Middle East Journal of Science

https://dergipark.org.tr/mejs

MEJS

e-ISSN:2618-6136

Research Article

CLOSED BKS-TYPE UNIVERSES AND DIRAC SPIN EFFECT IN THE RAINBOW GRAVITY

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Abstract: *The result related to astrophysical datasets suggest that our universe has recently entered a phase of accelerated expansion. This accelerated expansion is not a situation predicted by the general theory of relativity. Therefore, the emergence of alternative approaches to general relativity has become inevitable. Modifying general relativity and absolute parallelism theory are just two of these theories. In addition, with the discovery of gravitational waves, the need for a view that includes gravitational quantum contributions arose. In this context, rainbow gravity has an approach that also offers quantum contributions to the theory of general relativity and absolute parallelism. In this study, axial vector torsion is calculated for BKS-type universe models using the rainbow gravity formalism. With the calculations made, the vector part and axial vector part components of the torsion tensor are obtained. The spin process, which contributes to the Dirac particle, is also investigated using the rainbow gravitational theory. However, since the obtained axial vector fragment is in time-like form, it is concluded that the spin vector of the Dirac particle is constant. The axial part of the torsion tensor for general BKS-type universe models is calculated and presented in a table for some well-known rainbow functions.*

Keywords: *Absolute parallelism, rainbow gravity, BKS-type universes.*

Received: April 8, 2023 *Accepted: June 7, 2023*

1. Introduction

Torsion is the basis of absolute parallelism theory, which is used as an alternative to the general theory of relativity. While the dynamic basis quantity of the general theory of relativity is the metric tensor, the fundamental quantity of the absolute parallelism theory is tetrads. The unify of gravitational and electromagnetic interaction underlies the absolute parallelism theory by Einstein [1-2]. Weitzenböck connections must be considered when using torsion instead of curvature [3-4]. The torsion tensor describes the textural deformation of space-time, such as the axial-vector part showing how much the axial symmetry deviates from the spherical symmetry [5-6].

Although general relativity and absolute parallelism theories offer some answers to experimental and theoretical astrophysical results, they do not include quantum contributions. Especially with the discovery of gravitational waves, the need for quantum gravity theory has become an indisputable reality. In this context, one of the prominent theories in the literature is the rainbow formalism of gravity [7-8]. According to this formalism, the energy of a test particle creates an effect in the space-time fabric. Thus, a distribution relation with a variable of the form $\epsilon = \frac{E}{E_{Pl}}$ is defined as follows:

$$f_1^2(\epsilon)E^2 - f_2^2(\epsilon)p^2 = m^2. \quad (1)$$

Here $E, m,$ and p are the energy, mass, and momentum of the tested particle, respectively. Also E_{Pl} is represented by the energy of Planck. $f_1(\epsilon)$ and $f_2(\epsilon)$ are known as rainbow functions [7-8]. Recently, many studies have shown the effect of rainbow gravity. The rainbow gravity effect has been studied when the black hole is modified by a particle carrying energy and electric charge [9]. However, many studies have investigated the thermodynamic properties of black holes in the rainbow gravitational framework [10-17]. Various physical properties have been analyzed considering particle equations (Klein-Gordon, Dirac, Photon, etc.) within the framework of rainbow gravity [18-20]. In addition, the thermodynamic phase transition was investigated by applying a quantum correction to the space-time metric in the rainbow of gravity of the Schwarzschild black hole [21]. There are studies examining black string solutions [22] and investigating Hawking radiation from a modified Schwarzschild black hole [23] by considering rainbow gravity.

The Dirac equation can be written in Weitzenböck geometry as below:

$$[h_i{}^{\alpha}\tilde{\gamma}^i(\partial_\mu + \Gamma_\mu) + m]\Psi = 0 \quad (2)$$

where $h_i{}^{\alpha}$ is the tetrad field and $\tilde{\gamma}^i$ are the flat Dirac matrices [24]. The spin connection is represented by Γ_μ and defined as;

$$\Gamma_\mu = \frac{1}{8}[\tilde{\gamma}^i, \tilde{\gamma}^j]h_i{}^{\sigma}h_{j\sigma;\mu} \quad (3)$$

where ";" denotes covariant derivative. The relationship between spin connections (Γ_μ) and vector part (V_μ) and axial vector part (A_μ) is given by [1].

$$\Gamma_\mu = \frac{V_\mu}{2} - \frac{3i}{4}A_\mu\tilde{\gamma}_5. \quad (4)$$

Here V_μ and A_μ are defined as below respectively,

$$V_\mu = T^{\lambda}{}_{\lambda\mu}, \quad (5)$$

$$A_\mu = \frac{\varepsilon^{\mu\nu\alpha\beta}}{6}T_{\nu\alpha\beta}. \quad (6)$$

$\varepsilon^{\mu\nu\alpha\beta}$ is defined as an antisymmetric Levi-Civita tensor ($\varepsilon^{0123} = 1$) and is related to skew-symmetric tensor ($\delta^{\mu\nu\alpha\beta}$) as follows:

$$\delta^{\mu\nu\alpha\beta} = \frac{\varepsilon^{\mu\nu\alpha\beta}}{\sqrt{-g}}, \quad (7)$$

where g is a determinant of metric tensor ($g_{\mu\nu}$). Variation of the semiclassical spin vector ($\vec{S}$) of Dirac particle with time in terms of space-like axial vector torsion ($\vec{A}$) and spin vector is given by [25]

$$\frac{d\vec{S}}{dt} + \frac{3}{2}\vec{A} \times \vec{S} = \vec{0}. \quad (8)$$

This paper is organized as follows: considering the rainbow formalism, the Dirac spin effect in closed Bianchi Kantowski-Sachs type (BKS-Type) space-time models will be evaluated in the next section. Then calculations will be given in the results and discussion. Finally, the conclusion is devoted to the interpretations of the main results of our research.

2. Materials And Methods

According to the McCallum diagram [26], spatially and homogenous closed universes have been reorganized after the pioneering work of Bianchi [27].

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j \quad (9)$$

The list of BKS-type space-time line elements [28] could be given as

Table 1. The list of Kantowski-Sachs and Bianchi-type space-time line elements (0≤m≤1)

BKS-Types	$g_{ij}dx^i dx^j$
Kantowski-Sachs (KS)	$dx^2 + dy^2 + \sin^2 y\, dz^2$
Bianchi-I (B1)	$dx^2 + dy^2 + dz^2$
Bianchi-II (B2)	$dx^2 + dy^2 + (1 + y^2)dz^2 - 2ydxdz$
Bianchi-IV (B4)	$e^{2y}(1 + y^2)dx^2 + dy^2 + e^{2y}dz^2 - 2ye^{2y}dxdz$
Bianchi-V (B5)	$e^{2y}dx^2 + dy^2 + e^{2y}dz^2$
Bianchi-VI (B6)	$e^{2(m-1)y}dx^2 + dy^2 + e^{(m+1)}dz^2$
Bianchi-VII (B7)	$e^{2my}dx^2 + dy^2 + e^{2my}dz^2$
Bianchi-VIII (B8)	$(1 + 2\sinh^2 y)dx^2 + dy^2 + dz^2 + 2\sinh y\, dxdz$
Bianchi-IX (B9)	$dx^2 + dy^2 + dz^2 - 2\sin y\, dxdz$

Now a general form of BKS-type space-time metric could be written in the following format

$$ds^2 = -dt^2 + \mathrm{R}_1^2(y)dx^2 + dy^2 + \mathrm{R}_2^2(y)dz^2 - 2\mathrm{R}_3(y)dxdz. \tag{10}$$

Introducing rainbow functions to general BKS-type metric ($dt \to \frac{dt}{f_1}, dx^i \to \frac{dx^i}{f_2}$) creates the equation (10):

$$ds^2 = -\frac{1}{f_1^2(\epsilon)}dt^2 + \frac{\mathrm{R}_1^2(y)}{f_2^2(\epsilon)}dx^2 + \frac{1}{f_2^2(\epsilon)}dy^2 + \frac{\mathrm{R}_2^2(y)}{f_2^2(\epsilon)}dz^2 - 2\frac{\mathrm{R}_3(y)}{f_1^2(\epsilon)}dxdz. \tag{11}$$

The metric tensor and its reverse are written as follows:

$$g_{\mu\nu} = -\frac{1}{f_1^2(\epsilon)}\delta_\mu^0\delta_\nu^0 + \frac{\mathrm{R}_1^2(y)}{f_2^2(\epsilon)}\delta_\mu^1\delta_\nu^1 + \delta_\mu^2\delta_\nu^2 + \frac{\mathrm{R}_2^2(y)}{f_2^2(\epsilon)}\delta_\mu^3\delta_\nu^3 - \frac{\mathrm{R}_3(y)}{f_1^2(\epsilon)}(\delta_\mu^1\delta_\nu^3 + \delta_\mu^3\delta_\nu^1) \tag{12}$$

$$g^{\mu\nu} = -f_1^2(\epsilon)\delta_0^\mu\delta_0^\nu + \frac{f_2^2(\epsilon)\mathrm{R}_2^2(y)}{\mathrm{R}_1^2(y)\mathrm{R}_2^2(y)-\mathrm{R}_3^2(y)}\delta_1^\mu\delta_1^\nu + f_2^2(\epsilon)\delta_2^\mu\delta_2^\nu + \frac{f_2^2(\epsilon)\mathrm{R}_1^2(y)}{\mathrm{R}_1^2(y)\mathrm{R}_2^2(y)-\mathrm{R}_3^2(y)}\delta_3^\mu\delta_3^\nu$$
$$+\frac{f_2^2(\epsilon)\mathrm{R}_3^2(y)}{\mathrm{R}_1^2(y)\mathrm{R}_2^2(y)-\mathrm{R}_3^2(y)}\left(\delta_1^\mu\delta_3^\nu+\delta_3^\mu\delta_1^\nu\right). \tag{13}$$

Using $g_{\mu\nu} = \eta_{ij}h^i{}_\mu h^j{}_\nu$ relation, the tetrad components of the general BKS-type metric can be obtained in a matrix form as below:

$$h^i{}_\mu = \begin{pmatrix} \frac{1}{f_1} & 0 & 0 & 0 \\ 0 & \frac{R_1}{f_2} & 0 & -\frac{R_3}{f_2} \\ 0 & 0 & \frac{1}{f_2} & 0 \\ 0 & 0 & 0 & \frac{\Im}{f_2 R_1} \end{pmatrix}, \quad h_i{}^\mu = \begin{pmatrix} f_1 & 0 & 0 & 0 \\ 0 & \frac{f_2}{R_1} & 0 & 0 \\ 0 & 0 & f_2 & 0 \\ 0 & \frac{f_2 R_3}{r_1 \Im} & 0 & \frac{f_2 R_1}{\Im} \end{pmatrix}, \tag{14}$$

where we introduced the definition $\Im^2 = R_1^2 R_2^2 - R_3^2$.

The axial and vector part depends on the torsion tensor via the Weitzenböck connection ($\Gamma^\lambda{}_{\mu\nu}$) which is defined as follows [29]:

$$\mathrm{T}^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu}, \tag{15}$$

and

$$\Gamma^\lambda{}_{\mu\nu} = h_i{}^\lambda \partial_\nu h^i{}_\mu\,. \tag{16}$$

3. 3. Results and Discussions

Considering equation (16), the corresponding non-vanishing components of the Weitzenböck connection are found as follows:

$$\Gamma^{1}{}_{12} = \frac{\partial_y R_1}{R_1}, \tag{17}$$

$$\Gamma^{1}{}_{32} = \frac{\Im \partial_y R_3 - R_3 \partial_y \Im}{R_1^2 \Im}, \tag{18}$$

$$\Gamma^{3}{}_{32} = \frac{R_1 (R_1 \partial_y \Im - \Im \partial_y R_1)}{R_1^2 \Im}. \tag{19}$$

The non-zero components of the antisymmetric torsion tensor become:

$$\mathrm{T}^{1}{}_{12} = -\mathrm{T}^{1}{}_{21} = -\frac{\partial_y R_1}{R_1}, \tag{20}$$

$$\mathrm{T}^{1}{}_{23} = -\mathrm{T}^{1}{}_{32} = \frac{R_3 \partial_y \Im - \Im \partial_y R_3}{R_1^2 \Im}, \tag{21}$$

$$\mathrm{T}^{3}{}_{23} = -\mathrm{T}^{3}{}_{32} = \frac{\partial_y \Im}{\Im} - \frac{\partial_y R_1}{R_1}. \tag{22}$$

Taking account into equations (5-6) and (20-22), the non-vanishing vector part and the axial vector part of torsion are obtained as follows:

$$V_2(y) = -\frac{\partial_y \Im}{\Im} = \frac{R_3 \partial_y R_3 - R_1 R_2 (R_2 \partial_y R_1 + R_1 \partial_y R_2)}{R_1^2 R_2^2 - R_3^2}, \tag{23}$$

$$A_0(y) = \frac{f_1 f_2 (2R_3 \partial_y R_1 + R_1 \partial_y R_3)}{3 R_1 \Im} = \frac{f_1 f_2 (2R_3 \partial_y R_1 + R_1 \partial_y R_3)}{3 R_1 \left(R_1^2 R_2^2 - R_3^2\right)^{\frac{1}{2}}}. \tag{24}$$

According to equation (24) axial vector, part of the axial vector torsion behaves time-like form, and space-like form vanishes:

$$\vec{A}(y) = \vec{0}, \tag{25}$$

so the spin vector of the Dirac particle behaves as a constant.

3.1. Special cases

For a particular case discussion of our results, we will use some well-known rainbow functions in the literature. Table 2 shows some rainbow functions frequently encountered in the literature, and the corresponding axial part of the torsion tensor is given for some BKS-type models.

Table 2. Some popular rainbow functions and corresponding axial parts of the torsion tensor.

Cases	Rainbow Functions [30-32] f_1	f_2	BKS-Type	Axial Part
1	$(1-a_3\epsilon)^{-1}$	1	B2	$\frac{e^{2y}(1+2y)}{3\sqrt{1-(e^{4y}-2)y^2+y^4}(a_3\epsilon-1)}$
			B4	$\frac{(1-y^2)}{3(1+y^2)(a_3\epsilon-1)}$
			B6	0
			B8	$\frac{1-2\text{Sech}(2y)}{3-3a_3\epsilon}$
2	1	$1+\frac{\epsilon}{2}$	B2	$-\frac{e^{2y}(1+2y)(2+\epsilon)}{6\sqrt{1-(-2+e^{4y})y^2+y^4}}$
			B4	$\frac{(y^2-1)(2+\epsilon)}{6(1+y^2)}$
			B6	0
			B8	$\frac{1}{6}(2+\epsilon)[1-2\text{Sech}(2y)]$
3	$1+\frac{\epsilon}{2}$	$1+(2\epsilon)^{-1}$	B2	$-\frac{e^{2y}(1+2y)(2+\epsilon)(1+2\epsilon)}{12\sqrt{1-(-2+e^{4y})y^2+y^4}\epsilon}$
			B4	$\frac{(y^2-1)(2+\epsilon)(1+2\epsilon)}{12(1+y^2)\epsilon}$
			B6	0
			B8	$\frac{(1+2\epsilon)[1-2\text{Sech}(2y)]}{6\epsilon}$
4	$(1-a_4\epsilon)^{-1}$	$(1-a_4\epsilon)^{-1}$	B2	$-\frac{e^{2y}(1+2y)}{3\sqrt{1-(-2+e^{4y})y^2+y^4}(-1+\epsilon a_4)^2}$
			B4	$\frac{(y^2-1)}{3(1+y^2)(a_4\epsilon-1)^2}$
			B6	0
			B8	$\frac{1-2\text{Sech}(2y)}{3(a_4\epsilon-1)^2}$

4. Conclusions

Dirac spin effects for various space-time models are a frequently studied topic in the literature [33-36]. In particular, it plays an essential role in developing the theory of absolute parallelism, which is presented as an alternative to general relativity. As can be seen in equation (23), only one component of the vector part of axial vector torsion is non-zero. However, the vector part component has no dependency on the rainbow function. The axial vector part of the axial vector torsion has only a time-like form. However, it does not have a space-like piece. Since the spin vector of the Dirac particle depends on the space-like components of the axial vector torsion, the variation of the spin vector over time remains constant. However, dependence on rainbow functions is observed within the axial vector part. Therefore, the energy of the test particle affects the axial vector torsion. This effect is clearly shown in Table 2:

- For the Bianchi type VI model, the rainbow functions have no effect as the axial vector part is zero.
- According to case 1, the energy of the test particle exerts a reducing effect on the axial vector part for B2, B4, and B8 space-time models.

- For case 2, the energy of the test particle increases the axial vector part for B2, B4, and B8 space-time models.
- Considering the 3rd case, the energy of the test particle increases the axial vector part for B2 and B4 type space-time models and decreases for B8 type space-time model models.
- Finally, considering the 4th case, the energy of the test particle reduces the axial vector part for all the space-time models given in the table.

Ethical Statements

The author declares that this document does not require ethics committee approval or special permission.

Conflict of interest

Author(s) declare no conflict of interest.

Authors Contributions

The author makes all contributions to the manuscript.

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