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# On the Properties of *r*-Circulant Matrices Involving Generalized Fermat Numbers

Bahar KULOĞLU *[1], Engin ESER[2], Engin ÖZKAN[3]

## Abstract

$r$-circulant matrices have applied in numerical computation, signal processing, coding theory, etc. In this study, our main goal is to investigate the r-circulant matrices of generalized Fermat numbers which are shown by $GR_{a,b,n} = a^{2^n} + b^{2^n}$ We obtain the eigenvalues, determinants, sum identity of matrices. Also we find upper and lower bounds for the spectral norms of generalized Fermat r-circulant matrices. Beside these, we present $GR^*_{a,b,r}$ matrix in the form of the Hadamard product of two matrices as $GR^*_{a,b,r} = A.B$. In addition, we get the right and skew-right circulant matrices for $r = 1$ and $r = -1$. Finally, we examine their different norms (Spectral and Euclidean) and limits for matrix norms.

**Keywords**: Fermat number, generalized Fermat numbers, norm (spectral and Euclidean), eigenvalues, circulant matrices

## 1. INTRODUCTION

Special numbers were created by grouping prime numbers within themselves. The best known of these are Pierre de Fermat and Marin Mersenne. Fermat numbers included pseudoprimes. First, Fermat conjectured that all numbers produced by $2^k + 1$ ($k$ is a nonnegative integer) are prime numbers ($k$ must be the power of 2).

It can be easily seen that the first 5 numbers are prime, but when it comes to the 6th number there is a problem. Later, Euler proved that this number has factors. So, it was a composite number. Fermat had made a calculation error. Now an open question: Are there any other numbers like this? Mersenne numbers in the form $2^k - 1$ were studied in antiquity because of their connection with perfect numbers. Euclid-Euler theorem gives this connection.

Later, Francois Proth studied Fermat numbers. Proth gave a generalized version of Fermat numbers in the form $k2^n + 1$ where $2^n > k$ for $n, k \in \mathbb{Z}^+$ and $k$ is an odd number. The Proth numbers are also known

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as $k$-Fermat numbers [1]. From this, it can be easily seen that the general forms of Proth numbers without any restrictions are $k$-Fermat numbers. The first terms of these numbers are as follows:

3, 5, 13, 17, 41, 97, 113, 193, 241, 257, 353, 449, 577, 641, 673, 769, 929, 1153, 1217, 1409, 1601, 2113, 2689, 2753, 3137, 3329, 3457, 4481, 4993, 6529, 7297, 7681, 7937, 9473, 9601, 9857

(Proth numbers are referenced in the On-Line Encyclopedia of Integer Sequences in OEIS as: A080076 [2])

Generalized Fermat numbers [3], denoted by $\{GR_{a,b,n}\}$ is defined by

$$GR_{a,b,n} = a^{2^n} + b^{2^n}, \quad (1)$$

where $a$ and $b$ are any relatively prime integers, $a > b > 0$, $a \geq 2$, $n \geq 0$. The first terms of these numbers as follow:

| $n$ | 0 | 1 | 2 | 3 | … |
|---|---|---|---|---|---|
| $GR_{a,b,n}$ | $a + b$ | $a^2 + b^2$ | $a^4 + b^4$ | $a^8 + b^8$ | … |

These matrices are shown by $C_r$ . Their inverse, conjugate, transposes, sums, and multiplications have been calculated in [4].

The algebraic properties of these matrices have been investigated in [5, 6]. For any $r$ complex number except zero, we can define those matrices. We can analyze its eigenvalues, Euclidean norm, spectral norm, determinants and inverse. $C_r$ is determined by its first-row element and $r$. Given $r = +1$ and $r = -1$ in recurrence relation, we obtain its eigenvalues and determinant for right and skew right circulant matrices.

Circulant matrices and $r$-circulant matrices including such as Fibonacci, Pell, Pell-Lucas numbers have been great interest. In several studies, eigenvalues, determinants, norms, bounds and inverses for these matrices are found. For instance, Bueno [7] presented eigenvalues and the determinant of the right circulant matrices with Pell numbers and Pell-Lucas numbers.

Solak [8] presented norms for circulant matrices including Fermant and Mersenne numbers. S.-Q., Shen et al. [9, 10] presented the bounds for $r$-circulant matrices with Fibonacci and Lucas numbers and also presented the spectral norms with $k$-Fibonacci and $k$-Lucas numbers. Zheng et al. [11] showed the exact inverse of circulant matrices with Fermat and Mersenne numbers. Bozkurt [12] solved the determinants of these matrices using matrix decomposition. Kumari et al. studied on the properties of the $r$-circulant matrices involving Mersenne and Fermat numbers [13].

Marin has also discussed examples of concrete usage areas of studies similar to those in [14, 15]. Pucanovic et al. by taking the $r-$circulant matrix and Chebyshev polynomials as the subject, handled the inputs of the r-circulant matrices by constructing them from Chebyshev polynomials [16]. In this paper, we study $r-$circulant matrices with generalized Fermat numbers.

**Lemma 1.1** [17, 18] Let $X = (x_{ij})$ be a matrix.

The Frobenius or Euclidean norm of $X$ is defined as

$$\|X\|_{E=F} = \sqrt{\sum_{i=1}^{a}\sum_{j=1}^{b}|x_{ij}|^2}.$$

The column norm of $X$ is defined as

$$\|X\|_1 = \max_{1 \leq j \leq n} \sum_{j=1}^{n} |x_{ij}|.$$

The row norm of $X$ is defined as

$$\|X\|_\infty = \max_{1 \le j \le m} \sum_{j=1}^{n} |x_{ij}|.$$

The spectral norm of a matrix $X$ defined as

$$\|X\|_2 = \sqrt{\gamma(X^*X)} = \mu_{max}(X).$$

where $\gamma(X^*X)$ denote the eigenvalues of $(X^*X)$ and $X^*$ is the conjugate transpose of $X$ and $\mu_{max}(X)$ is the largest singular value of $X$.

Matrix $X$ has the relationship between the norm values given below:

$$\frac{1}{\sqrt{n}}\|X\|_F \le \|X\|_2 \le \|X\|_F \quad (2)$$

**Lemma 1.2** [13] For $A = [a_{ij}] \in \mathbb{M}_{m,n}(\mathbb{C})$ , $B = [b_{ij}] \in \mathbb{M}_{m,n}(\mathbb{C})$ , If $C$ is the Hadamard product of A and B, then we get

$$\|C\|_2 \le m(A)n(B) \quad (3)$$

where $m(A) = max_{1\le i\le m}\sqrt{\sum_{j=1}^{n}|a_{ij}|^2}$ and $n(B) = max_{1\le j\le n}\sqrt{\sum_{j=1}^{m}|b_{ij}|^2}$.

**Definition 1.3** [19] For $r \in C - \{0\}$, a matrix $C_r$ is said to be $r$-circulant matrix if it is of the form and it is denoted by $C_r = Circ(r;\vec{c})$, where $\vec{c} = (c_0, c_1, \dots, c_{n-1})$ is the first row vector. For $r = 1$ and $r = -1$, we get the right and skew-right circulant matrices, respectively

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \dots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_2 & rc_3 & rc_4 & \dots & c_0 & c_1 \\ rc_1 & rc_2 & rc_3 & \dots & rc_{n1} & c_0 \end{bmatrix}$$

**Lemma 1.4** [19] Let $C_r$ be $r$-circulant matrices then its eigenvalues

$$\mu_i = \sum_{j=0}^{n-1} c_j\,(pw^{-i})^j, \quad i = 0,1,2,\dots,n-1$$

where $w$ is the $n$th root of unity and $p$ is the $n$th root of r.

**Lemma 1.5** [13] The Euclidean norm of $r$-circulant matrix $C_r$ is given by

$$\|C_r\|_E = \sqrt{\sum_{j=0}^{n-1}|C_j|^2[n - j(1 - |r|^2)]} \quad (4)$$

**Lemma 1.6** [7] For any $a$ and $b$ ,we get

$$\prod_{i=0}^{n-1}(a - bp_i w_{-i}) = a^n - rb^n \quad (5)$$

where $w$ is the n-th root of unity and $p$ is the n-th root of $r$.

**Lemma 1.7** [20] Determinant of the circulant matrix $C_n$ is

$$|C_n| = \prod_{i=0}^{n-1}\left(\sum_{j=0}^{n-1} c_j(t_l^j)\right)$$

where the entry $\{i,j\}$ is equal to the entry $\{i+l, j+l\}$ for $l = 1,2,\dots,$ $t_l = e^{\frac{2\pi l}{n}}$ are the $n$-th roots of unity and $p_i$ are $n^{th}$ root of of $r$. Using above lemmas, we can calculate the eigenvalues, the determinant, Euclidean norms, bounds for spectral norms of $r$-circulant matrices involving generalized Fermat numbers with arithmetic indices. We present many new identities for generalized Fermat numbers. In addition, r-circulant studies for Fermat and k-Fermat are handled in the most general format depending on the numbers $a$ , $b$ and $n$ on these number sequences.

## 2. GENERALIZED FERMAT $r$-CIRCULANT MATRIX

Let $u$ and $v$ be non-negative integers and $r \in C - \{0\}$ and show the Generalized Fermat $r$-circulant matrices with $GR_{a,b,r}$.

**Definition 2.1** Generalized Fermat $r$-circulant matrix is defined as
$GR_{a,b,r} = Circ(a,b,r;\vec{c})$ where first row vector is

$\vec{c} = (GR_{a,b,u}, GR_{a,b,u+v}, GR_{a,b,u+2v}, \dots, GR_{a,b,u+(n-1)v})$

ie., matrix of the form:

$$GR_{a,b,r} = \begin{bmatrix} GR_{a,b,u} & GR_{a,b,u+v} & \cdots & GR_{a,b,u+(n-1)v} \\ rGR_{a,b,u+(n-1)v} & GR_{a,b,u} & \cdots & GR_{a,b,u+(n-2)v} \\ \vdots & \vdots & \ddots & \vdots \\ rGR_{a,b,u+2v} & rGR_{a,b,u+3v} & \cdots & GR_{a,b,u+v} \\ rGR_{a,b,u+v} & rGR_{a,b,u+2v} & \cdots & GR_{a,b,u} \end{bmatrix}.$$

Let's show eigenvalues of $GR_{a,b,r}$ with $\lambda_i(GR_{a,b,r})$.

**Theorem 2.2** The eigenvalues of the generalized Fermat $r$-circulant matrix $GR_{a,b,r}$ are as follows.

$$\lambda_i(GR_{a,b,r}) = \sum_{j=0}^{n-1} \left(a^{2^{u+jv}} + b^{2^{u+jv}}\right)(pw^{-i})^j$$

**Proof.** If we substitute $a^{2^{u+jv}} + b^{2^{u+jv}}$ for $c_j$ in Lemma 1.4, the proof ends.

**Corollary 2.3**

$$\sum_{j=0}^{n-1} \left(a^{2^{u+jv}} + b^{2^{u+jv}}\right)(pw^{-i})^j = \sum_{j=0}^{n-1} \sqrt[n]{r^j(1)^{-ji}}\left(\left(a^{2^u}\right)^{(2^v)^j} + \left(b^{2^u}\right)^{(2^v)^j}\right)$$

**Proof.** Since $w$ is the $n$-th root of unity and $p$ is the $n$-th root of $r$, we get $(pw^{-i})^n = p^n(w^{-i})^n = r$.

$$\sum_{j=0}^{n-1} \left(a^{2^{u+jv}} + b^{2^{u+jv}}\right)(pw^{-i})^j = \left(\left(a^{2^u}\right)^{(2^v)^0} + \left(b^{2^u}\right)^{(2^v)^0}\right) + pw^{-i}\left(\left(a^{2^u}\right)^{(2^v)} + \left(b^{2^u}\right)^{(2^v)}\right)$$
$$+(pw^{-i})^2\left(\left(a^{2^u}\right)^{(2^v)^2} + \left(b^{2^u}\right)^{(2^v)^2}\right) + \cdots$$
$$+(pw^{-i})^{n-2}\left(\left(a^{2^u}\right)^{(2^v)^{n-2}} + \left(b^{2^u}\right)^{(2^v)^{n-2}}\right)$$
$$+(pw^{-i})^{n-1}\left(\left(a^{2^u}\right)^{(2^v)^{n-1}} + \left(b^{2^u}\right)^{(2^v)^{n-1}}\right)$$
$$= \sqrt[n]{r^0(1)^0}\left(\left(a^{2^u}\right)^{(2^v)^0} + \left(b^{2^u}\right)^{(2^v)^0}\right) + \sqrt[n]{r(1)^{-i}}\left(\left(a^{2^u}\right)^{(2^v)} + \left(b^{2^u}\right)^{(2^v)}\right)$$
$$+\sqrt[n]{r^2(1)^{-2i}}\left(\left(a^{2^u}\right)^{(2^v)^2} + \left(b^{2^u}\right)^{(2^v)^2}\right) + \cdots$$
$$+\sqrt[n]{r^{n-2}(1)^{-(n-2)i}}\left(\left(a^{2^u}\right)^{(2^v)^{n-2}} + \left(b^{2^u}\right)^{(2^v)^{n-2}}\right)$$
$$+\sqrt[n]{r^{n-1}(1)^{-(n-1)i}}\left(\left(a^{2^u}\right)^{(2^v)^{n-1}} + \left(b^{2^u}\right)^{(2^v)^{n-1}}\right)$$
$$= \sum_{j=0}^{n-1} \sqrt[n]{r^j(1)^{-ji}}\left(\left(a^{2^u}\right)^{(2^v)^j} + \left(b^{2^u}\right)^{(2^v)^j}\right).$$

If we simplify the operations for convenience, we obtain the following equation.

$$= \left(a^{2^u} + b^{2^u}\right) + pw^{-i}\left(\left(a^{2^u}\right)^{(2^v)} + \left(b^{2^u}\right)^{(2^v)}\right)$$
$$+(pw^{-i})^2\left(\left(a^{2^u}\right)^{(2^v)^2} + \left(b^{2^u}\right)^{(2^v)^2}\right) + \cdots$$
$$+\frac{rw^{2i}}{p^2}\left(\left(a^{2^u}\right)^{(2^v)^{n-2}} + \left(b^{2^u}\right)^{(2^v)^{n-2}}\right)$$
$$+\frac{rw^{i}}{p}\left(\left(a^{2^u}\right)^{(2^v)^{n-1}} + \left(b^{2^u}\right)^{(2^v)^{n-1}}\right).$$

**Corollary 2.4** For $r = 1$ and $r = -1$, we get eigenvalues for the generalized Fermat right circulant and skew-right circulant matrices respectively, so that

$$\lambda_i(GR_{a,b,1}) = \left(a^{2^u} + b^{2^u}\right)$$
$$+pw^{-i}\left(\left(a^{2^u}\right)^{(2^v)} + \left(b^{2^u}\right)^{(2^v)}\right)$$
$$+(pw^{-i})^2\left(\left(a^{2^u}\right)^{(2^v)^2} + \left(b^{2^u}\right)^{(2^v)^2}\right) + \cdots$$
$$+\frac{w^{2i}}{p^2}\left(\left(a^{2^u}\right)^{(2^v)^{n-2}} + \left(b^{2^u}\right)^{(2^v)^{n-2}}\right)$$

$$+\frac{w^i}{p}\left(\left(a^{2^u}\right)^{(2^v)^{n-1}}+\left(b^{2^u}\right)^{(2^v)^{n-1}}\right)$$

$$\lambda_i\left(GR_{a,b,-1}\right)=\left(a^{2^u}+b^{2^u}\right)$$
$$+pw^{-i}\left(\left(a^{2^u}\right)^{(2^v)}+\left(b^{2^u}\right)^{(2^v)}\right)$$
$$+\left(pw^{-i}\right)^2\left(a^{2^u}\right)^{(2^v)^2}+((b^{2u})^{2v})^2+\cdots$$
$$-\frac{w^{2i}}{\xi^2}\left(\left(a^{2^u}\right)^{(2^v)^{n-2}}+\left(b^{2^u}\right)^{(2^v)^{n-2}}\right)$$
$$-\frac{w^i}{\xi}\left(\left(a^{2^u}\right)^{(2^v)^{n-1}}+\left(b^{2^u}\right)^{(2^v)^{n-1}}\right)$$

where $\xi$ is the $n$-th root of -1.

**Theorem 2.5** For a positive integer $n$, we have

$$\sum_{i=0}^{n-1}\left(a^{2^u}+b^{2^u}\right)$$
$$+pw^{-i}\left(\left(a^{2^u}\right)^{(2^v)}+\left(b^{2^u}\right)^{(2^v)}\right)$$
$$+\left(pw^{-i}\right)^2\left(\left(a^{2^u}\right)^{(2^v)}+\left(b^{2^u}\right)^{(2^v)}\right)+\cdots$$
$$+\frac{rw^{2i}}{p^2}\left(\left(a^{2^u}\right)^{(2^v)^{n-2}}+\left(b^{2^u}\right)^{(2^v)^{n-2}}\right)$$
$$+\frac{rw^i}{p}\left(\left(a^{2^u}\right)^{(2^v)^{n-1}}+\left(b^{2^u}\right)^{(2^v)^{n-1}}\right)$$
$$=n\left(a^{2^u}+b^{2^u}\right).$$

**Proof.** To get the desired result, we need to know that the trace of any given square matrix is equal to the sum of the eigenvalues of that matrix. In that case

$$\sum_{i=0}^{n-1}\lambda_i\left(GR_{a,b,r}\right)=$$
$$\sum_{i=0}^{n-1}\left(a^{2^u}+b^{2^u}\right)$$
$$+pw^{-i}\left(\left(a^{2^u}\right)^{(2^v)}+\left(b^{2^u}\right)^{(2^v)}\right)$$
$$+\left(pw^{-i}\right)^2\left(\left(a^{2^u}\right)^{(2^v)}+\left(b^{2^u}\right)^{(2^v)}\right)+\cdots$$
$$+\frac{rw^{2i}}{p^2}\left(\left(a^{2^u}\right)^{(2^v)^{n-2}}+\left(b^{2^u}\right)^{(2^v)^{n-2}}\right)$$
$$+\frac{rw^i}{p}\left(\left(a^{2^u}\right)^{(2^v)^{n-1}}+\left(b^{2^u}\right)^{(2^v)^{n-1}}\right)$$

according to the expression given above,

$$\sum_{i=0}^{n-1}\lambda_i\left(GR_{a,b,r}\right)=n.GR_{a,b,u}$$
$$=n\left(a^{2^u}+b^{2^u}\right).$$

**The Sum Identity**

On setting $i=0, r=1$ and $p=1$ in eigenvalues for generalized Fermat numbers, the following identity is verified for these numbers.

$$\sum_{j=0}^{n-1}GR_{a,b,u+jv}$$
$$=\left(a^{2^u}+b^{2^u}\right)+\left(\left(a^{2^u}\right)^{(2^v)}+\left(b^{2^u}\right)^{(2^v)}\right)$$
$$+\left(\left(a^{2^u}\right)^{(2^v)^2}+\left(b^{2^u}\right)^{(2^v)^2}\right)^2+\cdots$$
$$+\left(\left(a^{2^u}\right)^{(2^v)^{n-2}}+\left(b^{2^u}\right)^{(2^v)^{n-2}}\right)$$
$$+\left(\left(a^{2^u}\right)^{(2^v)^{n-1}}+\left(b^{2^u}\right)^{(2^v)^{n-1}}\right)$$

## 3. NORM OF GENERALIZED FERMAT $r$-CIRCULANT MATRICES

When we take $u=0$ and $v=1$, we get

$$GR^*_{a,b,r}=\begin{bmatrix} GR_{a,b,0} & GR_{a,b,1} & \cdots & GR_{a,b,n-1} \\ rGR_{a,b,n-1} & GR_{a,b,0} & \cdots & GR_{a,b,n-2} \\ \vdots & \vdots & \ddots & \vdots \\ rGR_{a,b,2} & rGR_{a,b,3} & \cdots & GR_{a,b,1} \\ rGR_{a,b,1} & rGR_{a,b,2} & \cdots & GR_{a,b,0} \end{bmatrix},$$

with the help of the lemma given below, the sum of the squares of the generalized Fermat numbers will be used to obtain the norms of different matrices.

**Lemma 3.1** The finite sum of squares of the generalized Fermat numbers is given by

$$\sum_{j=0}^{n-1}\left(GR_{a,b,j}\right)^2=\sum_{i=0}^{n-1}(a^{2^i}+b^{2^i})^2$$

**Proof.**

$$\sum_{j=0}^{n-1}\left(GR_{a,b,j}\right)^2 = \sum_{j=0}^{n-1} a^{2^j} + 2(ab)^{2^j} + b^{2^j}$$
$$= a^{2^1} + a^{2^2} + \cdots + a^{2^n} + 2((ab)^1 + (ab)^2 + \cdots + (ab)^n) + b^{2^1} + b^{2^2} + \cdots + b^{2^n}$$
$$= \sum_{i=0}^{n-1}(a^{2^i} + b^{2^i})^2$$

**Theorem 3.2** The Euclidean norm for the generalized Fermat $r$-circulant matrices is given by

$$\left\|GR^*_{a,b,r}\right\|_E = \sum_{j=0}^{n-1}\left(GR_{a,b,j+1} + 2(ab)^{2^j}\right)[n - j(1 - |r|^2)]$$

**Proof.** By equation (4) we get

$$\left\|GR^*_{a,b,r}\right\|_E^{\ 2} = \sum_{j=0}^{n-1}\left(GR_{a,b,j+1} + 2(ab)^{2^j}\right)[n - j(1 - |r|^2)]$$

$$\sqrt{\sum_{i=0}^{n-1}(a^{2^i} + b^{2^i})^2} \le \left\|GR^*_{a,b,r}\right\|_2$$
$$\le \sqrt{a + b + |r|^2\left(\sum_{i=0}^{n-1}(a^{2^i} + b^{2^i})^2\right)}.\sqrt{\sum_{i=0}^{n-1}(a^{2^i} + b^{2^i})^2}$$

and for $|r| < 1$

$$|r|\sqrt{\sum_{i=0}^{n-1}(a^{2^i} + b^{2^i})^2} \le \left\|GR^*_{a,b,r}\right\|_2 \le \sqrt{n\sum_{i=0}^{n-1}(a^{2^i} + b^{2^i})^2}.$$

**Proof.** By Eq (4), the Euclidean norm is given as

$$\left\|GR^*_{a,b,r}\right\|_E^{\ 2} = \sum_{j=0}^{n-1}\left(GR_{a,b,j}\right)^2[n - j(1 - |r|^2)]$$

**State1.**

If $|r| \ge 1$, then from Lemma 3.1 we get

**Theorem 3.3** The bound for the spectral norm of the generalized Fermat $r$-circulant matrices is: for $|r| \ge 1$

$$\frac{\left\|GR^*_{a,b,r}\right\|_E}{\sqrt{n}} \ge \sqrt{\sum_{i=0}^{n-1}(a^{2^i} + b^{2^i})^2}$$

and from Eq. (1.2) we get

$$\left\|GR^*_{a,b,r}\right\|_2 \ge \sqrt{\sum_{i=0}^{n-1}(a^{2^i} + b^{2^i})^2}$$

Now to obtain the upper bound for the spectral norm, we write $GR^*_{a,b,r}$ in the form of the Hadamard product of two matrices.

$$A = \begin{bmatrix} GR_{a,b,0} & 1 & \dots & 1 & 1 \\ rGR_{a,b,n-1} & GR_{a,b,0} & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ rGR_{a,b,2} & rGR_{a,b,3} & \dots & GR_{a,b,0} & 1 \\ rGR_{a,b,1} & rGR_{a,b,2} & \dots & rGR_{a,b,n-1} & GR_{a,b,0} \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & GR_{a,b,1} & \dots & GR_{a,b,n-2} & GR_{a,b,n-1} \\ 1 & 1 & \dots & GR_{a,b,n-3} & GR_{a,b,n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & GR_{a,b,1} \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}$$

then clearly $GR^*_{a,b,r} = A \circ B$, where $\circ$ denotes the Hadamard product. Now,

$$u(A) = \max_{1 \le i \le n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2}$$

$$= \sqrt{\left(GR_{a,b,0}\right)^2 + |r|^2 \sum_{j=1}^{n-1} \left(GR_{a,b,j}\right)^2}$$

$$= \sqrt{a + b + |r|^2 \sum_{i=0}^{n-1} (a^{2^i} + b^{2^i})^2}$$

$$\sqrt{\sum_{i=0}^{n-1} (a^{2^i} + b^{2^i})^2} \le \left\|GR^*_{a,b,r}\right\|_2$$

$$\le \sqrt{a + b + |r|^2 \sum_{i=0}^{n-1} (a^{2^i} + b^{2^i})^2} \cdot \sqrt{\sum_{i=0}^{n-1} (a^{2^i} + b^{2^i})^2}.$$

**State 2.** If $|r| < 1$, then from Eq. (4) and Lemma 3.1 we get

$$\left\|GR^*_{a,b,r}\right\|_E^2 \ge \sum_{j=0}^{n-1} (n-j)|r|^2 \left|GR_{a,b,j}\right|^2 + |r|^2 \sum_{j=0}^{n-1} j \left|GR_{a,b,j}\right|^2 = n|r|^2 \left(\sum_{i=0}^{n-1} (a^{2^i} + b^{2^i})^2\right)$$

and

$$v(B) = \max_{1 \le j \le n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2}$$

$$= \sqrt{1 + \sum_{j=1}^{n-1} \left(GR_{a,b,j}\right)^2}$$

$$= \sqrt{1 + \sum_{i=0}^{n-1} (a^{2^i} + b^{2^i})^2}\,.$$

Thus, by Lemma 1.2, we get

$$\left\|GR^*_{a,b,r}\right\|_2 \le u(A)v(B) = \sqrt{a + b + |r|^2 \sum_{i=0}^{n-1} (a^{2^i} + b^{2^i})^2} \cdot \sqrt{\sum_{i=0}^{n-1} (a^{2^i} + b^{2^i})^2}$$

$$\frac{\left\|GR^*_{a,b,r}\right\|_E}{\sqrt{n}} \ge |r| \sqrt{\sum_{i=0}^{n-1} (a^{2^i} + b^{2^i})^2}$$

and from Eq. (2) we get

$$\left\|GR^*_{a,b,r}\right\|_2 \ge |r| \sqrt{\sum_{i=0}^{n-1} (a^{2^i} + b^{2^i})^2}.$$

We calculate the upper bound for the spectral norm of $GR^*_{a,b,r}$.

Let

$$N = \begin{bmatrix} GR_{a,b,0} & GR_{a,b,1} & \dots & GR_{a,b,n-2} & GR_{a,b,n-1} \\ GR_{a,b,n-1} & GR_{a,b,0} & \dots & \dots & GR_{a,b,n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ GR_{a,b,2} & GR_{a,b,3} & \dots & \dots & GR_{a,b,1} \\ GR_{a,b,1} & GR_{a,b,2} & \dots & \dots & GR_{a,b,0} \end{bmatrix}$$

hence, we have

$$M = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ r & 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & r & \dots & r & 1 \\ r & r & r & \dots & r & 1 \end{bmatrix}$$

then clearly $GR^*_{a,b,r} = M \circ N$, where $\circ$ denotes the Hadamard product. So,

$$s_1(M) = \max_{1 \le i \le n} \sqrt{\sum_{j=1}^{n} |m_{ij}|^2} = \sqrt{n}$$

$$t_1(N) = \max_{1 \leq j \leq n} \sqrt{\sum_{j=1}^{n} |n_{ij}|^2}$$

$$= \sqrt{\sum_{j=1}^{n} {GR_{a,b,j}}^2} = \sqrt{\sum_{i=0}^{n-1} (a^{2^i} + b^{2^i})^2}$$

hence, by Lemma 1.2, we have

$$\|GR^*_{a,b,r}\|_2 \leq s_1(M)t_1(N)$$
$$= \sqrt{n \sum_{i=0}^{n-1} (a^{2^i} + b^{2^i})^2}$$

thus

$$|r| \sqrt{\sum_{i=0}^{n-1} (a^{2^i} + b^{2^i})^2} \leq \|GR^*_{a,b,r}\|_2$$
$$\leq \sqrt{n \left( \sum_{i=0}^{n-1} (a^{2^i} + b^{2^i})^2 \right)}$$

as desired.

## 4. CONCLUSION

Based on the generalized Fermat numbers with similar recurrences, which have been studied less, the r-circulant matrices of these sequences were created. Euclidean, row and spectral norms, which are eigenvalues, determinants, and some special norm values, are discussed depending on this matrix. In addition, the lower and upper bounds of these matrices for the spectral norm were examined in closed form. In addition, right circulant and skew-right circulant matrices are examined for 1 and -1 values of $r$ depending on eigenvalues. Finally, some interesting results and sum properties are given.

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The authors of the paper declare that they comply with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that they do not make any falsification on the data collected. In addition, they declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

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