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Polyanalytic Nonlinear Boundary Value Problems for Upper Half Plane

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Abstract. Our aim is to improve the techniques of partial differential equations in complex variable.
In this article we consider the explicit representations for polyanalytic Dirichlet boundary value problems in the upper half plane. We also point out the solutions of some simple nonlinear equations.

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1. Introduction

The boundary value problems are known as Dirichlet, Neumann, Robin, Schwarz are inherited from Riemann and Riemann-Hilbert problems. In this article, our discussions are limited to Dirichlet problems. Dirichlet problems may be decomposed into two subgroups; either having a polyanalytic leading term as in Begehr and Kumar [7–9] or a polyharmonic leading term as in Kumar and Prakash [18, 19]. For both of the cases we know the explicit integral representations.

In this article, we restrict ourselves to the equations with polyanalytic leading terms. Let us also note that the boundary value problems have been investigated by many researchers in bounded domains, relevant examples are given for example by Çelebi and Gökgöz [15, 16], Aksoy and Çelebi [4, 5], Begehr, Du and Wang [10]. The results in unbounded domains are also obtained. We want to state just several of them. A short list of such investigations may give us Akel and Begehr [1, 2] and Aksoy-Begehr-Çelebi [3], Begehr and Gaertner [12], Chaudhary and Kumar [14], Gaertner [17].

We feel ourselves to improve these results as in general theory of partial differential equations in $\mathbb{C}$.

2. Preliminaries

Firstly, we recall the Gauss theorem:

Let $D \subset \mathbb{C}$ be a regular domain, that is a bounded domain with piecewise smooth boundary ∂D. Take $w \in C^1(D;\mathbb{C}) \cap C(\bar{D},\mathbb{C})$, then

$$\frac{1}{2i}\int_{\partial D} w(z)dz = \int_D w_{\bar{z}}(z)dxdy$$

and

$$-\frac{1}{2i}\int_{\partial D} w(z)d\bar{z} = \int_D w_z(z)dxdy,$$

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where $z = x + iy$. For the proof of these formulas see [13, 20].

Now let us take $\mathbb{H} = \{z : 0 < \operatorname{Im} z\}$ which is the upper half plane. Let $w \in W^{1,1}(\mathbb{H}; \mathbb{C}) \cap C(\overline{\mathbb{H}}; \mathbb{C})$. To apply the Gauss theorem we take the restricted domain

$$\mathbb{H}_R = \{z : 0 < \operatorname{Im} z, |z| < R\}.$$

Hence, we can apply Gauss theorem:

$$\frac{1}{2i} \int_{\partial \mathbb{H}_R} w(z) dz = \int_{\mathbb{H}_R} w_{\bar{z}}(z) dxdy.$$

Firstly, assuming

$$\lim_{R \to +\infty} RM(R, w) = 0,$$

where $M(R, w)$ is the maximum value of w in $\mathbb{H}_R$

$$M(R, w) = \max_{\substack{|z|=R \\ 0 \le \varphi \le \pi}} |w(z)|$$

gives us the Gauss theorem in $\mathbb{H}$:

$$\frac{1}{2i} \int_{-\infty}^{\infty} w(t) dt = \int_{\mathbb{H}} w_{\bar{z}}(z) dxdy.$$

Thus, we can state the Cauchy-Pompeiu representation for the upper half plane by the following:

Theorem 2.1 ([17]). *If* $w : \mathbb{H} \to \mathbb{C}$ *satisfies* $|w(x)| \le C|x|^{-\varepsilon}$ *for* $|x| > k$ *and* $w_{\bar{z}} \in L_1(\mathbb{H}; C)$*, then*

$$w(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} w(t) \frac{dt}{t - z} - \frac{1}{\pi} \int_{0 < Im\zeta} w_{\bar{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z},$$

$$w(z) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} w(t) \frac{dt}{t - \bar{z}} - \frac{1}{\pi} \int_{0 < Im\zeta} w_{\zeta}(\zeta) \frac{d\xi d\eta}{\overline{\zeta - z}},$$

where $z \in \mathbb{H}$.

Once we have obtained this Cauchy-Pompeiu representation given in Theorem 2.1 we may extend it to higher-order integral representations.

Theorem 2.2 ([17]). *Let for* $w \in W^{k,1}(\mathbb{H}, \mathbb{C}), \lim_{R \to +\infty} R^{\nu} M\left(\partial_{\bar{z}}^{\nu} w, R\right) = 0$*, where* $0 \le \nu \le k - 1$*. And also* $\bar{z}^{k-2} \partial_{\bar{z}}^{k} w \in L^1(\mathbb{H}, \mathbb{C})$ *when* $k \ge 2$*. Then,*

$$w(z) = \sum_{\nu=0}^{k-1} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\nu!} \frac{(\overline{z - \zeta})^{\nu}}{\zeta - z} \partial_{\bar{\zeta}}^{\nu} w(\zeta) d\zeta$$
$$- \frac{1}{\pi} \int_{\mathbb{H}} \frac{1}{(k-1)!} \frac{\overline{(z - \zeta)}^{k-1}}{\zeta - z} \partial_{\bar{\zeta}}^{k} w(\zeta) d\xi d\eta,$$

where $z \in \mathbb{H}$.

This result will enable us to find the representation of the solution for higher-order Dirichlet problem in the upper half plane, see [17].

Now, we want to define the integral representation of a complex function in the upper half plane $\mathbb{H}$, as in [14]. *Let* $\mathcal{F}_k$ *be the space of functions* w *in* $W^{k,1}(\mathbb{H}, \mathbb{C})$ *for which* $\lim_{R \to \infty} R^{\nu} M\left(\partial_{\bar{z}}^{\nu} w, R\right) = 0$, $0 \le \nu \le k - 1$ *where* $M\left(\partial_{\bar{z}}^{\nu} w, R\right) = \max_{\substack{|z|=R \\ 0 \le Imz}} \left|\partial_{\bar{z}}^{\nu} w(z)\right|$ *and* $\bar{z}^{k-2} \partial_{\bar{z}}^{k} w \in L^1(\mathbb{H}, \mathbb{C})$*. Then, using ([17], Theorem 4), it follows that* $w \in \mathcal{F}_k$ *is representable as*

$$w(z) = \sum_{\nu=0}^{k-1} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\nu!} \frac{(\overline{z - \zeta})^{\nu}}{(\zeta - z)} \partial_{\bar{\zeta}}^{\nu} w(\zeta) \mathrm{d}\zeta$$
$$- \frac{1}{\pi} \int_{\mathbb{H}} \frac{1}{(k-1)!} \frac{(\overline{z - \zeta})^{k-1}}{(\zeta - z)} \partial_{\bar{\zeta}}^{k} w(\zeta) \mathrm{d}\xi \mathrm{d}\eta, \quad \textit{for } z \in \mathbb{H}.$$

Let us also note that for $z = x + iy \in \mathbb{H}, \gamma \in L^p(\mathbb{R}, \mathbb{C}), p \geq 1$

$$\lim_{z \to t_0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\gamma(t) y}{|t - z|^2} \, \mathrm{d}t = \gamma(t_0).$$

Now, we can define the higher-order Pompeiu operators $T_{0,n}$ for $\mathbb{H}$, as studied in [11]. For $w \in \mathcal{F}_k$ we can state

$$T_{0,n} w(z) = \iint_{\mathbb{H}} \partial_{\bar{\xi}}^n w(\xi) \frac{(\overline{z - \zeta})^{n-1}}{(z - \zeta)} \mathrm{d}\xi \mathrm{d}\eta.$$

We assume that, one of the regularity conditions

$$\lim_{R \to \infty} R^n M(w, R) = 0$$

or

$$|w(z)| = O\left(|z|^{-n-\delta}\right) \qquad \text{as } z \to \infty$$

hold. Then, we can give the following lemma.

Lemma 2.3 ([17]). *Let w be a continuous function satisfying above regularity conditions. Then,*
(i) $T_{0,n}w(z)$ exists as a Lebesgue integral and is continuous in $\mathbb{H}$.
(ii) In the sense of Sobolev derivatives

$$T^k w(z) := \partial_{\bar{z}} T_{0,n} w(z) = T_{0,n-1} w(z).$$

Also, it is easy to show that $\left\| T_{0,n} \right\|$ is bounded and $\partial_{\bar{z}}^l T_{0,n}$ are compact operators for $0 \leq l \leq n$ using similar computations given in [6, 11]. In a similar way we may prove that the operator Π^n which is given by

$$T_{n,0} f(z) := \Pi^n f(z) = \partial_z^n T_{0,0} f(z)$$

is a bounded integral operator [6], [11].

We may start to discuss the polyanalytic Dirichlet problem in the upper half plane. Now, we state the existence and uniqueness theorem for Dirichlet problem in the upper half plane [14].

Theorem 2.4. *For $w \in \mathcal{F}_n$, the Dirichlet problem for the inhomogeneous polyanalytic equation in the upper half plane $\mathbb{H}$,*

$$\partial_{\bar{z}}^n w = f \quad \text{in } \mathbb{H}, \quad \partial_{\bar{z}}^{\nu} w = \gamma_\nu \quad \text{on } \mathbb{R}, \quad 0 \leq \nu \leq n - 1$$

is uniquely solvable for $f \in L_{p,2}(\mathbb{H}, \mathbb{C})$, $p > 2$ satisfying regularity conditions above and

$$t^\lambda \gamma_\lambda(t) \in L^p(\mathbb{R}, \mathbb{C}) \cap C(\mathbb{R}, \mathbb{C}), \quad 0 \leq \lambda \leq n - 1$$

if and only if for $z \in \mathbb{H}$

$$\sum_{\lambda=\nu}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda-\nu}}{(\lambda - \nu)!} \int_{-\infty}^{\infty} \gamma_\lambda(t)(t - z)^{\lambda-\nu} \frac{\mathrm{d}t}{t - \bar{z}} + \frac{(-1)^{n-\nu}}{(n - \nu - 1)!} \frac{1}{\pi} \int_{\mathbb{H}} f(\zeta)(\bar{\zeta} - z)^{n-1-\nu} \frac{\mathrm{d}\xi \mathrm{d}\eta}{(\zeta - \bar{z})} = 0.$$

Then, the solution is given by

$$w(z) = \sum_{\lambda=0}^{n-1} \frac{1}{2\pi i} \frac{(-1)^\lambda}{\lambda!} \int_{-\infty}^{\infty} \gamma_\lambda(t)(t - \bar{z})^\lambda \frac{\mathrm{d}t}{(t - z)}$$
$$+ \frac{(-1)^n}{(n - 1)!} \frac{1}{\pi} \int_{\mathbb{H}} f(\zeta)(\overline{\zeta - z})^{n-1} \frac{\mathrm{d}\xi \mathrm{d}\eta}{\zeta - z}.$$

From this representation we may define the relevant Pompeiu operator, choosing homogeneous boundary conditions:

$$T_{0,n}(\partial_{\bar{z}} w) = \frac{(-1)}{(n - 1)!} \frac{1}{\pi} \int_{\mathbb{H}} f(z) \frac{(\bar{\xi} - \bar{z})^{n-1}}{\xi - z} d\xi d\eta.$$

The general form of polyanalytic linear equation is

$$\partial_{\bar{z}}^k w + \sum_{j=1}^{k} \left[q_{1j}(z) \partial_z^j \partial_{\bar{z}}^{k-j} w + q_{2j} \partial_{\bar{z}}^j \partial_z^{k-j} \bar{w} \right] + \sum_{l=0}^{k} \sum_{m=0}^{l} \left[a_{ml}(z) \partial_z^m \partial_{\bar{z}}^{l-m} w + b_{ml}(z) \partial_z^m \partial_{\bar{z}}^{l-m} \bar{w} \right] = f(z), \ z \in \mathbb{H}. \tag{2.1}$$

Substituting the function w by $T^k w = g$ we get

$$\left(I + \widehat{\Pi} + \widehat{K}\right) g = f,$$

where

$$\widehat{\Pi} g = \sum_{j=1}^{k} \left(q_{1j} \Pi^j g + q_{2j} \overline{\Pi^j g}\right)$$

is bounded and

$$\widehat{K} g = \sum_{l=0}^{k} \sum_{m=0}^{l} \left[a_{ml} \partial_z^m T^{k-l+m} g + b_{ml} \partial_{\bar{z}}^m \overline{T^{k-l+m} g}\right]$$

is compact. Then, we use Fredholm alternative to arrive the solution of the Equation (2.1). This technique may be employed for Dirichlet, Neumann, Robin and Schwarz boundary value problems over any regular domain in $\mathbb{C}$.

3. Nonlinear Equations

In this section, we give some solution techniques of homogeneous Dirichlet problems for nonlinear equations.

3.1. First Kind of Nonlinear Boundary Value Problems. Let us start with the inhomogeneous Dirichlet problem having homogeneous conditions:

$$w_{\bar{z}} = f(z) w^{\alpha}, \ \alpha \in \mathbb{R}, \ z \in \mathbb{H}, \ \alpha \neq 1, 0$$
$$w(x) = 0.$$

This equation may be converted into

$$w^{-\alpha} w_{\bar{z}} = f(z)$$

or

$$\partial_{\bar{z}} \left(\frac{1}{1-\alpha} w^{1-\alpha}\right) = f(z).$$

Now, let us call

$$\frac{1}{1-\alpha} w^{1-\alpha} = w_1(z),$$

then

$$\partial_{\bar{z}} w_1 = f(z),$$
$$w_1(z) = T f(z),$$

or

$$\frac{1}{1-\alpha} w^{1-\alpha} = T f(z).$$

Thus,

$$w(z) = \left[(1-\alpha) T f(z)\right]^{\frac{1}{1-\alpha}}.$$

3.2. Second Kind of Nonlinear Boundary Value Problems. The inhomogeneous Dirichlet problem having homogeneous boundary conditions in this case is

$$w_{\bar{z}} + f_1(z) w = f_2(z) w^{\alpha}, \ \alpha \neq 0, 1$$
$$w(x) = 0.$$

Let us start with

$$\partial_{\bar{z}} \left(\frac{1}{1-\alpha} w^{1-\alpha}\right) + f_1(z) w^{1-\alpha} = f_2(z).$$

Defining $\frac{1}{1-\alpha} w^{1-\alpha} = w_1$, we get

$$\partial_{\bar{z}} w_1 + f_1(z)(1-\alpha) w_1 = f_2(z)$$
$$\partial_{\bar{z}} \left[w_1 e^{\int (1-\alpha) f_1(z) d\bar{z}}\right] = f_2(z) e^{\int (1-\alpha) f_1(z) d\bar{z}}.$$

Hence,

$$\frac{1}{1-\alpha} w^{1-\alpha} e^{\int (1-\alpha) f(z) d\bar{z}} = T\left(f_2(z) e^{\int (1-\alpha) f_1(z) d\bar{z}}\right)$$

or

$$w = \left[\frac{T\left(f_2(z)e^{\int(1-\alpha)f_1(z)d\bar{z}}\right)(1-\alpha)}{e^{\int(1-\alpha)f_1(z)d\bar{z}}} \right]^{\frac{1}{1-\alpha}},$$

$$w(z) = \left[e^{-\int(1-\alpha)f_1(z)d\bar{z}}(1-\alpha)T\left(f_2(z)e^{\int(1-\alpha)f_1(z)d\bar{z}}\right)\right]^{\frac{1}{1-\alpha}}.$$

3.3. **Third Kind of Nonlinear Boundary Value Problems.** The inhomogeneous Dirichlet problem with homogeneous boundary conditions in this case is

$$w_{\bar{z}} + f_1(z)w + f_2(z)w^2 = f_3(z) \tag{3.1}$$

$$w(x) = 0.$$

To find a solution we should eliminate $f_3(z)$. Thus, we assume that a function $w = u(z)$ satisfies (3.1). Then, start with $w = w_1(z) + u(z)$ to eliminate $f_3(z)$. Substitute this function in (3.1):

$$w_{1\bar{z}} + u_{\bar{z}} + f_1(z)(w_1 + u) + f_2(z)(w_1 + u)^2 = f_3(z)$$

and

$$w_{1\bar{z}} + f_1 w_1 + f_2\left(w_1^2 + 2uw_1\right) + \left(u_{\bar{z}} + f_1 u + f_2 u^2 - f_3\right) = 0$$

or

$$w_{1\bar{z}} + (f_1 + 2uf_2)w_1 + f_2 w_1^2 = 0.$$

Hence, (3.1) is reduced into second kind nonlinear boundary value problem.

3.4. **Nonlinear Boundary Value Problems.** Now, we take the Dirichlet boundary value problem in the upper half plane $\mathbb{H}$ for

$$\partial_{\bar{z}}^n w = F(z, w, D^{\alpha_1}w, ..., D^{\alpha_n}w) \tag{3.2}$$

with homogeneous boundary conditions, in which $D = (\partial_z, \partial_{\bar{z}}),\ \alpha_j = (k,l),\ \ |\alpha_j| = k + l = j;\ j = 0,1,2,...,n$ and

$$D^{(k,l)}w = \frac{\partial^{k+l}w}{\partial z^k \partial \bar{z}^l} = \partial_z^k \partial_{\bar{z}}^l w$$

for $k, l \in \mathbb{N}_0$ and $0 \le k + l \le n$ with the restriction $(k,l) \ne (0,n)$. To simplify the notation we use

$$\{D^{\alpha_1}w, D^{\alpha_2}w, ..., D^{\alpha_n}w\} = \left\{\partial_z^k \partial_{\bar{z}}^l w\right\} = \{w_{k,l}\}.$$

Thus, the equation (3.2) may be written as

$$\partial_{\bar{z}}^n w = F(z, \{w_{k,l}\}).$$

We assume that, the function F holds the following conditions [3];

(1) The function $F(z, \{w_{k,l}\})$ is continuous with respect to all of its variables,

(2) For any multiple $\left(z, \left\{w_{k,l}^*\right\}\right)$ where $\left\{w_{k,l}^*\right\} \in L^p(\mathbb{H}),\ p > 2$,
$F\left(z, \left\{w_{k,l}^*\right\}\right) \in L^p(\mathbb{H})$ holds

(3) The function $F(z, \{w_{k,l}\})$ satisfies the Lipschitz condition with respect to the variables $\{w_{k,l}\}$.

Besides we know that we can convert

$$\partial_{\bar{z}}^n w = F(z, \{w_{k,l}\})$$

into the integral operator

$$w = T_{0,n}F = \frac{(-1)}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{H}} F(\xi,\eta)\frac{(\bar{\xi}-\bar{z})^{n-1}}{\xi - z}d\xi d\eta.$$

Thus, we find the solution.

4. Conclusion

In this article, we have given some techniques for the solutions of nonlinear complex partial differential equations.

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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