambda\theta C_x(\bar{X}C_x\omega_2 + \bar{R}_x\rho_{xr_x}C_r\omega_3\rho_{xr_x}) + \bar{Z}\omega_1(8 + \theta\lambda C_x (3\theta C_x - 4C_y\rho_{yx}))\Big]. \tag{8}$$

The MSE of $\hat{\hat{Z}}_{pr}$under the first order of approximation are respectively given

$$\begin{aligned} MSE(\hat{\hat{Z}}_{pr}) &= \bar{Z}^2 + \lambda\bar{X}C_x^2\omega_2(-\bar{Z} + \bar{X}\omega_2) + \lambda\bar{R}_x^2C_r^2\omega_3^2 + \lambda\bar{R}_xC_xC_r(-\bar{Z} + 2\bar{X}\omega_2)\omega_3\rho_{xr_x} \\ &+ \bar{Z}^2\omega_1^2\left[1 + \lambda\left\{C_y^2 + C_x - 2C_y\rho_{yx}\right\}\right] + \frac{1}{4}\bar{Z}\omega_1\Big[-8\bar{Z} + \lambda C_x\Big\{C_x(-3\bar{Z}\theta + 8\bar{X}\omega_2) + \\ & 8\bar{R}_xC_r\omega_3\rho_{xr_x} + 4C_y(\bar{Z} - 2\bar{X}\omega_2)\rho_{xy}\Big\} - 8\bar{R}_x\lambda C_yC_r\omega_3\rho_{yr_x}\Big]. \end{aligned} \tag{9}$$

The optimum values of ω_1, ω_2, and ω_3, obtained by minimizing equation (9) respectively, given by

$$\omega_{1(opt)} = \frac{8 - \lambda\theta^2 2C_x^2}{8\left(1 + \lambda C_z^2\left(1 - Q^2_{z.xrx}\right)\right)},$$

$$\omega_{2(opt)} = \frac{\bar{Z}\left[\lambda\theta^3C_x^3\left(-1 + \rho^2_{xr_x}\right) + \left(-8C_z + \lambda\theta^2 2C_x^2C_z\right)\left(\rho_{zx} - \rho_{xr_x}\rho_{zr_x}\right) + 4\theta C_x\left(-1 + \rho^2_{xr_x}\right)\left(-1 + \lambda C_z^2\left(1 - Q^2_{z.xr_x}\right)\right)\right]}{8\bar{X}C_x\left(-1 + \rho^2_{xr_x}\right)\left(1 + \lambda C_z^2\left(1 - Q^2_{z.xr_x}\right)\right)},$$

and

$$\omega_{3(opt)} = \frac{\bar{Z}\left(8 - \lambda\theta^2C_x^2\right)C_z\left(\rho_{xr_x}\rho_{zx} - \rho_{zr_x}\right)}{8\bar{R}_xC_r\left(-1 + \rho^2_{xr_x}\right)\left(1 + \lambda C_z^2\left(1 - Q^2_{z.xr_x}\right)\right)},$$

$$Q^2_{z.xr_x} = \frac{\rho^2_{zx} + \rho^2_{zr_x} - 2\rho_{zx}\rho_{zr_x}\rho_{xr_x}}{1 - \rho^2_{xr_x}}.$$

Substitute the value of $\omega_1, \omega_2, \omega_3$ in equation (9), and we get the minimum MSE given by

$$MSE_{\min}(\hat{\hat{Z}}_{pr}) = \frac{\lambda\bar{Z}^2\left\{64C_y^2(1 - Q^2_{z.xr_x}) - \lambda\theta^4C_x^4C - 16\lambda\theta^2C_x^2C_y^2(1 - Q^2_{z.xr_x})\right\}}{64\left\{1 + \lambda C_y^2(1 - Q^2_{z.xr_x})\right\}}. \tag{10}$$

Efficiency comparison

We present the mathematical comparison of the proposed estimator with existing estimators under Model-I as

i) By equations (5) and (10)

$$MSE_{\min}(\hat{\hat{Z}}_{pr}) \leq MSE(\hat{t}_{GRR}).$$

ii) By equations (2) and (10)

$$MSE_{\min}(\hat{\hat{Z}}_{pr}) \leq MSE(\hat{\mu}_{RS}).$$

iii) By equations (4) and (10)

$$MSE_{\min}(\hat{\hat{Z}}_{pr}) \leq MSE(\hat{\mu}_{RN}).$$

iv) By equations (3) and (10)

$$MSE_{\min}(\hat{\hat{Z}}_{pr}) \leq MSE(\hat{\mu}_{RG}).$$

Real-life conifer tree data set 1

Real-life data set is used for numerical comparison. Detail is given as: Population Source: [12]. Let z be our study variable used in our estimator and model. We study the total height of the conifer tree. x is our auxiliary variable which is non-sensitive. We measure the circumference of the conifer tree at breast height. We assume three samples in our simulation study, $n = 100$, $n = 200$, and $n = 300$.

z: Total height of a conifer tree in feet;
x: Circumference of a conifer tree at breast height in cm
N=399, ρ_{XY} = 0.914981, ρ_{zr_x} = 0.983609, ρ_{xr_x} = 0.890219, μ_X = 285.125, μ_Y = 5182.64, σ_X = 310.1403, σ_Z = 3250.5050, C_z = 0.354194, C_x = 0.948459, C_r = 0.573765

Table 2. The MSE and PRE values of estimators for real-life data set 1

n	W	T	Estimator	Theoretical	PRE
50	0.3	0.3	$\hat{\mu}_{YS}$	55.909	100.00
			$\hat{\mu}_{RS}$	54.303	102.95
			$\hat{\mu}_{RG}$	10.708	522.11
			$\hat{\mu}_{RN}$	55.258	101.17
			$\hat{t}_{GRR}$	9.7486	573.51
			$\hat{Z}_{pr}^{(1)}$	5.8744	951.74
			$\hat{Z}_{pr}^{(2)}$	5.8344	958.275
			$\hat{Z}_{pr}^{(3)}$	5.781	967.11
			$\hat{Z}_{pr}^{(4)}$	5.8344	958.27
			$\hat{Z}_{pr}^{(5)}$	5.8135	961.72
			$\hat{Z}_{pr}^{(6)}$	5.8135	961.72
			$\hat{Z}_{pr}^{(7)}$	5.7810	967.11
			$\hat{Z}_{pr}^{(8)}$	5.8135	961.72
			$\hat{Z}_{pr}^{(9)}$	5.8344	958.27
	0.5	0.5	$\hat{\mu}_{YS}$	56.033	100.00
			$\hat{\mu}_{RS}$	54.303	103.18
			$\hat{\mu}_{RG}$	10.718	522.75
			$\hat{\mu}_{RN}$	55.259	101.40
			$\hat{t}_{GRR}$	9.7507	574.66
			$\hat{Z}_{pr}^{(k)*}$	5.8744	953.85
	0.7	0.7	$\hat{\mu}_{YS}$	55.909	100.00
			$\hat{\mu}_{RS}$	54.303	102.95
			$\hat{\mu}_{RG}$	10.729	521.08
			$\hat{\mu}_{RN}$	55.259	101.17
			$\hat{t}_{GRR}$	9.7486	573.51
			$\hat{Z}_{pr}^{(k)*}$	5.8744	951.74
100	0.3	0.3	$\hat{\mu}_{YS}$	23.404	100.00
			$\hat{\mu}_{RS}$	21.798	107.36
			$\hat{\mu}_{RG}$	24.409	95.88
			$\hat{\mu}_{RN}$	22.7538	102.86
			$\hat{t}_{GRR}$	24.0141	97.459
			$\hat{Z}_{pr}^{(1)}$	20.516	114.07
			$\hat{Z}_{pr}^{(2)}$	21.841	107.15
			$\hat{Z}_{pr}^{(3)}$	20.943	111.75
			$\hat{Z}_{pr}^{(4)}$	21.811	107.15
			$\hat{Z}_{pr}^{(5)}$	21.49	108.81
			$\hat{Z}_{pr}^{(6)}$	21.48	108.91
			$\hat{Z}_{pr}^{(7)}$	20.43	111.75
			$\hat{Z}_{pr}^{(8)}$	21.89	108.91
			$\hat{Z}_{pr}^{(9)}$	21.841	107.15
	0.5	0.5	$\hat{\mu}_{YS}$	23.528	100.00
			$\hat{\mu}_{RS}$	21.798	107.93
			$\hat{\mu}_{RG}$	22.754	103.41
			$\hat{\mu}_{RN}$	21.754	108.41
			$\hat{t}_{GRR}$	25.015	94.072
			$\hat{Z}_{pr}^{(k)*}$	20.516	114.68
	0.7	0.7	$\hat{\mu}_{YS}$	23.528	100.00
			$\hat{\mu}_{RS}$	21.7984	107.93
			$\hat{\mu}_{RG}$	22.7540	103.41
			$\hat{\mu}_{RN}$	22.754	103.41
			$\hat{t}_{GRR}$	24.015	97.972
			$\hat{Z}_{pr}^{(k)*}$	20.516	114.68
150	0.3	0.3	$\hat{\mu}_{YS}$	13.853	100.00
			$\hat{\mu}_{RS}$	12.046	113.33
			$\hat{\mu}_{RG}$	2.5196	541.88

		$\hat{\mu}_{RN}$	13.002	105.00
		$\hat{t}_{GRR}$	13.779	99.082
		$\hat{Z}_{pr}^{(1)}$	2.2938	595.22
		$\hat{Z}_{pr}^{(2)}$	2.1938	631.22
		$\hat{Z}_{pr}^{(3)}$	2.3938	578.70
		$\hat{Z}_{pr}^{(4)}$	2.4938	555.22
		$\hat{Z}_{pr}^{(5)}$	2.4938	595.22
		$\hat{Z}_{pr}^{(6)}$	1.4938	595.22
		$\hat{Z}_{pr}^{(7)}$	4.4938	312.22
		$\hat{Z}_{pr}^{(8)}$	3.4938	396.22
		$\hat{Z}_{pr}^{(9)}$	2.0938	595.22
0.5	0.5	$\hat{\mu}_{YS}$	13.777	100.00
		$\hat{\mu}_{RS}$	12.046	114.36
		$\hat{\mu}_{RG}$	2.5220	546.26
		$\hat{\mu}_{RN}$	13.002	105.95
		$\hat{t}_{GRR}$	13.778	99.98
		$\hat{Z}_{pr}^{(k)*}$	2.2942	600.50
0.7	0.7	$\hat{\mu}_{YS}$	13.777	100.00
		$\hat{\mu}_{RS}$	13.046	105.60
		$\hat{\mu}_{RG}$	2.5220	546.26
		$\hat{\mu}_{RN}$	13.002	105.95
		$\hat{t}_{GRR}$	13.278	103.75
		$\hat{Z}_{pr}^{(k)*}$	2.2942	600.50

Using Table 2, we find the percentage relative efficiency of existing and proposed estimators, shown in the graph. Black boxes indicate the existing estimator, and red boxes indicate the proposed estimator. This graph shows our proposed estimator is more efficient compared to the existing estimator.

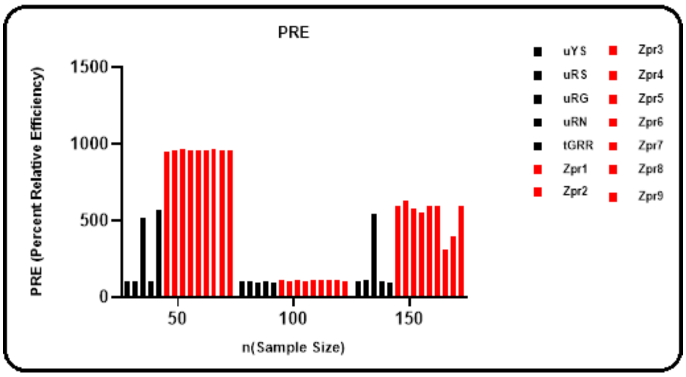

Figure 1. PRE of Real-Life data set 1

Percentage relative efficiency

We have computed the percentage relative efficiency of proposed estimators at different combinations of existing estimators. We used the following expression to obtain the percentage relative performance. Results are presented above tables.

$$PRE = \frac{MSE(\hat{\mu}_{YS})}{MSE(\hat{\mu}_{Q})} * 100, \tag{11}$$

where $Q = RS, RG, NR, GRR$.

Simulation study

We consider a multivariate normal population with multiple covariance matrices to represent (Y, X) distribution. The normal distribution is followed by scrambling variables. The standard deviation is equal to 100 percent of the standard deviation of X when the mean is 0. We generate two populations. For population 1, N is 1000 and mean is 2. For population 2, N is 1000 and mean is 3. Correlation is always positive for both populations. After Simulation, we standardize the study variable and auxiliary variable and rank of the auxiliary variable. The Simulation is conducted to evaluate the performance of the proposed estimator. For this study, we have generated the population size N=1000 from a standard normal distribution using the MVRNORM package in software R, where study and auxiliary variables are correlated with a correlation given below. The whole simulation process starting from the drawing sample from variable Y and auxiliary variable X from the normal population and calculating the estimates, was repeated 10000 times. The reported generalized response is calculated using the formula $Z = Y + KS$. The covariance matrices are given by

Table 3. Data Summary I

Population 1	**Population 2**
N = 1000	N = 1000
μ= [2,2]	μ= [3,3]
$\Sigma = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$	$\Sigma = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$
ρ_{yx}=0.37144	ρ_{yx}=0.388517

Table 4. The MSE and PRE values of estimators for Population 1&2

				Population 1			**Population 2**		
n	W	T	Estimator	Theoretical	Empirical	PRE	Theoretical	Empirical	PRE
200	0.3	0.3	$\hat{\mu}_{YS}$	0.01600	0.01697	100.00	0.03194	0.01425	100.00
			$\hat{\mu}_{RS}$	0.05133	0.01770	31.17	0.03160	0.01425	175.60
			$\hat{\mu}_{RG}$	0.01610	0.01679	101.28	0.02347	0.01527	136.07
			$\hat{\mu}_{RN}$	0.05065	0.01787	31.58	0.09302	0.01439	210.07
			$\hat{t}_{GRR}$	0.00700	0.02390	233.44	0.01988	0.16697	160.68
			$\hat{Z}_{pr}^{(1)}$	0.00470	0.00018	340.4	0.01815	0.01130	242.60
			$\hat{Z}_{pr}^{(2)}$	0.00446	0.00058	366.78	0.0178	0.01190	179.07
			$\hat{Z}_{pr}^{(3)}$	0.00400	0.00020	409.17	0.018	0.00720	190.02
			$\hat{Z}_{pr}^{(4)}$	0.00447	0.00089	366.78	0.01425	0.00570	224.14
			$\hat{Z}_{pr}^{(5)}$	0.00429	0.00039	381.63	0.01418	0.01000	225.24
			$\hat{Z}_{pr}^{(6)}$	0.00429	0.00105	381.63	0.01425	0.00470	224.91
			$\hat{Z}_{pr}^{(7)}$	0.00400	0.00110	409.17	0.01381	0.00023	231.04
			$\hat{Z}_{pr}^{(8)}$	0.00429	0.00065	381.63	0.01417	0.01010	225.45
			$\hat{Z}_{pr}^{(9)}$	0.00447	0.00066	366.78	0.01426	0.00070	223.91
	0.5	0.5	$\hat{\mu}_{YS}$	0.01700	0.01697	100.00	0.03346	0.01425	100.00
			$\hat{\mu}_{RS}$	0.05133	0.01770	31.49	0.01317	0.01425	107.98
			$\hat{\mu}_{RG}$	0.01610	0.01679	111.07	0.02112	0.01527	142.34
			$\hat{\mu}_{RN}$	0.05084	0.01787	31.53	0.02350	0.01439	200.88
			$\hat{t}_{GRR}$	0.03240	0.02390	233.44	0.01988	0.16697	168.25
			$\hat{Z}_{pr}^{(k)*}$	0.00700	0.00067	255.44	0.01815	0.00980	242.60
	0.7	0.7	$\hat{\mu}_{YS}$	0.01600	0.01697	100.00	0.03194	0.01425	100.00
			$\hat{\mu}_{RS}$	0.05133	0.01770	31.91	0.01317	0.01425	107.98
			$\hat{\mu}_{RG}$	0.01620	0.01679	101.28	0.02354	0.01527	135.72
			$\hat{\mu}_{RN}$	0.05103	0.01787	31.23	0.02112	0.01439	142.34
			$\hat{t}_{GRR}$	0.00700	0.02390	233.44	0.01988	0.16697	160.68
			$\hat{Z}_{pr}^{(k)*}$	0.00470	0.00046	340.4	0.01815	0.00136	242.60
500	0.3	0.3	$\hat{\mu}_{YS}$	0.01030	0.00209	100.00	0.01395	0.00615	100.00
			$\hat{\mu}_{RS}$	0.50725	0.00419	2.036	0.00482	0.00378	289.56
			$\hat{\mu}_{RG}$	0.00404	0.00427	255.19	0.00586	0.00420	237.81
			$\hat{\mu}_{RN}$	0.05004	0.00406	20.64	0.00768	0.00377	181.61
			$\hat{t}_{GRR}$	0.00175	0.02391	588.90	0.00497	0.16688	280.82
			$\hat{Z}_{pr}^{(1)}$	0.00112	0.00021	925.80	0.00455	0.00029	306.99
			$\hat{Z}_{pr}^{(2)}$	0.00117	0.00023	885.61	0.00427	0.00050	326.51
			$\hat{Z}_{pr}^{(3)}$	0.00116	0.00014	885.61	0.00430	0.00094	324.41
			$\hat{Z}_{pr}^{(4)}$	0.00115	0.00047	893.95	0.00427	0.00017	326.51
			$\hat{Z}_{pr}^{(5)}$	0.00115	0.00011	893.95	0.00430	0.00650	324.41
			$\hat{Z}_{pr}^{(6)}$	0.00114	0.00018	908.15	0.00430	0.00017	324.78
			$\hat{Z}_{pr}^{(7)}$	0.00115	0.00047	893.95	0.00427	0.00371	326.51
			$\hat{Z}_{pr}^{(8)}$	0.00114	0.00013	881.63	0.00430	0.00022	324.78
			$\hat{Z}_{pr}^{(9)}$	0.00116	0.00016	885.61	0.00430	0.00530	324.41
	0.5	0.5	$\hat{\mu}_{YS}$	0.01191	0.00209	100.00	0.01547	0.00615	100.00
			$\hat{\mu}_{RS}$	0.05072	0.00419	20.30	0.00482	0.00378	321.03

			$\hat{\mu}_{RG}$	0.00406	0.00427	293.74	0.00588	0.00420	263.30
			$\hat{\mu}_{RN}$	0.05024	0.00406	20.71	0.00687	0.00377	225.35
			$\hat{t}_{GRR}$	0.00176	0.02391	678.57	0.00497	0.16688	311.23
			$\hat{Z}_{pr}^{(k)*}$	0.00112	0.00017	1067.7	0.00455	0.00047	340.35
	0.7	0.7	$\hat{\mu}_{YS}$	0.01033	0.00209	100.00	0.01395	0.00615	100.00
			$\hat{\mu}_{RS}$	0.50725	0.00419	2.0365	0.00482	0.00378	289.56
			$\hat{\mu}_{RG}$	0.00406	0.00427	254.20	0.00588	0.00420	237.19
			$\hat{\mu}_{RN}$	0.05043	0.00406	20.48	0.00604	0.00377	230.77
			$\hat{t}_{GRR}$	0.00175	0.02391	588.90	0.00497	0.16688	280.82
			$\hat{Z}_{pr}^{(k)*}$	0.00112	0.00047	925.80	0.00454	0.00064	306.99
700	0.3	0.3	$\hat{\mu}_{YS}$	0.00912	0.00086	100.00	0.01395	0.00239	100.00
			$\hat{\mu}_{RS}$	0.00206	0.00175	442.02	0.00842	0.00164	165.81
			$\hat{\mu}_{RG}$	0.00162	0.00178	563.27	0.00235	0.00170	594.52
			$\hat{\mu}_{RN}$	0.00499	0.00166	182.26	0.01128	0.00171	123.70
			$\hat{t}_{GRR}$	0.00070	0.02388	1299.83	0.00298	0.16685	468.05
			$\hat{Z}_{pr}^{(1)}$	0.00047	0.00013	1957.36	0.00181	0.00050	767.23
			$\hat{Z}_{pr}^{(2)}$	0.00047	0.00017	1938.1	0.00178	0.00044	583.42
			$\hat{Z}_{pr}^{(3)}$	0.00046	0.00010	1957.36	0.00178	0.00030	581.97
			$\hat{Z}_{pr}^{(4)}$	0.00047	0.00060	1938.1	0.00178	0.00038	583.42
			$\hat{Z}_{pr}^{(5)}$	0.00047	0.00014	1945.2	0.00178	0.00022	581.97
			$\hat{Z}_{pr}^{(6)}$	0.00046	0.00067	1945.29	0.00178	0.00350	582.23
			$\hat{Z}_{pr}^{(7)}$	0.00046	0.00021	1957.36	0.00177	0.00062	582.23
			$\hat{Z}_{pr}^{(8)}$	0.00046	0.00031	1945.29	0.00178	0.00120	583.42
			$\hat{Z}_{pr}^{(9)}$	0.00047	0.00032	1938.1	0.00178	0.00010	581.97
	0.5	0.5	$\hat{\mu}_{YS}$	0.01070	0.00086	100.00	0.01547	0.00239	100.00
			$\hat{\mu}_{RS}$	0.00506	0.00175	214.15	0.00842	0.00164	183.82
			$\hat{\mu}_{RG}$	0.00162	0.00178	659.78	0.00235	0.00170	658.27
			$\hat{\mu}_{RN}$	0.00501	0.00166	213.35	0.01046	0.00171	147.87
			$\hat{t}_{GRR}$	0.00070	0.02388	1524.14	0.00199	0.16685	778.09
			$\hat{Z}_{pr}^{(k)*}$	0.00047	0.00016	2297.22	0.00182	0.00085	850.59
	0.7	0.7	$\hat{\mu}_{YS}$	0.00912	0.00086	100.00	0.01395	0.00239	100.00
			$\hat{\mu}_{RS}$	0.00506	0.00175	180.23	0.00841	0.00164	165.81
			$\hat{\mu}_{RG}$	0.00163	0.00178	561.07	0.00235	0.00170	592.99
			$\hat{\mu}_{RN}$	0.00503	0.00166	180.12	0.00964	0.00171	144.70
			$\hat{t}_{GRR}$	0.00070	0.02388	1299.83	0.00199	0.16685	702.05
			$\hat{Z}_{pr}^{(k)*}$	0.00047	0.00022	1957.36	0.00182	0.00052	767.23

*where assumes the values from 1 to 9.

In Table 4, according to population 1 we make a graph of PRE in which red boxes show the proposed estimator, and black boxes show the existing estimator. Percentage Relative Efficiency of Proposed and Existing estimator through simulation is given in that graph which shows that our proposed estimator is efficient compared to another estimator. By using population 2 we make a graph. Percentage Relative Efficiency of Proposed and Existing estimator through simulation is given in that graph which shows that our proposed estimator is efficient compared to another estimator. N shows sample size. Black boxes show the existing estimator, and Red boxes show the proposed estimator.

(a) Population 1 **(b)** Population 2

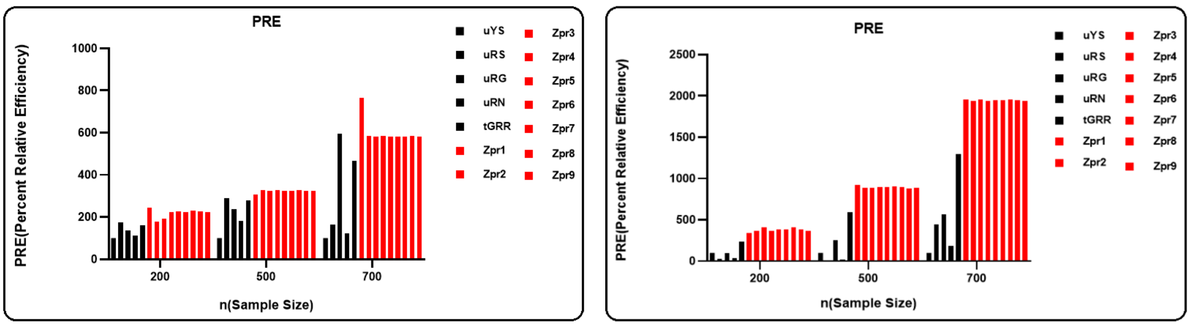

Figure 2. PRE of simulated data summary I

Generalized exponential-type estimator using two auxiliary variables for generalized two-stages optional scrambled response

Motivated by [13], we suggest a generalized exponential-type estimator for population mean using Model I, as

$$\hat{t}_{GRE} = \bar{z}\exp\left[\alpha\left(\frac{\bar{X}^{1/h} - \bar{x}^{1/h}}{\bar{X}^{1/h} + (a-1)\bar{x}^{1/h}} + \frac{\bar{U}^{1/h} - \bar{u}^{1/h}}{\bar{U}^{1/h} + (b-1)\bar{u}^{1/h}}\right)\right]. \tag{12}$$

According to a first-order approximation, the bias and mean square error (MSE) $\hat{t}_{GRE}$ are given

$$Bias(\hat{t}_{GRE}) = \lambda\bar{Z}\left[\frac{\alpha C_x^2}{ah^2}(1-\frac{1}{a}) + \frac{\alpha C_u^2}{bh^2}(1-\frac{1}{b}) - \frac{\alpha}{h}(\frac{C_{zx}}{a} + \frac{C_{zx}}{b}) + \frac{\alpha^2}{2h^2}(\frac{C_u^2}{b^2} + \frac{C_x^2}{a^2} + \frac{2C_{xu}}{ab}\right]. \tag{13}$$

By taking a square and applying expectation from equation (13), we obtained

$$MSE(\hat{t}_{GRE}) = \lambda\bar{Z}^2\left[C_z^2 + \frac{\alpha C_x^2}{a^2b^2} + \frac{\alpha C_u^2}{b^2h^2} - \frac{2\alpha C_{zx}}{ah} - \frac{2\alpha C_{zx}}{bh} + \frac{2\alpha^2 C_{xu}}{abh^2}\right]. \tag{14}$$

To obtain the minimum MSE, we need to estimate the value of a and b. By equation (14), we obtain $\hat{a}_{(opt)} = \frac{\alpha}{h}\left[\frac{C_{xu}^2 - C_u^2 C_x^2}{C_{xu}C_{zu} - C_u^2 C_{zu}}\right]$ and $\hat{b}_{(opt)} = \frac{\alpha}{h}\left[\frac{C_{xu}^2 - C_u^2 C_x^2}{C_{xu}C_{zx} - C_{zu}^2 C_x^2}\right]$.

Using $\hat{a}_{(opt)}$ and $\hat{b}_{(opt)}$, we get a minimum MSE of ($\hat{t}_{GRE}$) as

$$MSE(\hat{t}_{GRE}) = \lambda\bar{Z}^2\left[C_z^2 + \frac{C_x^2 C_{zu}^2 + C_u^2 C_{zu}^2 - 2C_{xu}^2 C_{zx}^2 C_{zu}^2}{C_{xu}^2 - C_u^2 C_x^2}\right]. \tag{15}$$

Efficiency comparison

We present the mathematical comparison of the proposed estimator using two auxiliary variables with the existing estimators under Model-I as

i) By equation (5) and (15)

$$MSE_{\min}(\hat{t}_{GRE}) \le MSE(\hat{t}_{GRR}).$$

ii) By equation (2) and (15)

$$MSE_{\min}(\hat{t}_{GRE}) \le MSE(\hat{\mu}_{RS}).$$

iii) By equation (4) and (15)

$$MSE_{\min}(\hat{t}_{GRE}) \le MSE(\hat{\mu}_{RN}).$$

iv) By equation (3) and (15)

$$MSE_{\min}(\hat{t}_{GRE}) \le MSE(\hat{\mu}_{RG}).$$

Real-life apple tree data set 2

We used data from 204 villages in Turkey's Black Sea Region, including apple production amount in 1999 (as the main variety), number of apple trees in 1999 (as the first auxiliary variety), and apple income and sales in 1998, to estimate the standard and new estimators (as the second auxiliary variety) (Source: Republic of Turkey's National Bureau of statistics)]. The MSE and PRE are calculated and simulated by the proposed model's generalized exponential type ratio estimator compared to the RRT ratio estimator for population 3.

We assume three samples in our simulation study, n= 50, 100, and 120.

N=204, $\rho_{XZ} = 0.71$, $\rho_{xu} = 0.83$, $\rho_{zu} = 0.94$, $\sigma_{zx} = 77372777$, $\sigma_{zu} = 5684276$, $\sigma_{xu} = 94636084$, $\mu_X = 26441$, $\mu_Z = 966$, $\mu_U = 1014$, $\sigma_X = 45402.78$, $\sigma_Z = 2389.76$ and $\sigma_U = 2521.40$.

Table 5. The MSE and PRE values of estimators for Real-life dataset 2

n	*W*	*T*	**Estimator**	**Theoretical**	**PRE**
50	0.3	0.3	$\hat{\mu}_{YS}$	86278.68	100.00
			$\hat{\mu}_{RS}$	86234.28	100.05
			$\hat{\mu}_{RG}$	42787.88	201.64
			$\hat{\mu}_{RN}$	86234.28	100.05
			$\hat{t}_{GRR}$	42765.76	201.74
			$\hat{t}_{GRE}$	22350.49	386.02
	0.5	0.5	$\hat{\mu}_{YS}$	86286.93	100.00
			$\hat{\mu}_{RS}$	86234.28	100.06
			$\hat{\mu}_{RG}$	42788.51	201.65
			$\hat{\mu}_{RN}$	86234.28	100.06
			$\hat{t}_{GRR}$	42765.89	201.76
			$\hat{t}_{GRE}$	22350.49	386.06
	0.7	0.7	$\hat{\mu}_{YS}$	86278.68	100.00
			$\hat{\mu}_{RS}$	86234.28	100.05
			$\hat{\mu}_{RG}$	42789.13	201.63
			$\hat{\mu}_{RN}$	86234.28	100.05
			$\hat{t}_{GRR}$	42765.76	201.74
			$\hat{t}_{GRE}$	22350.49	386.02
100	0.3	$\hat{\mu}_{YS}$	42875.44	100.00	
			$\hat{\mu}_{RS}$	42830.05	100.10
			$\hat{\mu}_{RG}$	21252.26	201.74
			$\hat{\mu}_{RN}$	42830.05	100.10
			$\hat{t}_{GRR}$	21241.27	201.84
			$\hat{t}_{GRE}$	17351.62	247.09
	0.5	0.5	$\hat{\mu}_{YS}$	42883.68	100.00
			$\hat{\mu}_{RS}$	42830.05	100.12
			$\hat{\mu}_{RG}$	21252.57	201.78
			$\hat{\mu}_{RN}$	42830.05	100.12
			$\hat{t}_{GRR}$	21241.33	201.88
			$\hat{t}_{GRE}$	17351.62	247.14
	0.7	0.7	$\hat{\mu}_{YS}$	42875.44	100.00
			$\hat{\mu}_{RS}$	42830.05	100.10
			$\hat{\mu}_{RG}$	21252.88	201.73
			$\hat{\mu}_{RN}$	42830.05	100.10
			$\hat{t}_{GRR}$	21241.27	201.84
			$\hat{t}_{GRE}$	17351.62	247.097
120	0.3	0.3	$\hat{\mu}_{YS}$	19460.53	100.00
			$\hat{\mu}_{RS}$	19414.6	100.23
			$\hat{\mu}_{RG}$	19634.35	100.81
			$\hat{\mu}_{RN}$	19414.6	100.23
			$\hat{t}_{GRR}$	19629.37	116.93
			$\hat{t}_{GRE}$	14654.87	132.79
	0.5	0.5	$\hat{\mu}_{YS}$	19468.78	100.00
			$\hat{\mu}_{RS}$	19414.6	100.27
			$\hat{\mu}_{RG}$	19634.49	102.07
			$\hat{\mu}_{RN}$	19414.6	100.27
			$\hat{t}_{GRR}$	19629.40	102.18
			$\hat{t}_{GRE}$	14654.87	132.84
	0.7	0.7	$\hat{\mu}_{YS}$	19460.53	100.00
			$\hat{\mu}_{RS}$	19414.6	100.23
			$\hat{\mu}_{RG}$	19634.63	100.12
			$\hat{\mu}_{RN}$	19414.6	100.23
			$\hat{t}_{GRR}$	19629.37	102.09
			$\hat{t}_{GRE}$	14654.87	132.79

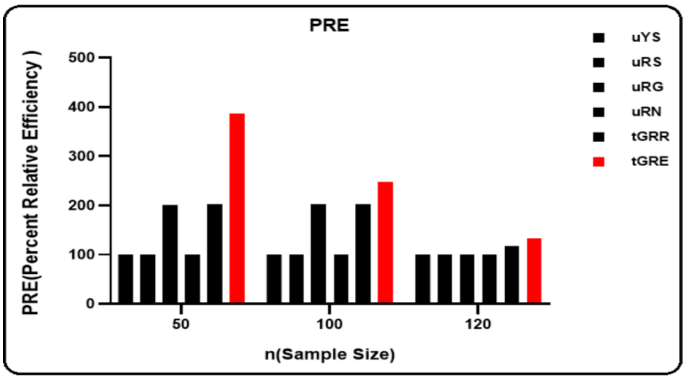

Figure 3. PRE of Real-life data set 2

Simulation study

We used R studio Version 1.3.1093 for the coding and simulation study. To describe the (Y, X) distribution, we assume a multivariate normal population with distinct covariance matrices. We generate random variables by following a multivariate normal distribution. The normal distribution is followed by scrambling variables. With a mean of 0 and an SD equal to 100% of the standard deviation of variable X. The reported generalized response is calculated using the formula Z = Y+ KS. The covariance matrices are given as:

Table 6. Data summary II

Population 1	Population-2
N =1000	N =1000
μ= [2,2,2]	μ= [3,3,3]
$\sum = \begin{pmatrix} 8 & 1 & 2.5 \\ 1 & 2 & 1.5 \\ 2.5 & 1.5 & 2 \end{pmatrix}$	$\sum = \begin{pmatrix} 2 & 1 & 2.2 \\ 1 & 3 & 1.2 \\ 2.2 & 1.2 & 3 \end{pmatrix}$
ρ_{xy}=0.27451	ρ_{xy}=0.40759
ρ_{xy}= 0.721011	ρ_{xy}= 0.39536

Table 7. The MSE and PRE values of estimators for Population 1&2

				Population 1			Population 2		
n	*W*	*T*	Estimator	Theoretical	Empirical	PRE	Theoretical	Empirical	PRE
200	0.3	0.3	$\hat{\mu}_{YS}$	0.03519	0.03140	100.00	0.01358	0.07440	100
			$\hat{\mu}_{RS}$	0.01755	0.00843	200.51	0.13800	0.02654	93.792
			$\hat{\mu}_{RG}$	0.03083	0.00866	114.13	9.79250	0.01910	112.69
			$\hat{\mu}_{RN}$	0.01438	0.00810	244.71	0.13640	0.01118	99.95
			$\hat{t}_{GRR}$	0.02892	0.02628	121.66	0.06609	0.14720	20.52
			$\hat{t}_{GRE}$	0.00523	0.00135	672.87	0.01135	0.01098	119.64
	0.5	0.5	$\hat{\mu}_{YS}$	0.03111	0.03140	100.00	0.01472	0.07440	100.00
			$\hat{\mu}_{RS}$	0.01754	0.00843	177.32	0.13870	0.02654	67.61
			$\hat{\mu}_{RG}$	0.03081	0.00866	101.00	0.01207	0.01910	121.89
			$\hat{\mu}_{RN}$	0.01302	0.00810	238.98	0.13706	0.01118	10.73
			$\hat{t}_{GRR}$	0.02890	0.02628	107.65	0.06613	0.14720	20.32
			$\hat{t}_{GRE}$	0.00523	0.00135	595.04	0.01135	0.01098	129.66
	0.7	0.7	$\hat{\mu}_{YS}$	0.03519	0.03140	100.00	0.01358	0.07440	100.00
			$\hat{\mu}_{RS}$	0.01754	0.00843	200.51	0.13870	0.02654	94.792
			$\hat{\mu}_{RG}$	0.03086	0.00866	114.01	0.01209	0.01910	112.27
			$\hat{\mu}_{RN}$	0.01619	0.00810	217.34	0.13772	0.01118	9.862
			$\hat{t}_{GRR}$	0.02892	0.02628	121.66	0.06600	0.14720	20.57
			$\hat{t}_{GRE}$	0.00523	0.00135	672.87	0.01135	0.01098	119.64
500	0.3	0.3	$\hat{\mu}_{YS}$	0.01185	0.00788	100.00	0.00787	0.00175	100.00
			$\hat{\mu}_{RS}$	0.00579	0.00216	204.65	0.13300	0.00641	59.918
			$\hat{\mu}_{RG}$	0.00771	0.00204	153.73	0.00301	0.00273	261.24
			$\hat{\mu}_{RN}$	0.00895	0.02631	132.25	0.13069	0.00274	69.023
			$\hat{t}_{GRR}$	0.00723	0.00213	163.88	0.01652	0.14720	43.75
			$\hat{t}_{GRE}$	0.00131	0.00037	906.36	0.00284	0.00025	277.35
	0.5	0.5	$\hat{\mu}_{YS}$	0.01262	0.00788	100.00	0.00900	0.00175	100.00
			$\hat{\mu}_{RS}$	0.00579	0.00216	218.04	0.13300	0.00641	69.773
			$\hat{\mu}_{RG}$	0.00771	0.00204	163.70	0.00301	0.00273	298.40
			$\hat{\mu}_{RN}$	0.00805	0.02631	156.75	0.13135	0.00274	68.586

			$\hat{t}_{GRR}$	0.00723	0.00213	174.59	0.01653	0.14720	54.545
			$\hat{t}_{GRE}$	0.00130	0.00037	965.66	0.00283	0.00025	317.41
	0.7	0.7	$\hat{\mu}_{YS}$	0.01185	0.00788	100.00	0.00787	0.00175	100.00
			$\hat{\mu}_{RS}$	0.00579	0.00216	204.65	0.13300	0.00641	59.188
			$\hat{\mu}_{RG}$	0.00771	0.00204	153.57	0.00302	0.00273	260.26
			$\hat{\mu}_{RN}$	0.00714	0.02631	165.76	0.13201	0.00274	59.63
			$\hat{t}_{GRR}$	0.00723	0.00213	163.88	0.01652	0.14720	54.545
			$\hat{t}_{GRE}$	0.00131	0.00037	906.36	0.00284	0.00025	277.35
700	0.3	0.3	$\hat{\mu}_{YS}$	0.01185	0.00328	100.00	0.00673	0.00090	100.00
			$\hat{\mu}_{RS}$	0.00579	0.00088	204.65	0.13186	0.00277	51.037
			$\hat{\mu}_{RG}$	0.00771	0.00094	153.73	0.00120	0.00114	558.34
			$\hat{\mu}_{RN}$	0.00895	0.00083	132.25	0.12955	0.00121	51.947
			$\hat{t}_{GRR}$	0.00723	0.02629	163.88	0.00661	0.14728	101.96
			$\hat{t}_{GRE}$	0.00131	0.00010	906.36	0.00114	0.00014	592.78
	0.5	0.5	$\hat{\mu}_{YS}$	0.01262	0.00328	100.00	0.00786	0.00090	100.00
			$\hat{\mu}_{RS}$	0.00579	0.00088	218.04	0.13186	0.00277	59.65
			$\hat{\mu}_{RG}$	0.00771	0.00094	163.70	0.00120	0.00114	651.43
			$\hat{\mu}_{RN}$	0.00805	0.00083	156.75	0.13021	0.00121	60.41
			$\hat{t}_{GRR}$	0.00723	0.02629	174.59	0.00661	0.14728	121.21
			$\hat{t}_{GRE}$	0.00130	0.00010	965.66	0.00114	0.00014	692.91
	0.7	0.7	$\hat{\mu}_{YS}$	0.01185	0.00328	100.00	0.00673	0.00090	100.00
			$\hat{\mu}_{RS}$	0.00579	0.00088	204.65	0.13180	0.00277	51.03
			$\hat{\mu}_{RG}$	0.00772	0.00094	153.57	0.00120	0.00114	556.24
			$\hat{\mu}_{RN}$	0.00714	0.00083	165.76	0.13087	0.00121	51.423
			$\hat{t}_{GRR}$	0.00723	0.02629	163.88	0.00660	0.14728	101.51
			$\hat{t}_{GRE}$	0.00131	0.00010	906.36	0.00114	0.00014	592.78

The results are represented in Tables 2, 4, 5, and 7. Tables 2 and 5 are used for real-life data sets in which we find theoretical values and percentage relative efficiency. It observed that the percentage relative efficiency of the proposed estimators ($\hat{\hat{Z}}_{pr}^{(k)*}$, $\hat{t}_{GRE}$) according to the model I is better as compared to the existing estimator ($\hat{\mu}_{YS}$,$\hat{\mu}_{RS}$,$\hat{\mu}_{RG}$,$\hat{\mu}_{RN}$,$\hat{t}_{GRR}$). Tables 4 and 8 are used for artificial data. It also shows higher PRE as compared to other ratio estimators. Graphical representation also shows that our proposed estimator's percentage relative efficiency is greater than existing estimators. We used table 7 (population 1) for PRE representations. Red boxes show the proposed estimator, and Black boxes show existing estimators. N shows sample size. Percentage Relative Efficiency of Proposed and Existing estimator through simulation is given in that graph which shows that our proposed estimator is efficient compared to another estimator. We follow population 2 and make a graph for percentage relative efficiency. Red boxes show the proposed estimator, and Black boxes show existing estimators. N shows sample size. Percentage Relative Efficiency of Proposed and Existing estimator through simulation is given in that graph which shows that our proposed estimator PRE values are efficient compared to another estimator.

Population 1 Population 2

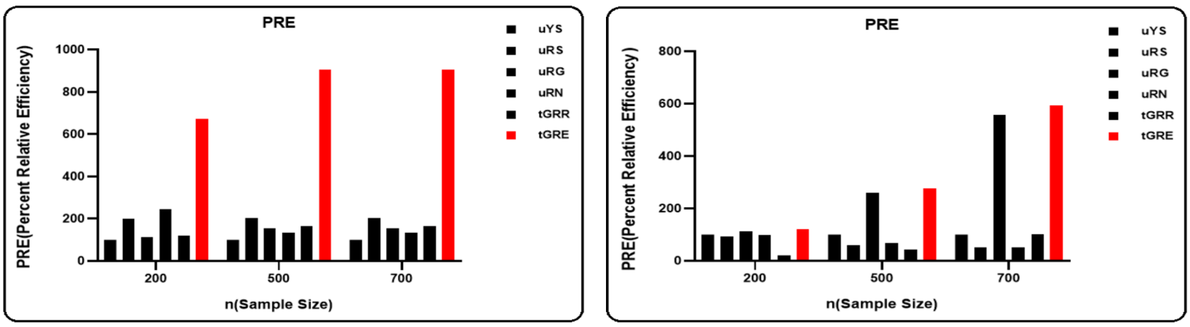

Figure 4. PRE of simulated data summary II

4 Proposed model II

Mean estimator for generalized quantitative randomized response

[14] proposed a quantitative randomized response model. In this study, the study variable Y_i is represented by a mean μ_Y and unknown variance σ_y^2. We assumed that $\theta_1 = E(S)$ variance is equal to $\sigma_1^2 = V(S)$ and their probability lies between $0 \leq p \leq 1$.

$$Z_{\alpha i} = \left\{ \begin{array}{ll} Y_i & \text{with probability p} \\ Y_i S & \text{with probability } (1-\text{p}) \end{array} \right\}. \quad (16)$$

We get

$$\hat{\mu}_y^* = \frac{\bar{Z}^*}{p + (1-p)\theta_1},$$

where

$$\bar{Z}_\alpha = \sum_{i=1}^{n} Z_{\alpha i}/n\ ,\ \ V(\hat{\mu}_x^*) = \frac{\sigma_Z^2}{n(p + (1-p)\theta_1)^2},$$

where

$$\sigma_{Z^*}^2 = \mu_y^2(1 + C_x^2)(p + (1-p)\theta_1^2(1 + C_1^2)) - \mu_x^2(p + (1-P)\theta_1)^2,$$

with

$$C_x = \frac{\sigma_y}{\mu_Y}\ and\ C_1 = \frac{\sigma_1}{\theta_1}.$$

Generalized exponential ratio type estimator using one auxiliary variable for quantitative randomized response

Motivated by [9], [10], and [11], proposed a difference- ratio-type-exponential estimator $\hat{\hat{Z}}_{\alpha i}$ that is given by

$$\hat{\hat{Z}}_{\alpha i} = \{\omega_1\bar{z} + \omega_2(\bar{X} - \bar{x}) + \omega_3(\bar{R}_x - \bar{r}_x)\}\exp\left(\frac{a(\bar{X} - \bar{x})}{a(\bar{X} - \bar{x}) + 2b)}\right), \tag{17}$$

$$\hat{\hat{Z}}_{\alpha i} = \omega_1\bar{Z}S_1 - \omega_1\bar{Z}S_1e_3 - \omega_2\bar{X}e_1 - \omega_3\bar{R}_xe_2\left\{1 - \frac{e_1}{2} + \frac{3e_1^2}{8} + ...\right\}. \tag{18}$$

By expanding the equation (10) and keeping terms only up to order two e_1s we can write

$$\begin{aligned}(\hat{\hat{Z}}_{pr} - \bar{Z}) = &- \bar{Z} + \bar{Z}S_1\omega_1 + \omega_1\bar{Z}S_1e_0 - 1/2\theta\bar{Z}e_1S_1\omega_1 - \bar{X}\omega_2e_1 - \bar{R}_xe_2\omega_3 - 1/2\theta\bar{Z}e_1e_0S_1\omega_1\\ &+ 3/8\theta^2\bar{Z}S_1e_1^2\omega_1 + 1/2\bar{X}\theta e_1^2\omega_2 + 1/2\bar{R}_xe_2e_1\omega_3.\end{aligned} \tag{19}$$

By taking expectation, we can find the bias $\hat{\hat{Z}}_{\alpha i}$ under first approximation, which is given as

$$\begin{aligned}Bias(\hat{\hat{Z}}_{\alpha i}) = &\frac{1}{8}[-8\bar{Z} + 4\lambda\theta C_x(\bar{X}C_x\omega_2 + \bar{R}_x\rho_{xr_x}C_r\omega_3 + \bar{Z}\theta_1\omega_1\{(8 + \lambda\theta C_x\\ &(3\theta C_x - 4C_y\rho_{yx})\}].\end{aligned} \tag{20}$$

The MSE of $\hat{\hat{Z}}_{\alpha i}$ under the first order of approximation are respectively given by

$$\begin{aligned}MSE(\hat{\hat{Z}}_{\alpha i}) = &\bar{Z}^2 + \lambda\bar{X}C_x^2\omega_2(-\bar{Z}\theta_1 + \bar{X}\omega_2) + \lambda\bar{R}_x^2C_r^2\omega_3^2 + \lambda\bar{R}_xC_xC_r(-\bar{Z}\theta_1 + 2\bar{X}\omega_2)\omega_3\rho_{xr_x} +\\ &\bar{Z}^2\theta_1\omega_1^2[1 + \lambda\{C_y^2 + C_x - 2C_y\rho_{yx})\}] + \frac{1}{4}\omega_1\theta_1[-8\omega_1\alpha\theta_2 + \lambda C_x\\ &\{C_x(-3\omega_1\bar{Z}\theta + 8\bar{X}\omega_2) + 8\bar{R}_xC_r\omega_3\rho_{xr_x} + 4C_y(\bar{Z}\theta_1 - 2\bar{X}\omega_2)\rho_{xy}\} - 8\bar{R}_x\lambda C_yC_r\omega_3\rho_{yr_x}].\end{aligned} \tag{21}$$

The optimum value of $\omega_1\ \omega_2\ \omega_3$, obtained by minimizing above equation respectively, given by $\omega_{1_{(opt)}} = \frac{8-\lambda\theta^2C_x^2}{8(1+\lambda C_z^2(1-Q_{z.xrx}^2))}$,

$$\omega_{2_{(opt)}} = \frac{\bar{Z}\left[\lambda\theta^3C_x^3\left(-1 + \rho_{xr_x}^2\right) + \left(-8C_z + \lambda2\theta^2C_x^2C_z\right)(\rho_{zx} - \rho_{xr_x}\rho_{zr_x}) + 4C_x\left(-1 + \rho_{xr_x}^2\right)\left(-1 + \lambda C_z^2\left(1 - Q_{z.xr_x}^2\right)\right)\right]}{8\bar{X}C_x\left(-1 + \rho_{xr_x}^2\right)\left(1 + \lambda C_z^2\left(1 - Q_{z.xr_x}^2\right)\right)},$$

and

$$\omega_{3(opt)} = \frac{\bar{Z}\left(8 - \lambda\theta^2C_x^2\right)C_z\left(\rho_{xr_x}\rho_{zx} - \rho_{zr_x}\right)}{8\bar{R}_xC_r\left(-1 + \rho_{xr_x}^2\right)\left(1 + \lambda C_z^2\left(1 - Q_{z.xr_x}^2\right)\right)},$$

where

$$Q^2_{z.xr_x} = \frac{\rho^2_{zx} + \rho^2_{zr_x} - 2\rho_{zx}\rho_{zr_x}\rho_{xr_x}}{1 - \rho^2_{xr_x}}.$$

Substitute the value ω_1 ω_2, ω_3 in equation (21), and we get the minimum MSE $\bar{x} = \bar{X}(1 + e_1)$given by

$$MSE_{\min}(\hat{\hat{Z}}_{\alpha i}) = \frac{\lambda \bar{Z}\theta_1 \left[64C_y^2(1 - Q^2_{z.xr_x}) - \lambda\theta^4 C_x^4 C - 16\lambda\theta^2 C_x^2 C_y^2(1 - Q^2_{z.xr_x})\right]}{64\left\{1 + \lambda C_y^2(1 - Q^2_{z.xr_x})\right\}}. \tag{22}$$

Efficiency comparison

We present the mathematical comparison of the proposed estimator with existing estimators under Model-II

v) By equations (5) and (22)

$$MSE_{\min}(\hat{\hat{Z}}_{\alpha i}) \leq MSE(\hat{t}_{GRR}).$$

vi) By equations (2) and (22)

$$MSE_{\min}(\hat{\hat{Z}}_{\alpha i}) \leq MSE(\hat{\mu}_{RS}).$$

vii) By equations (4) and (22)

$$MSE_{\min}(\hat{\hat{Z}}_{\alpha i}) \leq MSE(\hat{\mu}_{RN}) .$$

viii) By equations (3) and (22)

$$MSE_{\min}(\hat{\hat{Z}}_{\alpha i}) \leq MSE(\hat{\mu}_{RG}).$$

Simulation study

We generate the correlated scrambling variables S with parameters θ_1 and σ_1. The discrete uniform distribution is followed by scrambling variable, where $S = U(a_1, b_1)$. Scrambling variables followed uniform distribution. In other words, Stakes integer values between a_1 and b_1. The reported generalized response is calculated using the formula $Z = YS$. Z is our auxiliary variable. And Y is the study variable and S is the scramble variable.

Table 8. Data summary III

Population 1	**Population 2**
N = 1000	N = 1000
μ= [2,2]	μ= [3,3]
$\sum = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$	$\sum = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}$
ρ_{yx}=0.37144	ρ_{yx}=0.388517
$S_{\min} = 0$, $S_{\max} = 3$	$S_{\min} = 0$, $S_{\max} = 5$

Table 9. The MSE and PRE values of estimators for Population 1

				Population 1			Population 2		
n	W	T	Estimator	Theoretical	Empirical	PRE	Theoretical	Empirical	PRE
200	0.3	0.3	$\hat{\mu}_{YS}$	0.04768	0.00702	100.00	0.2981	0.0012	100.00
			$\hat{\mu}_{RS}$	0.57105	0.0085	115.01	0.2840	0.0096	104.97
			$\hat{\mu}_{RG}$	0.07370	0.0096	64.69	0.4047	0.0010	73.65
			$\hat{\mu}_{RN}$	0.05688	0.0078	83.82	0.2894	0.0095	103.06
			$\hat{t}_{GRR}$	0.04509	0.0062	105.95	0.2618	0.0029	113.85
			$\hat{\hat{Z}}^{(1)}_{pr}$	0.0414	0.00042	115.74	0.1686	0.00028	176.81
			$\hat{\hat{Z}}^{(2)}_{pr}$	0.04016	0.00035	118.72	0.1647	0.00011	177.79
			$\hat{\hat{Z}}^{(3)}_{pr}$	0.03810	0.0002	125.10	0.1639	0.00039	178.51
			$\hat{\hat{Z}}^{(4)}_{pr}$	0.04016	0.00014	118.72	0.1668	0.00029	177.72
			$\hat{\hat{Z}}^{(5)}_{pr}$	0.03930	0.00024	121.08	0.1683	0.00022	178.13

			$\hat{Z}_{pr}^{(6)}$	0.03938	0.00041	121.08	0.1676	0.00041	178.33
			$\hat{Z}_{pr}^{(7)}$	0.03811	0.00060	125.10	0.1659	0.00070	178.47
			$\hat{Z}_{pr}^{(8)}$	0.03937	0.00132	121.08	0.1623	0.00034	178.53
			$\hat{Z}_{pr}^{(9)}$	0.04016	0.0027	118.72	0.1616	0.00105	177.76
	0.5	0.5	$\hat{\mu}_{YS}$	0.04926	0.0010	100.00	0.2996	0.0029	100.00
			$\hat{\mu}_{RS}$	0.05710	0.0036	64.69	0.2894	0.0094	103.76
			$\hat{\mu}_{RG}$	0.07375	0.0031	66.799	0.2840	0.0098	105.49
			$\hat{\mu}_{RN}$	0.05694	0.0034	83.82	0.3718	0.0045	84.70
			$\hat{t}_{GRR}$	0.04509	0.0062	109.23	0.2618	0.0032	114.43
			$\hat{Z}_{pr}^{(k)*}$	0.04145	0.00049	118.83	0.1686	0.00234	177.71
	0.7	0.7	$\hat{\mu}_{YS}$	0.04760	0.0010	100.00	0.2981	0.0028	100.00
			$\hat{\mu}_{RS}$	0.05710	0.0042	64.69	6.2890	0.0014	4.74
			$\hat{\mu}_{RG}$	0.07370	0.0035	64.62	0.2840	0.0098	104.95
			$\hat{\mu}_{RN}$	0.05701	0.0042	83.82	6.3388	0.0025	4.70
			$\hat{t}_{GRR}$	0.04509	0.0063	105.74	0.2618	0.0032	113.85
			$\hat{Z}_{pr}^{(k)*}$	0.04145	0.00011	115.01	0.1686	0.00043	176.81
500	0.3	0.3	$\hat{\mu}_{YS}$	0.01810	0.00090	100.00	0.0805	0.0038	100.00
			$\hat{\mu}_{RS}$	0.05354	0.00089	33.83	0.5070	0.0034	15.87
			$\hat{\mu}_{RG}$	0.01840	0.00072	98.53	0.0710	0.0098	113.38
			$\hat{\mu}_{RN}$	0.05332	0.00061	33.95	0.3224	0.0035	24.96
			$\hat{t}_{GRR}$	0.01120	0.00051	161.08	0.0754	0.0032	120.97
			$\hat{Z}_{pr}^{(1)}$	0.01038	0.000028	174.87	0.0665	0.000421	122.97
			$\hat{Z}_{pr}^{(2)}$	0.01026	0.000035	176.41	0.1647	0.00017	107.79
			$\hat{Z}_{pr}^{(3)}$	0.01010	0.00045	179.20	0.1639	0.00019	108.51
			$\hat{Z}_{pr}^{(4)}$	0.01036	0.00009	174.71	0.1668	0.00032	107.72
			$\hat{Z}_{pr}^{(5)}$	0.01030	0.00042	175.08	0.1683	0.00079	108.13
			$\hat{Z}_{pr}^{(6)}$	0.03938	0.00029	141.08	0.1676	0.00017	108.33
			$\hat{Z}_{pr}^{(7)}$	0.03811	0.000027	142.10	0.1659	0.00029	108.47
			$\hat{Z}_{pr}^{(8)}$	0.03937	0.00036	132.08	0.1623	0.00053	108.53
			$\hat{Z}_{pr}^{(9)}$	0.04016	0.00099	119.72	0.1616	0.00062	107.76
	0.5	0.5	$\hat{\mu}_{YS}$	0.01970	0.0010	100.00	0.0620	0.0048	132.26
			$\hat{\mu}_{RS}$	0.02354	0.00087	83.68	6.5070	0.0014	115.50
			$\hat{\mu}_{RG}$	0.01840	0.0089	107.08	0.0720	0.0098	113.88
			$\hat{\mu}_{RN}$	0.02339	0.00076	84.69	6.5894	0.0015	122.29
			$\hat{t}_{GRR}$	0.01120	0.00063	175.11	0.0665	0.0032	123.25
			$\hat{Z}_{pr}^{(k)*}$	0.01038	0.00057	190.13	0.0805	0.0024	100.00
	0.7	0.7	$\hat{\mu}_{YS}$	0.01810	0.00010	100.00	0.5070	0.0022	15.87
			$\hat{\mu}_{RS}$	0.02354	0.00071	84.391	0.0710	0.0098	113.35
			$\hat{\mu}_{RG}$	0.01840	0.00091	98.44	0.5560	0.0035	14.22
			$\hat{\mu}_{RN}$	0.02345	0.00092	83.397	0.0654	0.0032	122.98
			$\hat{t}_{GRR}$	0.01120	0.00063	161.08	0.0565	0.0018	143.75
			$\hat{Z}_{pr}^{(k)*}$	0.01030	0.00005	174.87	0.0620	0.0048	132.26
700	0.3	0.3	$\hat{\mu}_{YS}$	0.01220	0.000201	100.00	0.0369	0.0034	100.00
			$\hat{\mu}_{RS}$	0.02283	0.0089	83.10	0.0550	0.0042	67.01
			$\hat{\mu}_{RG}$	0.00737	0.0086	166.24	0.0284	0.0018	130.17
			$\hat{\mu}_{RN}$	0.02260	0.0092	84.21	6.6000	0.0035	56.02
			$\hat{t}_{GRR}$	0.00450	0.0063	271.78	0.0261	0.0071	141.22
			$\hat{Z}_{pr}^{(1)}$	0.00415	0.00042	294.92	0.0201	0.000962	184.50
			$\hat{Z}_{pr}^{(2)}$	0.00401	0.00086	218.72	0.1247	0.0001	177.79
			$\hat{Z}_{pr}^{(3)}$	0.04810	0.00040	225.10	0.1339	0.00014	178.51
			$\hat{Z}_{pr}^{(4)}$	0.04016	0.00021	218.72	0.1368	0.0011	177.72
			$\hat{Z}_{pr}^{(5)}$	0.03930	0.00073	221.08	0.1483	0.00168	178.13
			$\hat{Z}_{pr}^{(6)}$	0.03938	0.00081	221.08	0.1676	0.0012	178.33
			$\hat{Z}_{pr}^{(7)}$	0.03811	0.00029	225.10	0.1559	0.0020	178.47
			$\hat{Z}_{pr}^{(8)}$	0.03937	0.00059	221.08	0.1623	0.0011	178.53
			$\hat{Z}_{pr}^{(9)}$	0.04016	0.00070	218.72	0.1616	0.0003	177.76
	0.5	0.5	$\hat{\mu}_{YS}$	0.01380	0.00102	100.00	6.5506	0.0014	100.00
			$\hat{\mu}_{RS}$	0.05283	0.0092	6.5506	6.5506	0.0014	58.78
			$\hat{\mu}_{RG}$	0.00737	0.0082	0.0284	0.0284	0.0042	135.5
			$\hat{\mu}_{RN}$	0.05260	0.0094	6.6329	6.6329	0.0028	58.0
			$\hat{t}_{GRR}$	0.00450	0.0063	0.0301	0.0301	0.0035	127.88

		$\hat{\hat{Z}}_{pr}^{(k)*}$	0.00416	0.00038	0.0262	0.0262	0.0071	147.01
0.7	0.7	$\hat{\mu}_{YS}$	0.01220	0.0010	100.00	6.6000	0.0034	100.00
		$\hat{\mu}_{RS}$	0.05283	0.0087	23.19	0.0262	0.0042	130.17
		$\hat{\mu}_{RG}$	0.00737	0.0088	166.09	0.0262	0.0018	130.17
		$\hat{\mu}_{RN}$	0.05274	0.0087	23.14	0.0201	0.0045	56.08
		$\hat{t}_{GRR}$	0.00450	0.0049	271.78	0.0262	0.0061	122.84
		$\hat{\hat{Z}}_{pr}^{(k)*}$	0.00416	0.00043	294.92	0.0201	0.00051	184.50

In Table 9 PRE for population 1 is given. Percentage Relative Efficiency of Proposed and Existing estimator through simulation is given in that graph which shows that our proposed estimator is efficient compared to another estimator. Black boxes show the existing estimator, and Red boxes show the proposed estimators. Using population I, we estimate PRE. In table 9 by using population 2 we find the percentage relative efficiency of estimators. Red boxes show the proposed estimator, and Black boxes show existing estimators. Percentage Relative Efficiency of Proposed and Existing estimator through simulation is given in that graph which shows that our proposed estimator is efficient compared to another estimator.

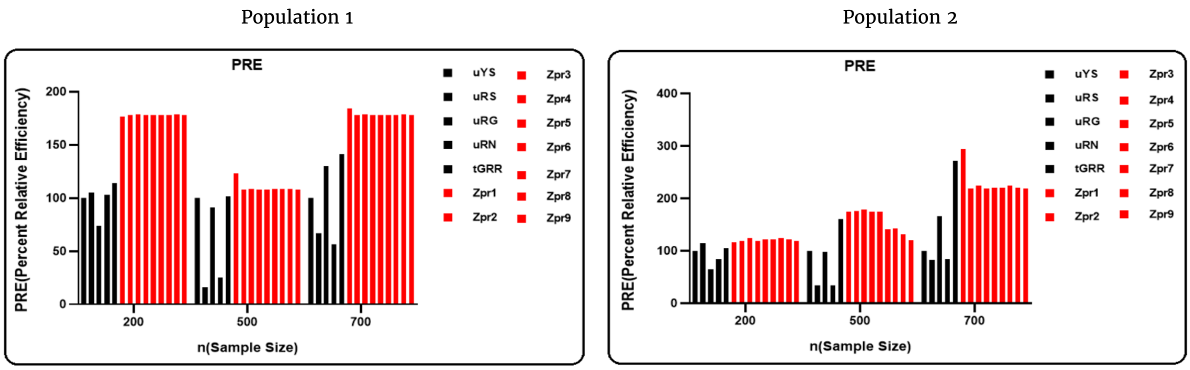

Figure 5. PRE of simulated data summary III

5 Generalized exponential-type estimator using two auxiliary variables for generalized quantitative randomize response

Motivated by [13], we suggest a generalized exponential-type estimator for population mean using Model II, as

$$\hat{t}_{GRE} = \bar{z}\exp\left[\alpha\left(\frac{\bar{X}^{1/h} - \bar{x}^{1/h}}{\bar{X}^{1/h} + (a-1)\bar{x}^{1/h}} + \frac{\bar{U}^{1/h} - \bar{u}^{1/h}}{\bar{U}^{1/h} + (b-1)\bar{u}^{1/h}}\right)\right] \tag{23}$$

$$\begin{aligned}(\hat{t}_{GRE} - \bar{Z}) = \bar{Z}S\Bigg[&\delta_z - \frac{\alpha\delta_z\delta_x}{ah} - \frac{\alpha\delta_x^2}{ah} - \frac{\alpha\delta_z\delta_u}{bh} + \frac{\alpha\delta_x}{ah} + \frac{\alpha\delta_x^2}{ah^2} - \frac{\alpha\delta_x^2}{a^2h^2} - \frac{\alpha\delta_u}{bh} + \frac{\alpha\delta_u^2}{bh^2} - \frac{\alpha\delta_u^2}{b^2h^2}\\ &+ \frac{\alpha^2\delta_x^2}{2a^2h^2} + \frac{\alpha^2\delta_u^2}{2b^2h^2} + \frac{\alpha^2\delta_x\delta_u}{abh^2}\Bigg].\end{aligned} \tag{24}$$

By taking expectations on both sides

$$Bias(\hat{t}_{GRE}) = \lambda\bar{Z}\theta_1\left[\frac{\alpha C_x^2}{ah^2}\left(1-\frac{1}{a}\right) + \frac{\alpha C_u^2}{bh^2}\left(1-\frac{1}{b}\right) - \frac{\alpha}{h}\left(\frac{C_{zx}}{a} + \frac{C_{zx}}{b}\right) + \frac{\alpha^2}{2h^2}\left(\frac{C_u^2}{b^2} + \frac{C_x^2}{a^2} + \frac{2C_{xu}}{ab}\right)\right]. \tag{25}$$

By taking a square and applying expectation from equation (20), we obtained

$$MSE(\hat{t}_{GRE}) = \lambda\bar{Z}^2\theta_1\left[C_z^2 + \frac{\alpha C_x^2}{a^2b^2} + \frac{\alpha C_u^2}{b^2h^2} - \frac{2\alpha C_{zx}}{ah} - \frac{2\alpha C_{zx}}{bh} + \frac{2\alpha^2 C_{xu}}{abh^2}\right]. \tag{26}$$

To obtain the minimum MSE, we need to estimate the value of a and b. By equation (22), we obtain

$$\hat{a}_{(opt)} = \frac{\alpha}{h}\left[\frac{C_{xu}^2 - C_u^2C_x^2}{C_{xu}C_{zu} - C_u^2C_{zu}}\right] \text{ and } \hat{b}_{(opt)} = \frac{\alpha}{h}\left[\frac{C_{xu}^2 - C_u^2C_x^2}{C_{xu}C_{zx} - C_{zu}^2C_x^2}\right].$$

Using $\hat{a}_{(opt)}$ and $\hat{b}_{(opt)}$, we get a minimum MSE of $(\hat{t}_{GRE})$ as

$$MSE(\hat{t}_{GRE}) = \lambda\bar{Z}^2\theta_1\left[C_z^2 + \frac{C_x^2C_{zu}^2 + C_u^2C_{zu}^2 - 2C_{xu}^2C_{zx}^2C_{zu}^2}{C_{xu}^2 - C_u^2C_x^2}\right]. \tag{27}$$

Efficiency comparison

We present the mathematical comparison of the proposed estimator using two auxiliary variables with the existing estimators under Model-II as

I By equation (5) and (27)

$$MSE_{\min}(\hat{t}_{GRE}) \leq MSE(\hat{t}_{GRR}).$$

II By equation (2) and (27)

$$MSE_{\min}(\hat{t}_{GRE}) \leq MSE(\hat{\mu}_{RS}).$$

III By equation (4) and (27)

$$MSE_{\min}(\hat{t}_{GRE}) \leq MSE(\hat{\mu}_{RN}).$$

IV By equation (3) and (27)

$$MSE_{\min}(\hat{t}_{GRE}) \leq MSE(\hat{\mu}_{RG}).$$

Simulation study for proposed generalized exponential type estimator using two auxiliary variables by model-II

To describe the (Y, X) distribution, we assume a multivariate normal population with distinct covariance matrices. We can generate the correlated scrambling variable S. With the chosen parameters, θ_1 and σ_1. The discrete uniform distribution is followed by scrambling variables. Where $S = U(a_1, b_1)$, in other words, S takes integer values between a_1 and b_1. The reported generalized response is calculated using the formula $Z = YS$.

Table 10. Data summary IV

Population 1	Population 2
N =1000	N =1000
μ= [2,2,2]	μ= [3,3,3]
$\sum = \begin{pmatrix} 8 & 1 & 2.5 \\ 1 & 2 & 1.5 \\ 2.5 & 1.5 & 2 \end{pmatrix}$	$\sum = \begin{pmatrix} 2 & 1 & 2.2 \\ 1 & 3 & 1.2 \\ 2.2 & 1.2 & 3 \end{pmatrix}$
ρ_{xy}=0.27451	ρ_{xy}=0.40759
ρ_{xy}= 0.721011	ρ_{xy}= 0.39536
$S_{\min} = 0$, $S_{\max} = 3$	$S_{\min} = 0$, $S_{\max} = 5$

Table 11. The MSE and PRE values of estimators for Population 1

				Population 1			Population 2		
n	W	T	Estimator	Theoretical	Empirical	PRE	Theoretical	Empirical	PRE
200	0.3	0.3	$\hat{\mu}_{YS}$	0.0153	0.11143	100.00	0.08297	1.81129	100.00
			$\hat{\mu}_{RS}$	1.2377	0.03221	81.24	0.02340	1.7110	52.45
			$\hat{\mu}_{RG}$	0.0426	0.02870	36.02	0.02630	0.0514	314.65
			$\hat{\mu}_{RN}$	1.2419	0.03008	91.23	0.02940	0.0493	44.45
			$\hat{t}_{GRR}$	0.0410	0.05720	37.40	0.02630	0.9229	314.86
			$\hat{t}_{GRE}$	0.0054	0.00716	283.43	0.01130	0.0171	733.54
	0.5	0.5	$\hat{\mu}_{YS}$	0.0182	0.11143	100.00	0.08376	1.81129	100.00
			$\hat{\mu}_{RS}$	1.2377	0.03221	41.47	0.0750	1.7110	111.68
			$\hat{\mu}_{RG}$	0.0427	0.02870	42.83	0.02630	0.0514	317.46
			$\hat{\mu}_{RN}$	1.2407	0.03008	51.47	0.0900	0.0493	93.066
			$\hat{t}_{GRR}$	0.0411	0.05720	44.51	0.02630	0.9229	317.82
			$\hat{t}_{GRE}$	0.0054	0.00716	337.41	0.01131	0.0171	740.51
	0.7	0.7	$\hat{\mu}_{YS}$	0.0153	0.11143	100.00	0.08290	1.81129	100.00
			$\hat{\mu}_{RS}$	1.2377	0.03221	31.237	0.0750	1.7110	110.53
			$\hat{\mu}_{RG}$	0.0427	0.02870	35.93	0.02640	0.0514	314.28
			$\hat{\mu}_{RN}$	1.2390	0.03008	102.0	0.02340	0.0493	354.27

			$\hat{t}_{GRR}$	0.0410	0.05720	37.40	0.02630	0.9229	314.86
			$\hat{t}_{GRE}$	0.0054	0.00716	283.43	0.01131	0.0171	733.54
500	0.3	0.3	$\hat{\mu}_{YS}$	0.0154	0.03042	100.00	0.08290	1.7778	100.00
			$\hat{\mu}_{RS}$	1.3189	0.01270	51.12	0.06600	1.6565	125.60
			$\hat{\mu}_{RG}$	0.0107	0.01281	144.09	0.0659	0.1408	125.79
			$\hat{\mu}_{RN}$	1.3231	0.01267	81.160	0.02140	0.1476	387.45
			$\hat{t}_{GRR}$	0.0103	0.05715	149.58	0.00658	0.9229	1259.46
			$\hat{t}_{GRE}$	0.0014	0.00384	1133.57	0.00280	0.0212	2934.16
	0.5	0.5	$\hat{\mu}_{YS}$	0.0182	0.03042	100.00	0.08370	1.7778	100.00
			$\hat{\mu}_{RS}$	1.3189	0.01270	138.5	0.06700	1.6565	124.60
			$\hat{\mu}_{RG}$	0.0107	0.01281	171.30	0.0659	0.1408	125.79
			$\hat{\mu}_{RN}$	1.3219	0.01267	138.2	0.02240	0.1476	388.45
			$\hat{t}_{GRR}$	0.0103	0.05715	178.02	0.00650	0.9229	1271.28
			$\hat{t}_{GRE}$	0.0014	0.00384	1349.48	0.00280	0.0212	2962.06
	0.7	0.7	$\hat{\mu}_{YS}$	0.0153	0.03042	100.00	0.08290	1.7778	100.00
			$\hat{\mu}_{RS}$	1.3189	0.01270	116.42	0.02640	1.6565	144.51
			$\hat{\mu}_{RG}$	0.0106	0.01281	143.70	0.0660	0.1408	125.13
			$\hat{\mu}_{RN}$	1.3207	0.01267	116.2	0.02840	0.1476	145.45
			$\hat{t}_{GRR}$	0.0103	0.05715	149.58	0.00658	0.9229	1259.46
			$\hat{t}_{GRE}$	0.0014	0.00384	1133.57	0.00282	0.0212	2934.1
700	0.3	0.3	$\hat{\mu}_{YS}$	0.0153	0.01373	100.00	0.01202	1.76471	100.00
			$\hat{\mu}_{RS}$	1.3352	0.00966	115 .00	0.02640	1.6297	414.51
			$\hat{\mu}_{RG}$	0.0042	0.00961	360.23	0.00260	0.91076	456.00
			$\hat{\mu}_{RN}$	1.3394	0.01030	114.60	0.02840	0.92118	441.45
			$\hat{t}_{GRR}$	0.0041	0.05715	373.94	0.00260	0.92313	456.30
			$\hat{t}_{GRE}$	0.0005	0.00241	2833.85	0.00113	0.2073	1063.05
	0.5	0.5	$\hat{\mu}_{YS}$	0.0182	0.01373	100.00	0.01280	1.76471	100.00
			$\hat{\mu}_{RS}$	1.3352	0.00966	136.9	0.02740	1.6297	44.51
			$\hat{\mu}_{RG}$	0.0043	0.00961	428.26	0.00263	0.91076	485.63
			$\hat{\mu}_{RN}$	1.3382	0.01030	136.0	0.00294	0.92118	435.37
			$\hat{t}_{GRR}$	0.0041	0.05715	445.04	0.00263	0.92313	486.18
			$\hat{t}_{GRE}$	0.0005	0.00241	3373.6	0.00113	0.2073	1132.79
	0.7	0.7	$\hat{\mu}_{YS}$	0.0153	0.01373	100.00	0.01202	1.76471	100.00
			$\hat{\mu}_{RS}$	0.0567	0.00966	269.84	0.00274	1.6297	438.68
			$\hat{\mu}_{RG}$	1.3350	0.00961	21.15	0.00264	0.91076	455.46
			$\hat{\mu}_{RN}$	0.0043	0.01030	359.24	0.02940	0.92118	457.03
			$\hat{t}_{GRR}$	1.3370	0.05715	71.14	0.00263	0.92313	456.30
			$\hat{t}_{GRE}$	0.0005	0.00241	2833.85	0.00113	0.2073	1063.05

The results are represented in Tables 9, and 11. Tables 9 and 11 are used for artificial data. It observed that the percentage relative efficiency of the proposed estimators ($\hat{\hat{Z}}_{pr}^{(k)*}$, $\hat{t}_{GRE}$) according to model II is better as compared to the existing estimator ($\hat{\mu}_{YS}$, $\hat{\mu}_{RS}$, $\hat{\mu}_{RG}$, $\hat{\mu}_{RN}$, $\hat{t}_{GRR}$). It also shows higher PRE as compared to other ratio estimators. Graphical representation also shows that our proposed estimator's percentage relative efficiency is greater than existing estimators. By using table 11 (population 1) we make a graph to represent the PRE of estimators. Red boxes show the proposed estimator, and Black boxes show existing estimators. n shows sample size. Percentage Relative Efficiency of Proposed and Existing estimator through simulation is given in that graph which shows that our proposed estimator PRE values are efficient compared to another estimator. By using table 11 (population 2) we represent the PRE of estimators. Red boxes show the proposed estimator, and Black boxes show existing estimators. N shows sample size. Percentage Relative Efficiency of Proposed and Existing estimator through simulation is given in that graph which shows that our proposed estimator PRE values are efficient compared to another estimator.

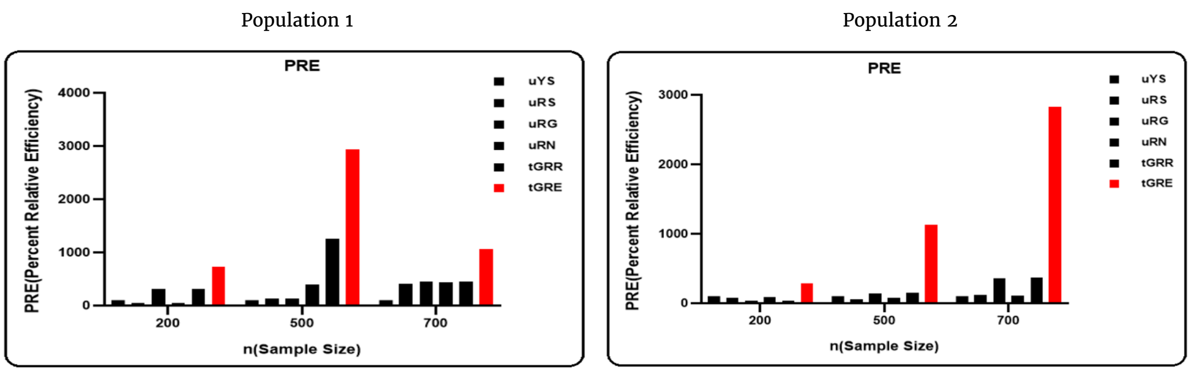

Figure 6. PRE of simulated data summary IV

6 Conclusion

For many scholars, figuring out the population means in a sample survey is important. However, more research on mean square error estimates using auxiliary data is required. In this study, the population mean is calculated using single or dual auxiliary variables and simple random sampling. For better results in the first estimator, we rank our auxiliary variable that is connected with the research variables. In comparison to an existing estimator, they provide efficient or better outcomes. They are also more efficient relative to the current estimator in terms of %.In the second proposed estimator; we use dual auxiliary variables to improve results. We compare a dual auxiliary variable with one auxiliary variable. Dual auxiliary variables show better results as compared to one auxiliary variable. The superiority of proposed MSE estimators over the usual MSE the expressions for least mean square errors using first-order approximation and the various unknown constant values. Using auxiliary data allows us to come up with more accurate population estimations. Using a single auxiliary variable, the performance of suggested estimators is compared to MSE estimators. The efficiency comparison makes estimators crucial. Other estimators that don't use auxiliary features are less effective than MSE estimators. The relative efficiency of the new estimators, which outperform the old estimators, is also calculated.

Declarations

Consent for publication

Not applicable.

Conflicts of interest

The authors declare that they have no conflict of interests.

Funding

No funding was used in this study.

Data Availability

The data used to support the findings of this study are included in the article.

Author's contributions

A.Z.: Conceptualization, Methodology, Software. S.M. (Saadia Masood): Data Curation, Writing-Original draft preparation. S.M. (Sumaira Mubarik): Visualization, Investigation. A.D.: Supervision, Software, Validation, Writing-Reviewing and Editing. All authors discussed the results and contributed to the final manuscript.

Acknowledgements

Not applicable.

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