ANKARA SCIENCE UNIVERSITY, RESEARCHER
Vol. 3, No. 1 July 2023
e-ISSN: 2717-9494
Research Article/Araştırma Makalesi
Doi: https://doi.org/10.55185/researcher.1281521

On Solutions of a Parabolic Partial Differential Equation of Neutral Type Including Piecewise Continuous Time Delay

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Received: 12 April 2023; Accepted: 28 July 2023

Reference/Atıf: Z. Özkan, D. Aruğaslan Çinçin, Marat Akhmet, "On Solutions of a Parabolic Partial Differential Equation of Neutral Type Including Piecewise Continuous Time Delay", Researcher, vol.3, no.1, pp.29-35, July 2023, doi:10.55185/researcher.1281521

Abstract

There have been very few studies on partial differential equations including piecewise constant arguments and generalized piecewise constant arguments. However, as far as we know, there is no study conducted on neutral type partial differential equations including piecewise constant argument of generalized type. With this motivation, we discuss the solution and analysis of a parabolic partial differential equation of neutral type including generalized piecewise constant delay. The aim of this study is to investigate detailed and well-defined qualitative properties of this equation. The formal solution of the handled equation is obtained by using the separation of variables method. Since there exist the piecewise constant arguments, we get an ordinary differential equation with respect to the time variable on each consecutive intervals and then apply the Laplace transform method using the unit step function and method of steps. With the help of the qualitative properties of the solutions of the obtained differential equation, unboundedness and oscillations of the solutions of the issue problem can be investigated.

Keywords: partial differential equation, neutral type, piecewise constant argument, laplace transformation, oscillation, unboundedness.

1. Introduction and Preliminaries

There are several mathematical models in the literature that use differential equations to investigate real-world situations. Most of these models include simply the present states of the processes, but in some circumstances, genuine issues cannot be portrayed realistically by these models since current and future states are heavily impacted by past states. From this point of view, functional differential equations enable to consider more practical mathematical models and meet the challenges encountered in many fields such as physics, medicine, economics, engineering. A careful examination of mathematical models including piecewise constant arguments is demonstrated in ([1]). Because difference and differential equations are so closely connected, differential equations including piecewise constant arguments are hybrid dynamical systems ([2]). Differential equations including piecewise constant arguments have attracted the interest of scientists in domains such as mathematics, physics, biology, engineering, economics, health, and others since the early 1980s ([3-5]). Some qualitative properties such as stability, oscillation, periodicity, boundedness/unboundedness and convergence of solutions to ordinary differential equations including piecewise constant arguments have been investigated in the literature ([6-23]). However, research on partial differential equations including piecewise constant arguments is scarce ([24-38]) when compared to ordinary differential equations with deviating arguments. In the studies ([24-38]), some qualitative properties were examined and problems were handled by the method of steps and transforming to difference equations. In 1991, the first fundamental work ([24]) was published for partial differential equations including piecewise constant arguments. It has been demonstrated that partial differential equations including piecewise constant time naturally exist when partial differential equations are approximated using piecewise constant arguments. For example, if the lateral heat change at discrete times is measured, in the equation

$$u_t(x,t) = a^2 u_{xx}(x,t) - bu(x,t) \qquad (1a)$$

which describes heat flow in a rod with both diffusion $a^2 u_{xx}(x,t)$ along the rod and heat loss (or gain) across the lateral sides of the rod, then the following differential equation including piecewise constant argument

$$u_t(x,t) = a^2 u_{xx}(x,t) - bu(x,t/h) \tag{1b}$$

where $h > 0$ is a constant, is obtained for $t \in [nh,(n+1)h]$, $n = 0,1,2, ...$, ([24]).

Wiener's book [34] is a good resource for both ordinary and partial differential equations including piecewise constant arguments. However, as far as we are aware, no research has been conducted on a neutral type equation including piecewise constant arguments of generalized type using the Laplace transform. With this reason, we address the initial boundary value problem

$$u_t(x,t) = a^2 u_{xx}(x,t) + bu_t\big(x,\beta(t)\big), 0 \le x \le 1, \quad \theta_0 \le t < \infty \tag{2}$$

under boundary conditions

$$u(0,t) = 0, \; u(1,t) = 0, \quad \theta_0 \le t < \infty \tag{3}$$

and initial condition

$$u(x,\theta_0) = u_0(x), \quad 0 \le x \le 1. \tag{4}$$

in the present research.

In this problem, a and b are nonzero real parameters, and $u\colon G \times [0,1] \to (-\infty,\infty)$, $\beta(t)$ is the generalized type piecewise constant function such that for $\theta_i \le t \le \theta_{i+1}$, $i = 0,1,2,\cdots$, $\beta(t) = \theta_i$, $|\theta_i| \to \infty$ as $i \to \infty$ and $u_0(x)$ is a continuous function on $[0,1]$. This equation is an example of neutral type since it contains the derivative u_t at different values of t.

From this point on, without losing generality, we will suppose that $\theta_0 = 0$ and there are two positive numbers $\underline{\theta}$ and $\overline{\theta}$ to satisfy $\underline{\theta} \le \theta_{i+1} - \theta_i \le \overline{\theta}$, $i = 0,1,2,\cdots$.

2. Existence of Solution

Before starting to solve the initial boundary value problem (2)-(4), let us define the properties of the solution $u(x,t)$ in G.

Definition 1 A function $u(x, t)$ is called a solution of the initial boundary value problem (2)-(4) in G if it satisfies the following three conditions:

(i) $u(x, t)$ is continuous in G,

(ii) u_t and u_{xx} exist and are continuous in G, there may be exceptional points (x, θ_i), $i = 0,1,2,\cdots$, where one-sided derivatives exist with respect to the second argument,

(iii) $u(x, t)$ satisfies Eq. (2) in G, with the possible exception of the points (x, θ_i), $i = 0,1,2,\cdots$, and conditions (3) and (4).

We can write $x(\beta(t))$ as a piecewise-defined function as follows

$$x\big(\beta(t)\big) = \begin{cases} x(0) & \text{if} \quad \theta_0 = 0 \le t \le \theta_1, \\ x(1) & \text{if} \quad \theta_1 \le t \le \theta_2, \\ & \vdots \\ x(n) & \text{if} \quad \theta_n \le t \le \theta_{n+1}, \\ & \vdots \end{cases}$$

Using this piecewise-defined function, we can write the following equality

$$x\big(\beta(t)\big) = x_0 u_0(t) + (x(\theta_1) - x_0)u_{\theta_1}(t) + (x(\theta_2) - x(\theta_1))u_{\theta_2}(t) + \cdots + \big(x(\theta_{n+1}) - x(\theta_n)\big)u_{\theta_{n+1}}(t) + \cdots.$$

We can rewrite $x\big(\beta(t)\big)$, using series, as follows

$$x\big(\beta(t)\big) = x(0) + \sum_{i=0}^{\infty}\big(x(\theta_{n+1}) - x(\theta_n)\big)u_{\theta_{n+1}}(t) \tag{5}$$

where $u_{\theta_{n+1}}(t)$ is the Heaviside step function defined as

$$u_n(t) = \begin{cases} 0 & \text{if} \quad t < n, \\ 1 & \text{if} \quad t \geq n. \end{cases}$$

Let us seek a solution in the form $u(x,t) = X(x)T(t)$. Then, taking partial derivatives of $u(x,t)$ and introducing partial derivatives of $u(x,t)$ into Equation 2, we have

$$\frac{T'(t) - bT'\big(\beta(t)\big)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\mu^2,$$

where μ is a real constant. From boundary conditions, we get $X(0) = 0, X(1) = 0$.

Then separation of variables gives a boundary value problem

$$\begin{gathered} X''(x) + \mu^2 X(x) = 0, \\ X(0) = 0, X(1) = 0, \end{gathered} \tag{6}$$

whose orthonormal set of solutions is given by

$$X_j(x) = 2\sin(\pi j x), j = 1, 2, 3, \cdots, \tag{7}$$

on [0, 1], and the following ordinary differential equation including piecewise constant argument of generalized type

$$T_j'(t) + a^2\pi^2 j^2 T_j(t) = bT_j'\big(\beta(t)\big). \tag{8}$$

In Equation 8, taking $t = \beta(t)$ with the condition $b \neq 1$, we obtain

$$T_j'\big(\beta(t)\big) + a^2\pi^2 j^2 T_j\big(\beta(t)\big) = bT_j'\big(\beta(t)\big).$$

If we solve for $T_j'\big(\beta(t)\big)$ from the last equality and introduce $T_j'\big(\beta(t)\big)$ into Equation 8, we get

$$T_j'(t) + a^2\pi^2 j^2 T_j(t) = \frac{ba^2\pi^2 j^2}{b-1} T_j\big(\beta(t)\big). \tag{9}$$

Now, using equality 5 we can write Equation 9 in an explicit way as

$$T_j'(t) + a^2\pi^2 j^2 T_j(t) = \frac{ba^2\pi^2 j^2}{b-1}\left(T_j(0) + \sum_{n=0}^{\infty}\Big(T_j(\theta_{n+1}) - T_j(\theta_n)\Big)u_{\theta_{n+1}}(t)\right).$$

Then, if we take Laplace transform of the last equality and solve for $\mathcal{L}\{T_j(t)\}$, we obtain

$$\mathcal{L}\{T_j(t)\} = \frac{1 + \dfrac{b}{(b-1)s}}{s + a^2\pi^2 j^2} T_j(0) + \frac{ba^2\pi^2 j^2}{b-1}\sum_{n=0}^{\infty}\Big(T_j(\theta_{n+1}) - T_j(\theta_n)\Big)\frac{e^{-\theta_{n+1}s}}{s(s + a^2\pi^2 j^2)}.$$

After applying inverse Laplace transform to last equality, we get the solution of Equation 9, and moreover Equation 8, as follows

$$T_j(t) = \left(\frac{b - e^{-a^2\pi^2 j^2 t}}{b-1}\right)T_j(0) + \frac{b}{b-1}\sum_{n=0}^{\infty}\Big(T_j(\theta_{n+1}) - T_j(\theta_n)\Big)\big(1 - e^{-a^2\pi^2 j^2 (t - \theta_{n+1})}\big)u_{\theta_{n+1}}(t). \tag{10}$$

Then, we have a proposition to obtain the solution given by the solution 10 through a nonrecursive relation.

Proposition 1 *The solution of Equation 9, and moreover Equation 8, on the interval* $[0, \infty)$ *is given by*

$$T_j(t) = \left(\frac{b - e^{-a^2\pi^2 j^2(t-\beta(t))}}{b-1}\right)\prod_{i=1}^{\sigma(t)}\left(\frac{b - e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})}}{b-1}\right)T_j(0) \tag{11}$$

where $\sigma(t)$ *denotes the number of discontinuity moments* θ_i *on the interval* $(0, t]$, *the numbers* $T_j(0)$ *and it is assumed that* $\prod_{i=1}^{0}(\cdot) = 1$.

Proof: Let $T_{nj}(t)$ be the solution of Equation 9, and moreover Equation 8, on an arbitrary interval $[\theta_n, \theta_{n+1})$. Then $T_{nj}(t)$ satisfies the following differential equation

$$T_{nj}'(t) + a^2\pi^2 j^2 T_{nj}(t) = \frac{ba^2\pi^2 j^2}{b-1}T_{nj}(\theta_n).$$

If we solve this equation, we obtain the general solution of the last equation as follows

$$T_{nj}(t) = \frac{b-e^{-a^2\pi^2 j^2(t-\theta_n)}}{b-1}T_{nj}(\theta_n).$$

Since solution $T_j(t)$ is continuous on the interval $[\theta_0, \infty)$, we have

$$T_{nj}(\theta_{n+1}) = T_{n+1,j}(\theta_{n+1}).$$

Then, with the help of this equality, we obtain

$$T_{nj}(t) = \frac{b - e^{-a^2\pi^2 j^2(t-\theta_n)}}{b-1}\prod_{i=1}^{n}\left(\frac{b - e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})}}{b-1}\right)T_{0j}(0).$$

Since $T_{nj}(t)$ represents the solution on an arbitrary interval $\theta_n \le t < \theta_{n+1}$ and $T_j(t)$ is continuous on the interval $[0, \infty)$, the solution on the interval $t \in [0, \infty)$ can be expressed in the following way

$$T_j(t) = \frac{b - e^{-a^2\pi^2 j^2(t-\beta(t))}}{b-1}\prod_{i=1}^{\delta(t)}\left(\frac{b - e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})}}{b-1}\right)T_j(0). \tag{12}$$

Hence, proposition is proved.

Then, we will obtain a nonrecursive relation in Equation 10 by writing $t = \theta_{n+1}$ and $t = \theta_n$ in the solution 11, respectively, and subtracting $T_j(\theta_n)$ from $T_j(\theta_{n+1})$ we will get the following equality

$$T_j(\theta_{n+1}) - T_j(\theta_n) = \left(\frac{1 - e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)}}{b-1}\right)\prod_{i=1}^{n}\left(\frac{b - e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})}}{b-1}\right)T_j(0). \tag{13}$$

After introducing Equality 13 into the solution 10, we can rewrite the solution of Equation 9, and moreover Equation 8, and consequently, the solution of Equation 9, and moreover Equation 8, as follows

$$T_j(t) = \left(\left(\frac{b - e^{-a^2\pi^2 j^2 t}}{b-1}\right) + \frac{b}{b-1}\sum_{n=0}^{\infty}\left(\frac{1 - e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)}}{b-1}\right)\prod_{i=1}^{n}\left(\frac{b - e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})}}{b-1}\right) \times \left(1 - e^{-a^2\pi^2 j^2(t-\theta_{n+1})}\right)u_{\theta_{n+1}}(t)\right)T_j(0). \tag{14}$$

So, the solutions of the equation of neutral type (2) satisfying the boundary conditions (3) are obtained as $u_j(x,t) = X_j(x)T_j(t),\ j = 1,2,\cdots$.

Since the Equation 2 is linear, with the superposition principle the solution of boundary value problem (2)-(3) on the region $[0,1] \times [0,\infty)$ is given by

$$\begin{aligned} u(x,t) &= \sum_{j=1}^{\infty} T_j(t)X_j(x) \\ &= \sum_{j=1}^{\infty}\left(\left(\frac{b - e^{-a^2\pi^2 j^2 t}}{b-1}\right) + \frac{b}{b-1}\sum_{n=0}^{\infty}\left(\frac{1 - e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)}}{b-1}\right)\prod_{i=1}^{n}\left(\frac{b - e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})}}{b-1}\right) \right. \\ &\quad \left. \times \left(1 - e^{-a^2\pi^2 j^2(t-\theta_{n+1})}\right)u_{\theta_{n+1}}(t)\right)T_j(0)\sqrt{2}\sin(\pi j x). \end{aligned} \tag{15}$$

Now, only the initial condition must be checked. To do this, putting $t = 0$ in the solution 15 gives

$$u(x,0) = u_0(x) = \sum_{j=1}^{\infty} T_j(0)\sqrt{2}\sin(\pi j x) \tag{16}$$

where

$$T_j(0) = \sqrt{2}\int_0^1 u_0(x)\sin(\pi j x)\,\mathrm{d}x. \tag{17}$$

Hence, Equality 15 with Equality 17 gives the solution of problem (2), (3), (4) in $[0,1] \times [0,\infty)$.

Theorem 1 *Assume that* $b > 1$. *Then each solution* $T_j(t)$ *of Equation 9, and moreover Equation 8), is monotone unbounded as* $t \to +\infty$.

Proof: For $b > 1$, we have

$$\frac{b - e^{-a^2\pi^2 j^2 t}}{b-1} > 1,$$

for all t. So,

$$\prod_{k=1}^{\delta(t)}\left(\frac{b - e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})}}{b-1}\right)$$

grows fastly as $t \to +\infty$, and proof can be obtained from the solution 12.

Theorem 2 *Assume that* $e^{-a^2\pi^2 j^2 \underline{\theta}} < b < \frac{1}{2}$. *Every function* $T_j(t)$ *tends to zero and has oscillation as* $t \to +\infty$.

Proof: From the inequality $e^{-a^2\pi^2 j^2 \underline{\theta}} < b < \frac{1}{2}$, we get the following inequalities

$$-1 < \frac{b - e^{-a^2\pi^2 j^2(\theta_i - \theta_{i-1})}}{b-1} < 0.$$

These inequalities give oscillation property and tending to zero property of the solution 12.

Contribution of Researchers

All researchers have contributed equally to writing this paper.

Conflicts of Interest

The authors declare no conflict of interest.

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