Research Article GU J Sci, Part A, 10(2): 232-240 (2023) 10.54287/gujsa.1280707

Gazi University
Journal of Science
PART A: ENGINEERING AND INNOVATION
http://dergipark.org.tr/gujsa

Identities for a Special Finite Sum Related to the Dedekind Sums and Fibonacci Numbers

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Keywords

Hardy Sums
Dedekind Sums
Simsek Sums
Fibonacci Numbers

Abstract

The origin of this article is to achieve original equations related to the special finite sum C(μ,β;1), which is connected with Dedekind, Hardy, Simsek, and many other finite sums. By using the analytic properties of this sum, many useful identities are established between the C(μ,β;1) sum and other well-known finite sums. Through the use of these identities, the reciprocity law of this sum is obtained. Furthermore, another reciprocity law of the sum C(μ,β;1) is presented for μ and β are particular Fibonacci numbers. This remarkable result establishes a connection between number theory and analysis.

Cite

Cetin, E. (2023). Identities for a Special Finite Sum Related to the Dedekind Sums and Fibonacci Numbers. *GU J Sci, Part A*, *10*(2), 232-240. doi:10.54287/gujsa.1280707

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Article Process

Submission Date	10.04.2023
Revision Date	15.05.2023
Accepted Date	17.05.2023
Published Date	26.06.2023

1. INTRODUCTION

The special finite sums have been widely studied by mathematicians due to their extensive applications in many other scientific fields. Lately, scientific developments have shown the importance of interdisciplinary studies. Therefore, the purpose of this research is to get beneficial connections for special finite sums. Cetin et al. (2014) introduced the sum $C(\mu,\beta;1)$, and by using the relationships between this particular sum and other familiar sums, new identities are obtained. At the end, a formula expressing the reciprocal relationship of the sum $C(\mu,\beta;1)$ has been provided, when μ and β are special Fibonacci numbers. This significant finding enables a fresh perspective for establishing new connections of analysis and number theory.

Definitions and notations concerning special finite sums are given below:

Let $\mathbb{N}=\{1,2,3,\cdots\}$, and $\mathbb{Z}=\mathbb{N}\cup\{0,-1,-2,-3,\cdots\}$. In this paper, we consider finite sums fundamentally involving the sawtooth function. This function is defined by

$$((y))=\begin{cases} y-[y]-\frac{1}{2}, & if\ y\notin\mathbb{Z}, \\ 0, & otherwise, \end{cases}$$

where $[y]$ denotes the largest integer $\leq y$ (Apostol, 1976; Berndt & Dieter, 1982). In the 19th century, Richard Dedekind made significant contributions to the fields of number theory and algebra. Based on the sawtooth function, he defined the Dedekind sums as follows:

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$$s(\mu,\beta)=\sum_{\gamma=1}^{\beta-1}\left(\left(\frac{\mu\gamma}{\beta}\right)\right)\left(\left(\frac{\gamma}{\beta}\right)\right), \tag{1}$$

assuming $\mu\in\mathbb{Z}$, and $\beta\in\mathbb{N}$ (Dedekind, 1892). The Dedekind sums have studied by many mathematicians and have invaluable properties (Rademacher & Grosswald, 1972; Goldberg, 1981; Sitaramachandrarao, 1987; Meyer, 2005; Simsek, 1993). However, this paper will provide limited information about the Dedekind sums according to the needs of the main results of this study. The next reciprocity law is a highly regarded property of Dedekind sums, and is widely studied:

$$s(\mu,\beta)+s(\beta,\mu)=-\frac{1}{4}+\frac{1}{12}\left(\frac{\mu}{\beta}+\frac{\beta}{\mu}+\frac{1}{\mu\beta}\right),$$

where μ and β are coprime positive integers (Dedekind, 1892). Although the first proof of the reciprocity law was provided by Richard Dedekind, many mathematicians subsequently proved it using different methods. One of the original proofs was given by the famous mathematician Hardy, who used contour integration (Hardy, 1905). In the same work, he also defined some new arithmetical sums, called the Hardy sums. These sums have many significant properties and are closely linked to the Dedekind sums and other famous special finite sums. Recently, Simsek (2003; 2006; 2009a) gave p-adic q-Hardy sums, and also generating functions for these sums. The Hardy sums are also deeply related to the finite $C(\mu,\beta;1)$ sum. Therefore, it is now time to introduce some of the Hardy sums and their essential properties that will be needed in the following sections. The Hardy sums $S(\mu,\beta)$ and $s_5(\mu,\beta)$ are expressed by

$$S(\mu,\beta)=\sum_{\gamma mod\beta}(-1)^{\gamma+1+\left[\frac{\mu\gamma}{\beta}\right]}, \tag{2}$$

$$s_5(\mu,\beta)=\sum_{\gamma mod\beta}(-1)^{\gamma+\left[\frac{\mu\gamma}{\beta}\right]}\left(\left(\frac{\gamma}{\beta}\right)\right), \tag{3}$$

where $\mu\in\mathbb{Z}$ and $\beta\in\mathbb{N}$ (Hardy, 1905).

The sum $s_5(\mu,\beta)$ is also given by

$$s_5(\mu,\beta)=\frac{1}{\beta}\sum_{\gamma=1}^{\beta-1}\gamma(-1)^{\gamma+\left[\frac{\mu\gamma}{\beta}\right]}, \tag{4}$$

when μ and β are odd integers (Berndt & Goldberg, 1984). Hardy (1905) proposed the following theorem of reciprocity for the $s_5(\mu,\beta)$ sum:

Theorem 1.1. Let $(\mu,\beta)=1$, and $\mu,\beta\in\mathbb{N}$. Assuming μ and β are both odd numbers, then

$$s_5(\mu,\beta)+s_5(\beta,\mu)=\frac{1}{2}-\frac{1}{2\mu\beta} \tag{5}$$

and if $\mu+\beta$ is odd then

$$s_5(\mu,\beta)=s_5(\beta,\mu)=0. \tag{6}$$

The following theorem shows the relations between Dedekind sum and the $s_5(\mu,\beta)$ sum (Pettet & Sitaramachandrarao, 1987).

Theorem 1.2. For $(\mu,\beta)=1$, the following equalities hold true:

$$s_5(\mu,\beta)=-10s(\mu,\beta)+4s(2\mu,\beta)+4s(\mu,2\beta), \quad if\ \mu+\beta\ is\ even, \tag{7}$$

$$s_5(\mu,\beta)=0, \quad if\ \mu+\beta\ is\ odd.$$

For the sum $S(\mu,\beta)$, the reciprocity law was proved by Hardy (1905) as given below:

Theorem 1.3. Let $(\mu,\beta)=1$, and $\mu,\beta\in\mathbb{N}$. Then

$$S(\mu,\beta)+S(\beta,\mu)=1 \quad if\ \mu+\beta\ is\ odd. \tag{8}$$

In consideration of (8), Apostol and Vu (1982) derived the next theorem.

Theorem 1.4. Let μ, β are odd simultaneously, and $(\mu,\beta)=1$, then

$$S(\mu,\beta)=S(\beta,\mu)=0. \tag{9}$$

The upcoming theorems establish a connection between the $s(\mu,\beta)$ sums and the $S(\mu,\beta)$ sums.

Theorem 1.5. Let $(\mu,\beta)=1$. Then

$$S(\mu,\beta)=8s(\mu,2\beta)+8s(2\mu,\beta)-20s(\mu,\beta), \quad if\ \mu+\beta\ is\ odd, \tag{10}$$

$$S(\mu,\beta)=0, \quad if\ \mu+\beta\ is\ even$$

(Pettet & Sitaramachandrarao, 1987).

Theorem 1.6. Let $(\mu,\beta)=1$. Then

$$S(\mu,\beta)=4s(\mu,\beta)-8s(\mu+\beta,2\beta), \tag{11}$$

if $\mu+\beta$ is odd and $\beta\in\mathbb{N}$ (Sitaramachandrarao, 1987).

The Simsek sum $Y(\mu,\beta)$ is defined by

$$Y(\mu,\beta)=4\beta\sum_{\gamma mod \beta}(-1)^{\gamma+\left[\frac{\mu r}{\beta}\right]}\left(\left(\frac{\gamma}{\beta}\right)\right) \tag{12}$$

(Simsek, 2009b).

An equation that connecting the sum $Y(\mu,\beta)$ and the sum $s_5(\mu,\beta)$ was provided as follows:

$$Y(\mu,\beta)=4\beta s_5(\mu,\beta) \tag{13}$$

(Simsek, 2009b).

The reciprocal relationship associated with the Simsek sum can be stated as follows:

$$\mu Y(\mu,\beta) + \beta Y(\beta,\mu) = 2\mu\beta - 2. \tag{14}$$

Main motivation of this study is to give several identities containing connections between the Fibonacci numbers and the certain family of the Finite sums. Therefore, we need the following properties of these numbers. Lately, Simsek (2023) described the following common expression

$$\frac{\sum_{v=0}^{k} q_v(y_v)t^v}{\sum_{v=1}^{j} p_v(y_v)t^v + 1} = \sum_{m=0}^{\infty} S_m(p_1(y_1), \cdots, p_j(y_j);\ q_0(y_0), q_1(y_1), \dots, q_k(y_k))t^m. \tag{15}$$

where $p_1(y_1), \cdots, p_j(y_j)$, $q_0(y_0), q_1(y_1), \dots, q_k(y_k)$ are any polynomials. For $j = 2$, setting $P_1(y_1) = P_2(y_2) = -1$ in Equation (15), we have the Fibonacci numbers. That is

$$F_m = S_m(-1,-1,0,\cdots,0;0,1,0\cdots,0) = S_m(-1,-1;0,1)$$

see for detail Simsek (2023). For the properties of the Fibaonacci numbers see also Koshy (2001). As a result, the following open question may arise:

Is it also possible to find and investigate new relationships among these new classes special of polynomials, numbers, and certain family finite sums including the $s(\mu,\beta)$ sums, the $S(\mu,\beta)$ sums, and the $Y(\mu,\beta)$ sums?

In 2005, Meyer showed that if $s(\mu,\beta) = s(\beta,\mu)$ then the pairs of integers $\{\mu,\beta\}$ is symmetric. He also gave many results for the Dedekind sums with the aid of this symmetric pair. The substitution of μ with F_{2m+1} and β with F_{2m+3} satisfies the symmetric pair property. In here F_k represents the k-th Fibonacci number. Thus, these type Fibonacci numbers satisfied this property.

In order to give our new results involving the symmetric pair, we need the next identity, which was given by Meyer (2005):

Theorem 1.7. Let $\mu,\beta \in Z$ with $(\mu,\beta) = 1$. If μ and β form a symmetric pair, then we have

$$s(\mu,\beta) = 0. \tag{16}$$

The author (Cetin, 2016a) gave the following formula involving a relation between the Simsek sum $Y(\mu,\beta)$ and the Fibonacci numbers:

$$\begin{aligned} &F_{2m+1}Y(F_{2m+1},F_{2m+3}) + F_{2m+3}Y(F_{2m+3},F_{2m+1}) \\ &\qquad = 2(F_{2m+1})^2 + 2(F_{2m+3})^2 - 4F_{2m+1}F_{2m+3}. \end{aligned} \tag{17}$$

The finite sum $C_1(\mu,\beta)$ is defined by

$$C_1(\mu,\beta) = \sum_{\gamma=1}^{\beta-1} (-1)^{r+\left[\frac{\mu\gamma}{\beta}\right]} \left(\left(\frac{\mu\gamma}{\beta}\right)\right) \tag{18}$$

where $\mu,\beta \in \mathbb{N}$ with $(\mu,\beta) = 1$ (Cetin et al., 2014).

This has many relations with other well known finite sums. Cetin (2016a; 2016b) gave some useful properties for this sum.

The finite sum $B_1(\mu,\beta)$ defined by

$$B_1(\mu,\beta)=\sum_{\gamma=1}^{\beta-1}(-1)^{\gamma+\left[\frac{\mu\gamma}{\beta}\right]}\left[\frac{\mu\gamma}{\beta}\right] \tag{19}$$

(Cetin et al., 2014).

The author gave many properties of the sum $B_1(\mu,\beta)$.

Relation among $B_1(\mu,\beta)$, $s_5(\mu,\beta)$, and $C_1(\mu,\beta)$ is given by the following theorem, which was proved by the author (Cetin, 2016a):

Theorem 1.8. If μ and β are relatively prime odd numbers and β is a positive integer, then

$$B_1(\mu,\beta)=\mu s_5(\mu,\beta)-C_1(\mu,\beta). \tag{20}$$

Many mathematicians investigated two and three term polynomial relations by reason of they are connected to the $s(\mu,\beta)$ sums, the $S(\mu,\beta)$ sums, and several othher special finite sums (Berndt & Dieter, 1982; Pettet & Sitaramachandrarao, 1987; Simşek, 1998). Cetin et al. (2014) defined new polynomials with the help of the *n* variable Carlitz polynomial. By taking partial derivative with respect to all variables, they gave some new equations involving some certain family of finite sums with open questions. They profoundly studied about equations and relations. Using them, they gave some original demonstration of the reciprocity laws for these sums.

In this paper, two term polynomial relation will be used for the next section. Therefore, it will be reminded as a corollary in below:

Corollary 1.9. Let $(\varphi,\rho)=1$, then

$$(t-1)\sum_{x=1}^{\varphi-1}t^{x-1}n^{\left[\frac{\rho x}{\varphi}\right]}+(n-1)\sum_{y=1}^{\rho-1}n^{y-1}t^{\left[\frac{\varphi y}{\rho}\right]}=t^{\varphi-1}n^{\rho-1}-1. \tag{21}$$

(21) is given by Berndt and Dieter (1982).

Cetin et al. (2014) introduced the sum $C(\mu,\beta;1)$ with the equation below:

Let $(\mu,\beta)=1$ and $\beta>0$, then

$$C(\mu,\beta;1)=\sum_{\gamma=1}^{\beta-1}\gamma(-1)^{\gamma+\left[\frac{\mu\gamma}{\beta}\right]}. \tag{22}$$

The relation between the sum $C(\mu,\beta;1)$ and the Hardy sum $S(\mu,\beta)$ is described by the following equality:

Let $\mu+\beta$ is odd, $\beta>0$, and $(\mu,\beta)=1$ (Cetin, 2018), then

$$C(\mu,\beta;1)=-\frac{\beta}{2}S(\mu,\beta). \tag{23}$$

One can express the connections the sum $C(\mu,\beta;1)$ with the Hardy sums $s_5(\mu,\beta)$ and $S(\mu,\beta)$ as follows (Cetin, 2023):

Let μ and β be integers with $(\mu,\beta)=1$. Then one can get the following identity:

$$C(\mu,\beta;1)=\beta s_5(\mu,\beta)-\frac{\beta}{2}S(\mu,\beta).$$

Let μ and β are positive odd integers. Then,

$$C(\mu,\beta;1)=\beta s_5(\mu,\beta). \tag{24}$$

2. RECIPROCITY LAWS FOR THE SUM $C(\mu,\beta;1)$

For the sum $C(\mu,\beta;1)$, a reciprocity law, and relations with other special sums are presented in this section. For this, we will use the Dedekind and the Hardy sums properties. With the aid of two term polynomial relation's partial derivatives, a different reciprocity law for the sum $C(\mu,\beta;1)$ is also obtained. And finally when μ and β are special Fibonacci numbers, a new reciprocity identity will be given for this sum.

The reciprocity laws for the sum $C(\mu,\beta;1)$ are given by the identities below.

Theorem 2.1. If $\mu+\beta$ is odd, $\beta>0$, and $(\mu,\beta)=1$, then

$$\mu C(\mu,\beta;1)+\beta C(\beta,\mu;1)=-\frac{\mu\beta}{2}$$

Proof. By combining (8) with (23), one can have,

$$-\frac{2}{\beta}C(\mu,\beta;1)-\frac{2}{\mu}C(\beta,\mu;1)=1.$$

If the last equation multiplied by $\mu\beta$, and divided by -2, then the required outcome is obtained.

Theorem 2.2. Let μ and β be positive and odd integers. Then the below equation holds true:

$$\mu C(\mu,\beta;1)+\beta C(\beta,\mu;1)=\frac{\mu\beta-1}{2}$$

Proof. To prove the theorem, the two term polynomial relation will be used. If the partial derivative of (21) is taken with regard to t, we get

$$\sum_{x=1}^{\mu-1}t^{x-1}n^{\left[\!\left[\frac{\beta x}{\mu}\right]\!\right]}+(t-1)\sum_{x=1}^{\mu-1}(x-1)t^{x-2}n^{\left[\!\left[\frac{\beta x}{\mu}\right]\!\right]}+(n-1)\sum_{y=1}^{\beta-1}\left[\!\left[\frac{\mu y}{\beta}\right]\!\right]n^{y-1}t^{\left[\!\left[\frac{\mu y}{\beta}\right]\!\right]-1}$$
$$=(\mu-1)t^{\mu-2}n^{\beta-1}.$$

Substitute $t=n=-1$, into the above equation, we have

$$S(\beta,\mu)+2C(\beta,\mu;1)+2B_1(\mu,\beta)=(\mu-1)(-1)^{\mu+\beta}. \tag{25}$$

Taking partial derivative of (21) with regard to n, we also get

$$(t-1)\sum_{x=1}^{\mu-1}\left[\!\left[\frac{\beta x}{\mu}\right]\!\right]t^{x-1}n^{\left[\!\left[\frac{\beta x}{\mu}\right]\!\right]-1}+(n-1)\sum_{y=1}^{\beta-1}(y-1)\,n^{y-2}t^{\left[\!\left[\frac{\mu y}{\beta}\right]\!\right]}+\sum_{y=1}^{\beta-1}n^{y-1}\,t^{\left[\!\left[\frac{\mu y}{\beta}\right]\!\right]}$$
$$=(\beta-1)t^{\mu-1}n^{\beta-2}.$$

Putting $t=n=-1$ in the above equation, we obtain

$$S(\mu,\beta)+2C(\mu,\beta;1)+2B_1(\beta,\mu)=(\beta-1)(-1)^{\mu+\beta} \tag{26}$$

is found. If the Equation (25) is multiplied by β, we get

$$\beta S(\beta,\mu)+2\beta C(\beta,\mu;1)+2\beta B_1(\mu,\beta)=\beta(\mu-1)(-1)^{\mu+\beta}. \tag{27}$$

Equation (26) is multiplied by μ, we obtain

$$\mu S(\mu,\beta)+2\mu C(\mu,\beta;1)+2\mu B_1(\beta,\mu)=\mu(\beta-1)(-1)^{\mu+\beta}. \tag{28}$$

Combining (27) and (28), allows us to assert the theorem.

Theorem 2.3. Let $\mu,\beta\in\mathbb{Z}$ with $\mu,\beta>0$, and $\{\mu,\beta\}$ is a symmetric pair. If $(\mu,\beta)=1$, $\mu=F_{6m-1}$ and $\beta=F_{6m+1}$ with $m\in\mathbb{N}$, where F_k is the k-th Fibonacci number then,

$$F_{6m-1}C(F_{6m-1},F_{6m+1};1)+F_{6m+1}C(F_{6m+1},F_{6m-1};1)$$
$$=\frac{(F_{6m-1})^2+(F_{6m+1})^2}{2}-F_{6m-1}F_{6m+1}$$

Proof. If we combine equation (13) with equation (24), then we have

$$4C(\mu,\beta;1)=Y(\mu,\beta).$$

If we use this last equation with equation (17), then we have the desired result.

3. IDENTITIES FOR THE SUM $C(\mu,\beta;1)$, THE SIMSEK SUM $Y(\mu,\beta)$, THE SUM $B_1(\mu,\beta)$, AND THE DEDEKIND SUMS

We now give relations among the sum $C(\mu,\beta;1)$, the Simsek sum $Y(\mu,\beta)$, the sum $B_1(\mu,\beta)$, and the Dedekind sums by the next theorems.

Theorem 3.1. If $\mu+\beta$ is odd, $\beta>0$, and $(\mu,\beta)=1$, then

$$C(\mu,\beta;1)=-4\beta s(\mu,2\beta)-4\beta s(2\mu,\beta)+10\beta s(\mu,\beta)$$

Proof. By combining (10) with (23) the proof is completed.

Theorem 3.2. If $\mu+\beta$ is odd, $\beta>0$, and $(\mu,\beta)=1$, then

$$C(\mu,\beta;1)=-2\beta s(\mu,\beta)+4\beta s(\mu+\beta,2\beta)$$

Proof. By combining (11) with (23) the intended outcome is achieved.

Theorem 3.3. Let $\mu,\beta\in\mathbb{N}$ be odd integers. Then

$$C(\mu,\beta;1)=-10\beta s(\mu,\beta)+4\beta s(2\mu,\beta)+4\beta s(\mu,2\beta)$$

Proof. By combining (7) with (23) the given equation is satisfied.

Theorem 3.4. Let $\mu,\beta\in\mathbb{N}$ be odd integers. Then the following result is derived:

$$C(\mu,\beta;1)=\frac{\beta}{\mu}B_1(\mu,\beta)+\frac{\beta}{2\mu}-\frac{1}{2\mu}$$

Proof. If the Equation (20) is multiplied by β, then combining final equation with (24) and

$$C_1(\mu,\beta)=\frac{1}{2}-\frac{1}{2\beta}$$

(see Cetin (2016a)-Theorem 2.2.), we achieve the intended outcome.

4. CONCLUSION

In light of the above, this article has presented new identities related to Dedekind sums and Fibonacci numbers, and explored its connections with other well-known sums. Through this paper, a reciprocity law for the sum $C(\mu, \beta; 1)$ was discovered when μ and β are special Fibonacci numbers, adding a new dimension to the study of these special sums. This reciprocity law has the potential to deepen our understanding of the behavior of these sums.

In future studies, different proof of this law is investigated in diverse branches of mathematics, such as algebraic geometry and modular forms, as a results far-reaching implications maybe obtain.

Due to many important applications of the certain classes of finite sums in many branches of mathematics, our future goal is to derive new reciprocity laws for more general cases of certain finite sums with their proofs methods. To achieve this, our primary focus will be on solving the open questions posed by Cetin et al. (2014). Solving these questions would make a significant contribution to the theory of the certain finite sums. In addition, Simsek (2022) has also identified valuable open questions in his recent article, and addressing these questions would yield a substantial contribution to the theory of certain finite sums, complementing our primary objective of deriving new reciprocity laws for more common scenarios. Furthermore, attempts will be made to find new connections with the trigonometric representations provided by Simsek (2010), and Milovanović and Simsek (2020).

CONFLICT OF INTEREST

The author declares no conflict of interest.

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