SAKARYA ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ DERGİSİ
SAKARYA UNIVERSITY JOURNAL OF SCIENCE

e-ISSN: 2147-835X
Dergi sayfası: http://dergipark.gov.tr/saufenbilder

Geliş/Received
31.05.2016
Kabul/Accepted
07.12.2016

Doi
10.16984/saufenbilder.283775

İki-Boyutlu Konvektif Sınır Koşullu Erime Problemi İçin Nümerik Yaklaşım

Vildan Gülkaç[1*],

ÖZ

Bu çalışmada, daha önce çözdüğümüz, iki-boyutlu konvektif sınır koşullu erime probleminde, türevlerin bir kısmında açık yöntem kullanırken bir kısmında da kapalı yöntem kullanarak sonlu farklar oluşturulmuştur ve bu denklemlerin çözümü için bir iteratif yöntem geliştirilmiştir. Metod (x, y) koordinatlarında ikinci dereceden doğruluğa sahiptir. Bu metodla elde edilen sonuçlar, önceki araştırmacılar tarafından verilen sonuçlarla tamamen uyumludur.

Anahtar Kelimeler: Flux Limiters, LOD metodu, hareketli sınır problemleri, Strang splitting.

Numerical Approach for the two-dimensional heat equation problem with convective boundary conditions

ABSTRACT

In this work, we extended our earlier study on the solution of two-dimensional heat equation problem by considering a class of time-split finite difference methods. Operator splitting is used as a procedure for computing, some derivatives are computed explicitly and some of them computed implicitly during this procedure. The procedure is second order accurate in time and in (x, y) coordinates. The results of computing by present procedure are in totally compatible with the results obtained previously by other researches.

Keywords: Flux limiters, LOD method, ADI method, moving boundary problems, Strang splitting.

* Sorumlu Yazar / Corresponding Author
1 Kocaeli Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü, Kocaeli - vgulkac@kocaeli.edu.tr

1. INTRODUCTION

Convection is the convective heat transfer which is the transfer of heat from one place to another and is the dominant form of heat transfer in liquids and gases. Convection is often defined as a different method of heat transfer includes the combined processes of conduction (heat diffusion) and advection (heat transfer by bulk fluid flow).

Movement of a fluid can force convection by means other than buoyancy forces. Convection may also be forced by thermal expansion of fluids. Natural convection occurs due to fluid motion when the fluid is heated by natural buoyancy forces.

The convection heat transfer mode consists of one mechanism. Energy is transferred by bulk, or macroscopic motion of the fluid as well as, diffusion which is the energy transfer because of specific molecular motion. Large numbers of molecules are moving together with or as combinations at any instant, related with this motion. Such motion, causes the heat transfer in the presence of a temperature increase. As a result of the molecules in combinations continue their random motion of the total heat transfer is then because of the superposition of the energy transport by random motion of the molecules and by the bulk motion of the fluid. When mentioning this cumulative transport, it is common to use the term convection and the term advection when mentioning the transport because of the bulk fluid motion.

For a long time convection heat transfer and fluid flow in porous media examined numerically and experimentally because of the important uses, for example micro-thrusters, geothermal energy extraction, matrix or micro-porous heat exchangers, catalytic and chemical particle beds, packed-bed regenerators and heat transfer enhancement. Melting and solidification procedures are came across in a range of industrial applications. Several methods have been proposed [1-25] to solve these problems. Many researchers explored different numerical and analytical methods to obtain solutions for moving boundary problems. Some references for analytical methods can be found in Goodman [4], Rasmussen [5]. Some examples for numerical approaches are found in Creyer [6], Furzeland [7, 8], Aitchison [9, 10], Landau [11], Gupta and Kumar [12], Ferris and Hill [13], Beaubauff and Chapman [14], Duda et al [15]. Öziş and Gülkaç [1, 2] prepared a numerical method to solve the multi-dimensional phase change problems by an independent variable interchange which extension and as well as modification of Boadway's transformation. Gülkaç published [16, 17] two different numerical method for two dimensional moving problem. Minkowycz and Sparrow [18] , the local non-similarity solution method applied to solve for natural convection on a vertical cylinder for conditions. Some researchers commonly used saturated porous media or mixed convection heat transfer numerical simulations with Darcy equation models. As seen [19] and [20].

In this work, we prepared a new method to solve the phase-change problems by Rannacher method [21] which extension and as well as modification of Rannacher method. In this method Crank-Nicolson finite-difference equations are solved with a number of initial time-steps with iterative method. In section 2, we explained the model problem in this study. In section 3, the discretization in space and time is demonstrated. In section 4 the accuracy and stability of the method is analyzed. Numerical results and conclusions for the fusion problem are presented in section 5. Finally we compare the proposed work with other schemes and results show that the present work has much higher efficiency.

2. FORMULATION OF THE PROBLEM

Following researchers worked on this problem, Gupta and Kumar [12], Sparrow and Hsu [22], Öziş and Gülkaç [1], and Gülkaç [2].

The problems physical situation takes place in a containment vessel which contains a liquid phase-change medium at temperatureT_f. A circular tube passes through the containment vessel which a coolant flows through the tube with inlet temperatureT_0. The containment vessel's upper and lower boundaries are adiabatic. In this variation the primary factor is the axial temperature rise undergone by the coolant as it flows through the tube. Heat transfer to the coolant because of the energy released by freezing process and by the sub cooling of the solid causes the temperature increase. Other possible participating factors to the axial thickness variation are heat conduction in the solid and heat storage in the coolant. At the beginning the containment tube is filled with a liquid phase-change material at fusion temperature $T_f$describes the freezing process. No coolant is flowing and no coolant fluid is present within the tube at T_f . At t=0 time the coolant flow is started and maintained. T_f is always higher than the entering coolant temperature T_0 .T_0 is maintained at the stable value which is always smaller then T_f, which causes the freezing process to be initiated and continues as long as coolant is supplied to the vessel [12].

According to the [12], below equations shows the process mathematically,

if $T = T(r, z, \tau)$ shows the temperature distribution in the solid region at time ,

$$\frac{\partial T}{\partial \tau} = \alpha\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right], r_0 < r < R(r,\tau),$$
$$0 \le z \le L_z, \tau > 0 \quad (1)$$

$$h(T_w - T_v) = k\frac{\partial T}{\partial r}\ at\ r = r_0 \quad (2)$$

$$T(r,z,\tau) = T_f, \text{ at } r = R(z,\tau) \quad (3)$$

We can write the second condition on interface boundary from the mass conservation equation as below

$$k\left(\frac{\partial T}{\partial r} = \varphi L\vartheta_n\right), \text{ at } r = R(z,\tau) \quad (4)$$

n is the normal to the interface with regard to liquid. The velocity of the interface in the direction of n is ϑ_n. L_z is length of the tube along the z axis. T_v is coolant temperature and T_w is the wall temperature, which are both functions of z and τ.

Now, we expressed the variables and the parameters in physical units. Most researchers demonstrate the problems in non-dimensional variables as in Gupta and Kumar [12]. Denote eqns. (1)-(4) to the following [12],

$$Ste\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{1+y}\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2},\ 0 < y < s(x,t),$$
$$0 \le x \le L_x,\ t > 0 \quad (5)$$

$$Bi(W - V) = \left(\frac{\partial u}{\partial y}\right)_{solid},\ y=0 \quad (6)$$

$$u(x,y,t) = 0,\ y=s(x,t) \quad (7)$$

and the eqn. (4) may be written as;

$$\frac{\partial u}{\partial n} = \mu\vartheta^n \text{ at } y = s(x,t) \quad (8)$$

3. DISCRETIZATION OF PROBLEM

We can solve the two-dimensional heat equation (5) by splitting it into two one-dimensional eqns. Eqn. (5) rewritten as eqn. (9),

$$u_t(t,x,y) = \mathcal{M}(u) + \mathcal{R}(u) + \Im(u) \quad (9)$$

We consider a discretization of eqn. (9) and time on a structured and equidistant grid (t^n, x_i, y_j), $t^n = n\Delta t$, $x_i = ih, y_j = jh$. The operator $\mathcal{M}$ and $\mathcal{R}$ are approximated using centered second order finite differences

$$\mathcal{M}u_{i,j} = \frac{1}{Ste}\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \quad (10)$$

$$\mathcal{R}u_{i,j} = \frac{1}{Ste}\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \quad (11)$$

Such in [24] to prevent spurious oscillations we use flux limiters for the first order operator $\Im$.

$$\Im u_{i,j} = \frac{1}{Ste}\frac{1}{1+y_j}\chi\frac{\Delta_+ u_{i,j} - \left(f_{j+1/2} - f_{j-1/2}\right)}{k} \quad (12)$$

with

$$\Delta_+ u_{i,j} = u_{i,j+1} - u_{i,j},\ \chi = 1 + \frac{1}{2}\Delta t,$$

$$f_{i,j-\frac{1}{2}} = \frac{1}{2}\left(1 - \frac{\Delta t}{k}\frac{1}{1+y}\chi\right)\delta_{i,j-1/2},$$

$$f_{i,j+\frac{1}{2}} = \frac{1}{2}\left(1 - \frac{\Delta t}{k}\frac{1}{1+y}\chi\right)\delta_{i,j-1/2} \quad (13)$$

Here $\delta_{i,j+1/2} = \phi\left(\Delta_+ u_{i,j+1}, \Delta_+ u_{i,j}\right)$ and ϕ is the flux limiter function. We consider the min-mod-limiter defined by ϕ

$$(\alpha,\beta) = \begin{cases} 0, if\alpha\beta < 0 \\ \alpha,\ if\ |\alpha| < |\beta| \\ \beta, if\ |\beta| < |\alpha| \end{cases} \quad (14)$$

So that we will use a Strang operator splitting as seen [23] which let us the treat $\mathcal{M}$ and $\mathcal{R}$ implicitly in time and $\Im$ explicitly. The implicit scheme to solve $u_t - \mathcal{M}u = 0$ from t^{n-1} to t^n is as seen as in [24-25].

The eqn. (8) is not suitable for numerical solution in this form. Hence following Crank and Gupta [12], we may express eqn. (8) in more suitable expression as eqn. (15).

$$S_t = \frac{1}{\mu}\left\{(u_x)^2 + \left(u_y\right)^2\right\}/u_y \quad (15)$$

According to Öziş and Gülkaç [1], to calculate the coolant temperature distribution along the axis of the cylinder, one may need the equation combining the energy equation, can be written as

$$2(W - V) = w\frac{\delta V}{\delta t} + \frac{1}{S_t}\frac{\delta V}{\delta x} t>0. \quad (16)$$

Hence to compute numerically at time level t_n as eqn. (9).

$$1.\ \tilde{u}^n_{i,j} = u^{n-1}_{i,j} - \frac{\Delta t}{4}\left(\mathcal{M}u^{n-1}_{i,j} + \mathcal{M}\tilde{u}^n_{i,j}\right), \quad (17)$$

$$2.\ u^n_{i,j} = \frac{1}{3}\left(4\tilde{u}^n_{i,j} - u^{n-1}_{i,j} - \Delta t\mathcal{M}u^n_{i,j}\right) \quad (18)$$

The explicit scheme to solve $u_t + \mathcal{R}u + \Im u = 0$ from t^{n-1} to $t^n$is

$$u^n_{i,j} = u^{n-1}_{i,j} + \Delta t\Im u^{n-1}_{i,j} + \Delta t\mathcal{R}u^{n-1}_{i,j} \quad (19)$$

Following [1], we can write eqn. (15) and (16) such as eqns. (20), (21), (22).

$$3.\ S^n_{i,j} = \frac{\Delta t}{\mu}\left\{\frac{\Delta y}{(\Delta x)^2}\frac{\left(u^{n-1}_{i+1,j} - u^{n-1}_{i-1,j}\right)^2}{u^{n-1}_{i,j+1} - u^{n-1}_{i,j-\frac{1}{2}}} + \frac{u^{n-1}_{i,j+1} - u^{n-1}_{i,j-1}}{\Delta y}\right\} + S^{n-1}_{i,j} \quad (20)$$

$$4.\quad V^n_i\left[\frac{\Delta x}{1+Biu^n_{i,j1}} + \frac{w\Delta x}{2\Delta x\Delta} + \frac{1}{S^n_{i,j}}\right] = \frac{1}{2S^n_{i,j}}V^n_{i-1} + \frac{w\Delta x}{2\Delta t}V^n_i + \frac{\Delta x}{1+Biu^n_{i,j1}} \quad (21)$$

$$5.\ W^n_i = u^{n-1}_{i,j1} + Biu^n_{i,j1}V^n_i \quad (22)$$

Fluid enters the cylinder at a constant temperature, say unity, we have $V^n_0 = 1$.

Flux limiter causes $\Im$ to be nonlinear, however it is applied explicitly in eqn. (19) to $u^{n-1}_{i,j}$ with low additional computational complexity. So that oscillations in solution diminish.

The solution to eqn. (9) is progressed one time step Δt with Strang splitting in two alternative ways. The first algorithm is:

1. Compute $u_{i,j}^{n}$ at $t^{n-1/2}$ using eqns. (17), (18) with time step $\Delta t/2$ and $u_{i,j}^{n-1}$ as initial data.
2. Compute $\tilde{u}_{i,j}^{n}$ at t^{n} using eqn. (19) with time step Δt in eqns. (11), (12), (13) and (16) and $u_{i,j}^{n-1/2}$ as initial data.
3. Compute $u_{i,j}^{n}$ at t^{n} using eqns. (17), (18) with time step $\Delta t/2$ and $\tilde{u}_{i,j}^{n}$ as initial data.
4. Compute $S_{i,j}^{n}$ at t^{n} using $u_{i,j}^{n}$.
5. Compute V_{i}^{n} at t^{n} using $u_{i,j}^{n}$ and $S_{i,j}^{n}$.
6. Compute W_{i}^{n} at using $u_{i,j}^{n}$ and V_{i}^{n}.

The second algorithm is:

1. Compute $u_{i,j}^{n-1/2}$ at $t^{n-1/2}$ using eqn. (19) with time step $\Delta t/2$ in eqns. (11), (12), (13) and (19) and $u_{i,j}^{n-1}$ as initial data.
2. Compute $\tilde{u}_{i,j}^{n}$ at t^{n} using eqns. (17), (18) with time step Δt and $u_{i,j}^{n-1/2}$ as initial data.
3. Compute $u_{i,j}^{n}$ at t^{n} using eqn. (19) with time step $\Delta t/2$ in eqns.(11), (12), (13) and (19) and $\tilde{u}_{i,j}^{n}$ as initial data.
4. Compute $S_{i,j}^{n}$ at t^{n} using $u_{i,j}^{n}$.
5. Compute V_{i}^{n} at t^{n} using $u_{i,j}^{n}$ and $S_{i,j}^{n}$.
6. Compute W_{i}^{n} at using $u_{i,j}^{n}$ and V_{i}^{n}.

Both algorithms are one step schemes where the solution $u_{i,j}^{n-1}$ at t^{n-1} is integrated to $u_{i,j}^{n}$ at t^{n} and than $u_{i,j}^{n}$ integrated to $u_{i,j}^{n+1}$ at t^{n+1}.

The scheme to two-(x, y)-dimensions is straight forward for a Cartesian grid. A banded direct solver or a Krylov subspace iteration method is used to solve the implicit part. $\mathfrak{I}$ is applied to the solution to compute the approximation of the first derivatives and the approximation of the second derivatives are computed by applying $\mathcal{R}$ at t^{n-1} initially in one coordinate direction and later in the other coordinate direction. Like in [23] to update the solution, those contributions are summed and multiplied by Δt. Operator splitting method can be used by first advancing the solution in time by the x-derivative and then by the y-derivative either consecutively or using Strang splitting [23] or ADI method [16, 17].

4. ACCURACY AND STABILITY

The accuracy of eqn. (9) with $\mathcal{M}$, $\mathfrak{I}$and $\mathcal{R}$ in eqns. (10), (11), (12) is analyzed respectively by considered the approximation of the first derivative. The solution

$$u_t - \frac{1}{Ste}u_{xx} = 0 \quad (23)$$

is progress from t^{n-1} to t^{n} by expanding the solution in a Taylor series in time

$$u^n = u^{n-1} + \Delta t u_t^{n-1} + \frac{1}{2}\Delta t^2 u_{tt}^{n-1} + \frac{1}{6}\Delta t^3 u_{ttt}^{n-1} + O(\Delta t^4) \quad (24)$$

The terms $u_t^{n-1} + \frac{1}{2}\Delta t u_{tt}^{n-1}$ approximate u_t at $t^{n+1/2}$ in eqn. (22). Then u_t and u_{tt} in this expression are replaced by derivatives in x using eqn. (21).

Introduce the x-derivatives into eqn. (22) to obtain

$$u^n = u^{n-1} + \Delta t(\frac{1}{Ste})u_x^{n-1} + \frac{1}{2Ste^2}\Delta t^2 u_{xx}^{n-1} + \frac{1}{6}\Delta t^3 u_{ttt}^{n-1} + O(\Delta x^4) \quad (25)$$

$$u^n = u^{n-1} + \Delta t\frac{1}{Ste}u_x^{n-1} + \frac{1}{2Ste^2}\Delta t^2 u_{yy}^{n-1} + O((\Delta t^3)) \quad (26)$$

The approximation of u^n of formal second order in Δt in eqn. (26).

Similarly, the solution

$$u_t - \frac{1}{Ste}\frac{1}{1+y}u_y - \frac{1}{Ste}u_{yy} = 0 \quad (27)$$

is progress from t^{n-1} to t^{n} by expanding the solution in a Taylor series in time

$$u^n = u^{n-1} + \Delta t u_t^{n-1} + \frac{1}{2}\Delta t^2 u_{tt}^{n-1} + \frac{1}{6}\Delta t^3 u_{ttt}^{n-1} + O(\Delta t^4) \quad (28)$$

The terms $u_t^{n-1} + \frac{1}{2}\Delta t u_{tt}^{n-1}$ approximate u_t at $t^{n+1/2}$ in eqn. (22). Then u_t and u_{tt} in this expression are replaced by derivatives in y using eqn. (21).

Introduce the y-derivatives into eqn. (24) to obtain

$$u^n = u^{n-1} + \frac{\Delta t}{Ste(1+y)}u_y^{n-1} + \frac{1}{2}\Delta t^2\left(\frac{1}{Ste^2(1+y)}u_y^{n-1} + \left(\frac{1}{Ste}\frac{1}{1+y}\right)^2 u_{yy}^{n-1}\right) + \frac{1}{6}\Delta t^3 u_{ttt}^{n-1} + O(\Delta t^4)$$

$$u^n = u^{n-1} + \Delta t\frac{1}{Ste}\frac{1}{1+y}\left(1 + \frac{1}{2}\Delta t\frac{1}{Ste}\right)u_y^{n-1} + \frac{1}{2}(\Delta t\frac{1}{Ste}\frac{1}{1+y})^2 u_{yy}^{n-1} + O(\Delta t)^3$$

$$u^n = u^{n-1} + \Delta t\frac{1}{Ste}\frac{1}{1+y}\chi u_y^{n-1} + \frac{1}{2}\left(\Delta t\frac{1}{Ste}\frac{1}{1+y}\chi\right)^2 u_{yy}^{n-1} + O(\Delta t)^3 \quad (29)$$

The approximation of u^n is of formal second order in Δt in eqn. (23) and the same factor $\frac{1}{Ste}\frac{1}{1+y}\chi$ is multiplying both u_y and u_{yy} as assumed in the derivation of the limiters as seen as [24]. It is noted in [24] that ignoring the correction χ for a variable coefficient in front of u_y formally decreases the order to accuracy but in practice the difference in the error is small. Then the two terms depending on u_y^{n-1} and u_{yy}^{n-1} are approximated as in eqn. (24).

The eqns. (17), (18) and (19) for the diffusive part of eqn. (9) is of second order accuracy in time as in [23]. Combining two second order accurate schemes in operator splitting as in the first algorithm or the second algorithm in section 3 is also second order accurate in time as in [23].

Suppose that the solution is smooth such that the limiter in eqn. (14) is $\delta_{i,j+1/2} = \Delta_+ u_{i,j}$ for j and j-1. Then $\mathfrak{I}u_{i,j}^{n-1/2} + \mathcal{R}u_{i,j}^{n-1/2}$ in eqns. (17) and (18) are

$$\mathfrak{I}u_{i,j}^{n-1/2} + \mathcal{R}u_{i,j}^{n-1/2} = -\frac{\frac{1}{Ste}\left(\frac{1}{1+y_j}\right)\chi}{2k}\left(u_{i,j+1}^{n-1} - u_{i,j-1}^{n-1}\right) - \frac{\Delta t\left(\frac{1}{Ste}\frac{1}{1+y_j}\chi\right)^2}{2k^2}\left(u_{i,j+1}^{n-1} - 2u_{i,j}^{n-1} + u_{i,j-1}^{n-1}\right) + u^{n-1} + \Delta t\frac{1}{Ste}\frac{1}{1+y}\chi u_y^{n-1} + \frac{1}{2}\left(\Delta t\frac{1}{Ste}\frac{1}{1+y}\chi\right)^2 u_{yy}^{n-1} + O(\Delta t)^3 \quad (30)$$

In eqn. (23) the approximation of the y-derivatives is second order.

To compute u(T), we can combine the last half step with the first half step in the first algorithm or the second algorithm into one full step with Δt. This is possible of all inner time steps except for the first one at t=0 and the last one to react t=T as seen in [23, 24].

5. NUMERICAL RESULTS AND CONCLUSIONS

We offer the new method to solve moving boundary problems with convective boundary conditions using time-split finite difference methods. Operator splitting is used as a procedure for computing, some derivatives are computed explicitly and some of them computed implicitly during this procedure. By using the new method Eqns. (9)-(15), (16) are discredited. Using Strang-splitting method, the discredited two-dimensional heat transfer equations with convective boundary conditions are solved. These methods perform as well as for the fusion problem with convective boundary conditions. The stability analysis is also investigated. The initial conditions from Sparrow and Hsu [22] used for the short-time solution. In addition to make comparison, the values of some parameters are taken equal with [12, 22] i.e., Ste=1.0, Bi=5.0, St=0.003, and w=0, $\Delta x = 10.0, \Delta y = 0.1$ and 0.05 and $\Delta t = 0.0001$ and 0.0005. L_x, the length of the cylinder, is also taken to be 100.0. A comparison of the values of the interface position obtained by other numerical results in the previous studies shown in Figure 1. Figure 2 shows the temperature distribution at the wall (u=0) and Figure 3 shows the temperature distribution at the coolant, along with the values from [12] along the axis of the cylinder at different times. In addition, obtained results prove that the present method can be applied to non-linear problems. Moreover, the new method may applied to three-dimensional problems. The present method's computationally efficiency is proved by the numerical results.

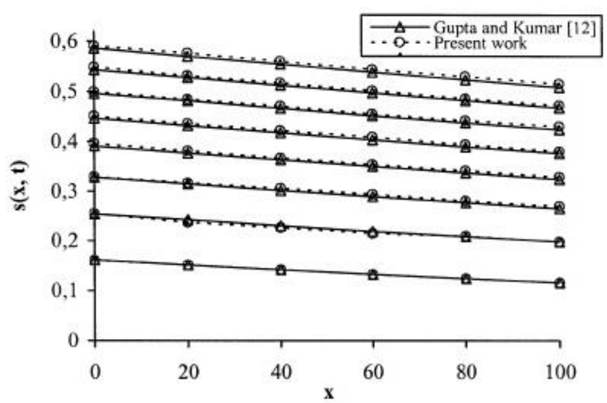

Figure 1. The interface positions against time along with comparative values from [12].

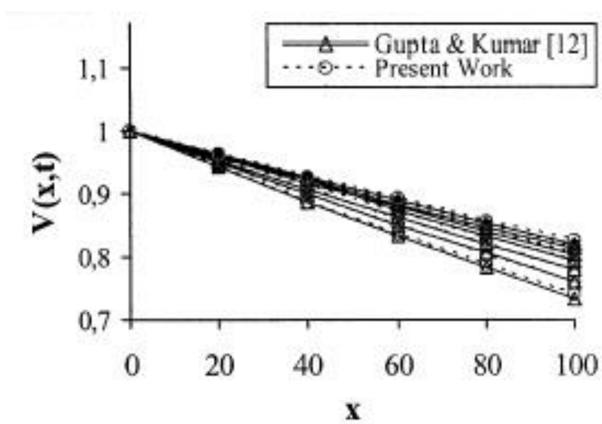

Figure 2. Graph displaying the temperature distribution in the coolant, along the axis of the cylinder at various times.

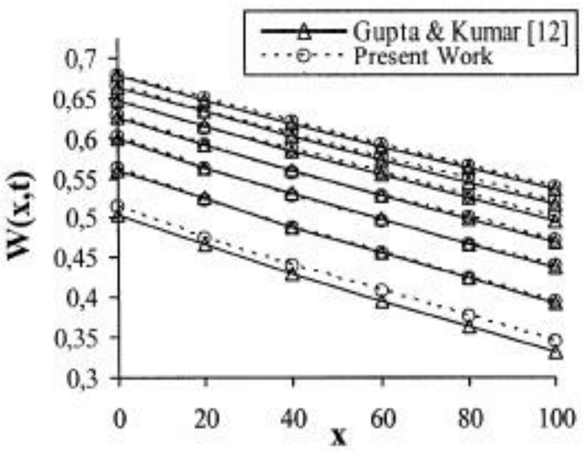

Figure 3. Graph displaying the temperature distribution in the wall, along the axis of the cylinder at various times.

REFERENCES

[1] T. Öziş , V. Gülkaç, "Application of variable interchange method for solution of two-dimensional fusion problem with convective boundary conditions," Numerical Heat Transfer, Part A, vol. 44, no. 1, 44, pp. 85-95, 2003.

[2] V. Gülkaç, "A Numerical Solution of Two-Dimensional Fusion Problem with Convective Boundary Conditions," International Journal for Computational Methods in Engineering Science and Mechanics, vol. 11, no.1, pp. 20-26, 2010.

[3] J. Crank, Numerical Methods in Heat Transfer, John Wiley, 1981.

[4] T. R. Goodmann, Application of integral Methods to Transient Non-Linear Heat Heat-Transfer, in T. F. Irvine., Jr., and J. P. Harnett (eds.), Advances in Heat Transfer, pp. 51-122. Academic Press, New-York, 1964.

[5] H. Rasmussen, "An Approximate Method for Solving Two-Dimensional Stefan Problems," Lett. Heat Mass Transfer, vol. 4, pp. 273-277, 1997.

[6] C. W. Cryer, A Survey of Steady-State Porous Flow Free Boundary Problems, MRC Tech. Summary Report 1657, University of Wisconsin, Madison, 1976. vol.R. M. Furzeland , A Survey of the Formulation and Solution of Free and Moving Boundary Problems, Technical Report TR76, Department of Mathematics, Brunel University, London, England, 1977.

[7] R. M. Furzeland , "Symposium on Free and Moving Boundary Problems in Heat Flow and Diffusion," Bull. Inst. Maths Applics, vol. 15, pp. 172-176, 1979.

[8] J. M. Aitchison, "Numerical Treatment of a Singularity in a Free Boundary Problem," Proceedings of the Royal Society of London, Series A, pp. 573-580, 1972.

[9] J. M. Aitchison, "The Numerical Solution of a Minimization Problem Associated with a Free Surface Flow," J. Inst. Maths Applics, vol. 20, pp. 33-44, 1977.

[10] H. G. Landau, "Heat Conduction in a Melting Solid," Qart. Appl. Math., vol. 8, pp. 81-94, 1950.

[11] R. S. Gupta, A. Kumar, "Isotherm Migration Method Applied to Fusion Problems with Convective Boundary Conditions," Int. J. Numer. Meth. Eng. vol. 26, pp. 2547-2558, 1988.

[12] D. H. Ferris and S. Hill, Report NA C45, National Physical Laboratory, Teddington, 1974.

[13] R. T. Beaubouef and A. J. Chapman, "Freezing of Fluids in Forced Flow", Int. J. Heat Mass Transfer, vol. 10, pp. 1581-1587, 1967.

[14] J. L. Duda , M. F. Malone, R. H. Noter, J. S. Vrentas, "Analysis of Two-Dimensional Diffusion Controlled Moving Boundary Problems," Int. J. Heat Mass Transfer, vol. 18, pp. 901-910, 1975.

[15] V. Gülkaç, T. Öziş, "On a LOD Method for Solution of Two-Dimensional Fusion Problem with Convective Boundary Conditions," International Communications in Heat and Mass Transfer, vol. 31, no.4, pp. 597-606, 2004.

[16] V. Gülkaç, "On the finite differences schemes for the numerical solution of two dimensional moving boundary problem," Applied Mathematics and Computation, vol. 168, no.1, pp. 549-556, 2005.

[17] W. J. Minkowycz and E. M. Sparrow, "Local Non-Similar Solutions for Natural Convection on a Vertical Cylinder," Journal of Heat Transfer, vol. 96, no. 2, pp. 178-183, 1974.

[18] P. Jiang, Z. Ren, "Numerical Investigation of Forced Convection Heat Transfer in Porous Media Using a Thermal Non-Equilibrium Model," International Journal of Heat and Fluid Flow, vol. 22, no. 1, pp. 102-110, 2001.

[19] J. L. Lage, "The Fundamental the Theory of Flow Through Permeable Media from Darcy to Turbulence," Transport Phenomena in Porous Media (D. M. Ingham & I. Pop. Eds.) Elsevier Science, Oxford, pp. 1-30, 1998.

[20] R. Rannacher, "Finite Element Solution of Diffusion Problems with Irregular Data," Numer. Math. vol. 43, no. 2, pp. 308-327, 1976.

[21] E. M. Sparrow, C. F. Hsu, "Analysis of Two-Dimensional Freezing an Outside of a Coolant Carying Tube," Int. J. Heat Mass Transfer, vol. 24, pp. 1345-1357, 1981.

[22] G. Strang, "On the construction and comparison of difference schemes," SIAM J. Numer. Anal. vol. 5, pp. 506-517, 1968.

[23] R.J. LeVeque, Finite Volume Methods for Hyperbolic Problems, in:Cambiridge Texts in Applied Mathematics. Cambiridge University Press, Cambridge, UK, 2002.

[24] R. E. Bank, W.M. Coughan, Jr. W. Fichtner, E. H. Grosse, D. J. Rose, R.K. Smith, “Transient Simulation of Silicon Devices and Circuits,” IEEE Trans. Comput. Aided De sign CAD- vol. 4, no. 4, pp. 436-450, 1985.

Conflict Interest

The author declare that there is no conflict of interests regarding the publication of this article.