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The Interpretation of the Apparent Motion of the Sun Through the Spherical Spiral

Küresel Spiral Aracılığıyla Güneşin Görünürdeki Hareketinin Yorumlanması

Deniz Güçler*, Nejat Ekmekçi

Ankara University, Faculty of Science, Department of Mathematics, Ankara, Türkiye

Abstract

This paper shows that the apparent motion of the Sun traces a spherical spiral in a limited section of the sphere. The rotation operator that produces the curve is in turn produced by a quaternion. The way the components of the quaternion correspond to the daily apparent motion of the Sun is shown. To achieve this, first, the spherical spiral formula is obtained using quaternions. It is known that the spherical spiral is a locus of a point P moving at constant angular speed ω along the meridian of a sphere while also rotating at constant angular speed cω around the polar axis where c > 2. Therefore, two rotations occur at the same time, and for each rotation, a quaternion can be defined. The directions of these quaternions form a 90° angle with each other. In this paper, it is shown that if this angle was 23° 27' then the curve that would form would coincide with the apparent motion of the Sun. The product of these quaternions gives the quaternion which produces the rotation operator that forms this curve. Afterward, it is shown that the quaternion rotation angle converted in time units displays the time the Sun needs to move from one point to another. On the other hand, the rotational axis for small angles is the same as the axis of the equatorial plane. The importance of this work is twofold: it gives the science of astronomy a new perspective regarding the interpretation of the apparent motion of the Sun, and at the same time it is an important example of a work that shows the convenience that the use of quaternions brings to other fields of science.

Keywords: Spherical spiral, quaternions, apparent motion of the sun, rotational motion

Öz

Bu makale, Güneş'in, günlük ve yıllık görünürdeki hareketlerinin birleşimi ile, kürenin sınırlı bir kesitinde bir spiral çizdiğini göstermektedir. Eğriyi oluşturan dönme operatörü bir kuaterniyon tarafından üretilmiştir. Kuaterniyon bileşenlerinin, Güneş'in günlük hareketine ne şekilde karşılık geldikleri gösterilmiştir. Bunun için öncelikle kuterniyonlar yardımıyla küresel spiralin formülü elde edilmiştir. Küresel spiralin, bir kürenin meridyeni boyunca sabit ω açısal hızıyla hareket ederken aynı zamanda, c > 2 olmak üzere, kutup ekseni etrafında cω açısal hızıyla dönen bir P noktasının yeri olduğunu göz önünde bulundurarak, her dönme hareketi için bir kuaterniyon tanımlanmıştır. Bu kuaterniyonların yönleri birbirleriyle 90° açı oluşturur. Bu açı 23° 27' olsaydı, oluşacak eğri Güneş'in görünürdeki hareketiyle çakışırdı. Bu kuaterniyonların çarpımı, bu eğriyi oluşturan, dönme operatorünü üreten kuaterniyonu verir. Zaman birimine dönüştürülen kuaterniyonun dönüş açısı, Güneş'in bir noktadan diğerine hareket etmesi için gereken süreyi verir. Küçük açılar için dönme ekseni, ekvator düzleminin ekseni ile aynıdır. Bu çalışmanın önemi iki yonlüdür; Astronomi bilimine, Güneş'in görünürdeki hareketinin yorumlanmasında yeni bir bakış açısı kazandırıyor olması, ve aynı zamanda kuaterniyonların kullanımının, bilimin diğer alanlarına getirdiği kolaylığa önemli bir örnek sunmasıdır.

Anahtar Kelimeler: Küresel spiral, kuaterniyonlar, güneş'in görünürdeki hareketi, dönme hareketi

1. Introduction

It is known that a spherical spiral is a locus of a point P moving at constant angular speed ω along the meridian of a sphere while also rotating at constant angular speed cω around the polar axis, where $c > 2$. This means that a point P that is located in the sphere with radius $r = 1$ and with center 0 in a coordinate system XYZ moves with a constant angular velocity ω across the great circle which passes through points P and (0,0,1). Additionally, at the same time point P also moves parallel to plane XY with a constant angular velocity cω.

A body performs a circular motion around an axis called the rotation axis. Point P makes two circular motions simulta-

*Corresponding author: deniz.gucler@yahoo.com

Deniz Güçler orcid.org/0000-0003-0376-0294
Nejat Ekmekçi orcid.org/0000-0003-1246-2395

neously that each have a rotational axis. As a result, a spherical spiral is drawn. The first rotational motion of point P has as a rotational axis, axis Y. The second rotational motion of this point has as a rotational axis, axis Z. Between these two axes a 90° angle is found.

At this point three questions come to mind: If the angle between these two rotational axes is different from 90° will the curve that forms still be considered a spherical spiral? Is there an equivalent motion in nature? If there is, how can it be interpreted?

The purpose of this paper is to answer the above-mentioned questions. To achieve this, the author has benefited from references (Altman 1986; Delphenich 2012; Hacısalihoğlu 1983; Kuipers 1975; Kuipers 1998; Griffin 2017; Dong et al. 2020) to present and understand the rotation operators that are produced by quaternions and which will be used to express the rotation motions of point P. The author has benefited from references (Karaali 1985; Kızılırmak 1977; Kummer 1996; Lowenstein 2012; Motz and Duveen 1966; Woolard and Clemence1966), to present the problem and the apparent motion of the Sun. The information needed for the understanding of spherical spirals is found in reference **(**Fisher and Ziebur 1965). A previous paper has been published on the method of defining the apparent motion of the Sun with the help of quaternions. This paper, which is found in reference (Güçler et al. 2022) is used as an example of a motion in nature of the type that was mentioned above. Afterward, the required interpretation has been done.

The main idea that gave rise to this paper was first presented at the 19th International Geometry Symposium by the authors of this paper (Güçler and Ekmekci 2022).

2. Preliminaries

In this paper, rotational operators produced by quaternions have been used to calculate the rotation sequences and express the rotation motions. For this reason, basic knowledge has been provided below about quaternions and the rotation operators produced by them.

A quaternion is a hyper-complex number of rank 4. The most important rule of this invention of Hamilton is:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1}$$

i, j, and k are the components of the vector part of the quaternion and they will be used to represent the standard orthogonal base of R^3.

The quaternion can be thought of as a quadruple of real numbers. This makes it an element of R^4. Accordingly, quaternion m can be expressed as below where m_0, m_1, m_2 and m_3 are each a real number.

$$m = m_0 + \alpha = m_0 + im_1 + jm_2 + km_3,\ \alpha = im_1 + jm_2 + km_3 \tag{2}$$

where m_0 is the scalar part and α is the vector part.

Multiplication of quaternions is done according to the following rule.

$$i_2 = j_2 = k_2 = ijk = -1 \text{ ve } ij = k = -ji, jk = i = -kj, ki = j = -ij \tag{3}$$

For $m = m_0 + \alpha_m = m_0 + im_1 + jm_2 + km_3$ *and* $n = n_0 + \alpha_n = n_0 + in_1 + jn_2 + kn_3$

$$m \times n = (m_0 + im_1 + jm_2 + km_3) \times (n_0 + in_1 + jn_2 + kn_3) \tag{4}$$

$$m \times n = m_0 n_0 - \langle \alpha_m, \alpha_n \rangle + m_0 \alpha_n + n_0 \alpha_m + \alpha_m \wedge \alpha_n \tag{5}$$

"$\langle , \rangle$ represents the scalar product of vectors, and $\wedge$ represents the cross product of vectors.

The complex conjugant of $m = m_0 + im_1 + jm_2 + km_3$ is $m^* = m_0 - im_1 - jm_2 - km_3$

Definition: The quaternion whose scalar part is zero is called a pure quaternion.

The unit quaternion $m = m_0 + \alpha$ satisfies the following equality $m_0^2 + |\alpha|^2 = 1$.

The quaternion that will be used as a rotation operator is:

$$m = m_0 + \alpha = \cos\phi + u \sin\phi \text{ and } m^* = m_0 - \alpha = \cos\phi - u\sin\phi \tag{6}$$

where $u = \alpha/|\alpha| = \alpha/\sin\phi$

Theorem 1: For any $q = q_0 + \boldsymbol{q} = \cos\phi + u\sin\phi$ unit quaternion (where q_0 is the scalar part and **q** is the vector part of the quaternion) and for any vector $v \in R^3$ the action of the operator

$$L_q(v) = q \times v \times q^*$$

on v may be interpreted geometrically as a rotation of the vector v through an angle 2φ about **q** as the axis of the rotation. According to Kuipers (1998).

Theorem 2: Suppose that q and p are unit quaternions that define the quaternion rotation operators:

$$Lq(u) = q \times u \times q^* \text{ and } L_p(v) = p \times v \times p^*$$

Then the quaternion product $p \times q$ defines a quaternion operator L_{pq} which represents a sequence of operators, L_q followed by L_p. The axis and the angles of rotation are those represented by the quaternion product, $r = p \times q$. According to Kuipers (1998).

Information must be provided on the apparent motion of the Sun as well. For this reason, the apparent yearly and daily motion of the Sun have been introduced below.

As it is known, the Earth rotates every day in a positive direction around its axis and parallel to the equatorial plane. This motion is reflected to an observer on Earth, as the apparent motion of the Sun, occurring in the negative direction.

It is also known that the Earth rotates every year around the Sun in the positive direction, in an ecliptic orbit found in the ecliptic plane. However, this motion appears to an observer on Earth, as if it was the Sun moving around the Earth during the year in a positive direction (Figure 1). The ecliptic plane intersects with the celestial equatorial plane and creates a angle. According to Karaali (1985).

Definition 2.1: The coordinate system defined by taking the celestial equator as the fixed plane, taking O& as the fixed line, and taking the center of the celestial sphere as the fixed point, is called the Celestial Equatorial Coordinate System.

3. The Interpretation of the Apparent Motion of the Sun Through the Spherical Spiral

As stated before, a spherical spiral is a locus of a point P moving at constant angular speed P along the meridian of a sphere while also rotating at constant angular speed cω around the polar axis, where . Because there exists a direct proportion between the velocities and the angles, point P will accordingly trace angles θ and cθ. Hereafter, it will be accepted that the motion will occur in a sphere with a radius $r = 1$.

Now let us determine the quaternions that will generate rotational operators that will perform each rotational motion. Since the first rotation takes place in the XZ plane in the positive direction with an angle θ, the direction of the quaternion used in the rotation operator is the same as the direction of the rotation axis, meaning $u_1 = j$. In this case, the quaternion generating the first operator, according to Theorem 1 is:

$$K_1 = \cos \theta/2 + j \sin \theta/2 \tag{7}$$

Since the second rotation takes place in the XY plane in the positive direction with an angle of cθ, the direction of the quaternion used in the rotation operator is the same as the direction of the rotation axis, meaning $u_2 = k$. In this case, the quaternion generating the second operator, according to Theorem 1 is:

$$K_2 = \cos (c\theta)/2 + k \sin (c\theta)/2 \tag{8}$$

The result of transferring vector v to the quaternion space is:

$$v = (1 ,0,0) \rightarrow w = 0 + 1i + 0j + 0k = i \tag{9}$$

The vector that is obtained is a pure quaternion. According to Theorem 2:

For $L_{K1}(w) = K_1 \times w \times K_2^*$, $L_{K2}(m) = K2 \times m \times K_2^*$, $m = K_1 \times w \times K_2^*$:

$$L(w) = (K_2 \times K_1)\ w\ (K_2 \times K_1)^* \tag{10}$$

If $K_2 \times K_1 = K$ and for $w = i$, $L(w) = K \times i \times K^*$ is shown. In this case:

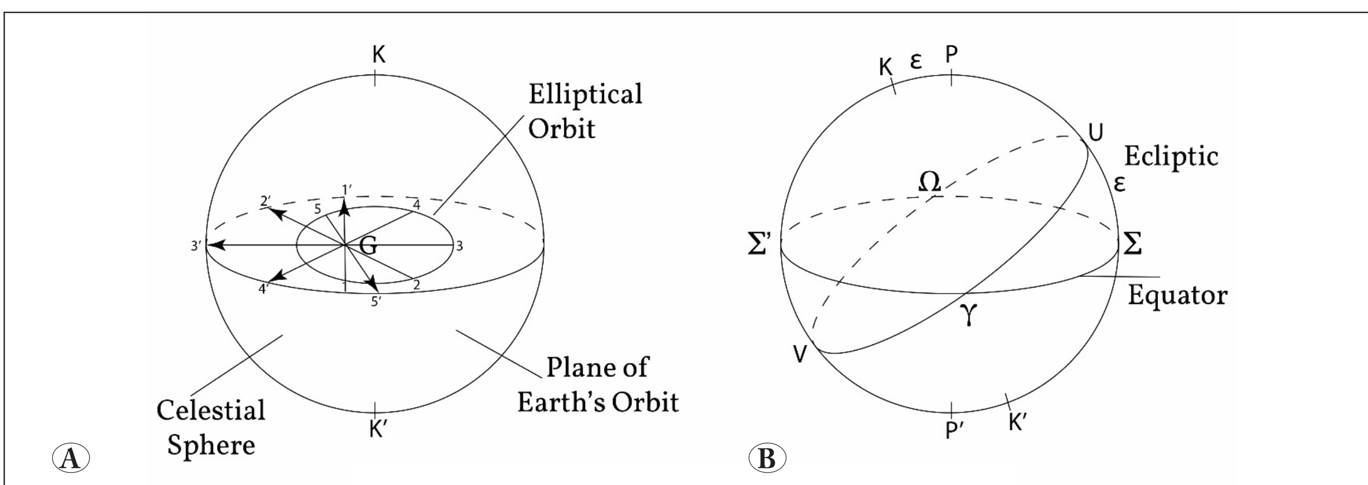

Figure 1. The elliptical orbit made by the actual motion of the Earth **(A)** the elliptical circle made by the annual motion of the Sun **(B)**.

$K = (\cos(c\theta)/2 + k \sin(c\theta)/2) \times (\cos\theta/2 + j \sin\theta/2)$ (11)

$K = \cos(c\theta)/2\ \cos\theta/2 - i \sin(c\theta)/2 \sin\theta/2 + j \cos(c\theta)/2 \sin\theta/2 + k \sin(c\theta)/2 \cos\theta/2$ (12)

Accordingly, when the rotation operator produced by the K quaternion is applied to $w = (i,0,0)$, the pure quaternion $W = (W_1, W_2, W_3)$ is obtained as shown below.

$W = K \times w \times K^*$ (13)

When calculations are made,

$W_1 = (\cos\theta \cos(c\theta))\, i$, $W_2 = (\sin(c\theta) \cos\theta)\, j$, $W_3 = (-\sin\theta)\, k$ (14)

is found. As a result of transferring pure quaternion W to the vector space, $V = (V_1, V_2, V_3)$ is obtained.

For $c > 2$, $0 \le \theta \le 2\pi$ where c is constant

$$\begin{cases} V_1 = X = \cos\theta\cos(c\theta) \\ V_2 = Y = \cos\theta\sin(c\theta) \\ V_3 = Z = -\sin\theta \end{cases} \quad (15)$$

is found. This is the parametric equation of the spherical spiral with the radius r = 1 (Figure 2).

It has been possible to express the spherical sphere equation with the help of the rotation operators. These operators are produced by the quaternions, the directions of which are found in the axes Y and Z.

But what if the angle between these two rotational axes is different from 90°? Can the curve that is formed still be considered a spherical spiral? Is there an equivalent motion in nature? If so, how can it be interpreted? To answer these questions, let us examine the apparent daily and yearly motion of the Sun (Figure 3). As seen in Figure 3, the angle between the two rotation axes is different from 90°.

In this paper, it is assumed the apparent motion of the Sun occurs in ideal conditions. So, it will be accepted that the apparent motion of the Sun occurs in a circular orbit with a constant angular velocity in the ecliptic plane.

Now let plane E represent the elliptic plane while plane XY represents the plane of the celestial equator, and angle $\varepsilon = 23°27'$ represents the angle which is the angle that is formed from the intersection of the celestial equatorial plane and the ecliptic plane (Figure 3). In this case, point represents (0,0,0)the Earth. In addition, the positive direction of axis X will represent the Aries constellation.

Let Q_1 be the quaternion that will realize the motion in the positive direction around axis N. Let Q_1^* be the quaternion that will realize the motion in the negative direction around axis Z. The starting point of the motion is $P = (1,0,0)$ which coincides with the Aries constellation. The vector OP that is found in the direction of the Earth-Aries constellation is $v = (1,0,0)$. First, let this vector be transferred to the quaternion space so:

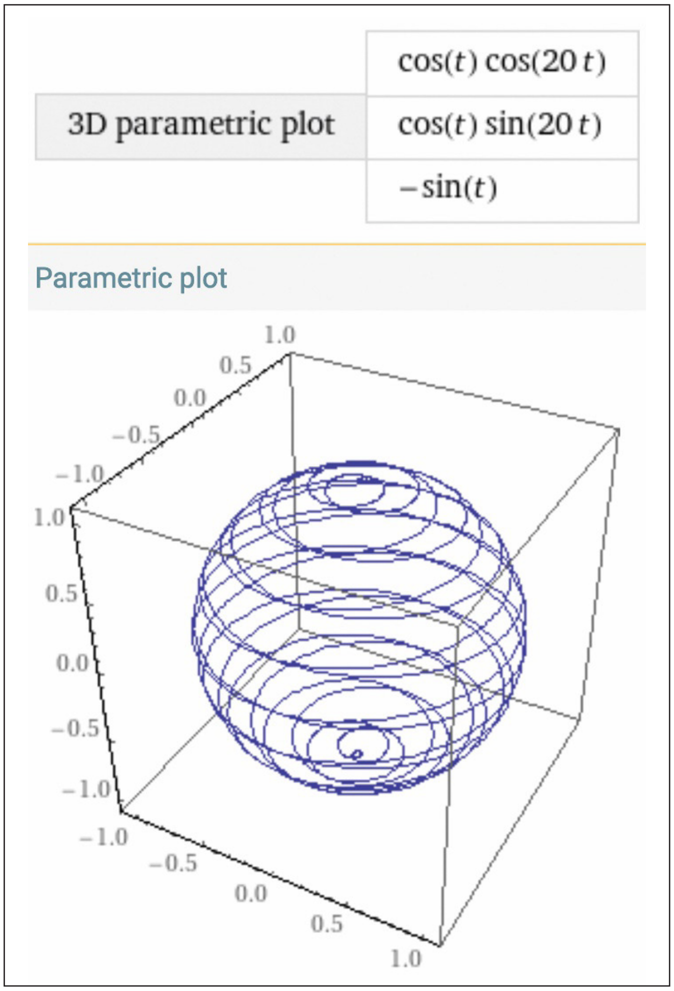

Figure 2. Spherical Spiral, c = 20

$v = (1,0,0)$ *vector* $\rightarrow \omega = 0 + i + 0j + 0k = i$ (16)

corresponds to a pure quaternion. The first rotation motion will be realized around axis $u = -j \sin\varepsilon + k \cos\varepsilon$ with θ angle. The second rotation motion will be realized around axis k with a $(c\theta)$ angle in a negative direction. In this case, the Q_1 and Q_2^* quaternions that will operate as rotation operators, for $a = \sin\varepsilon$ and $b = \cos\varepsilon$, are:

$Q_1 = \cos\theta/2 - j\, a \sin\theta/2 + k\, b \sin\theta/2$ *and* $Q_2^* = \cos(c\theta)/2 - k \sin(c\theta)/2$ (17)

According to Theorem 2 and if $Q_2^* \times Q_1 = Q$ and $w = i$ then

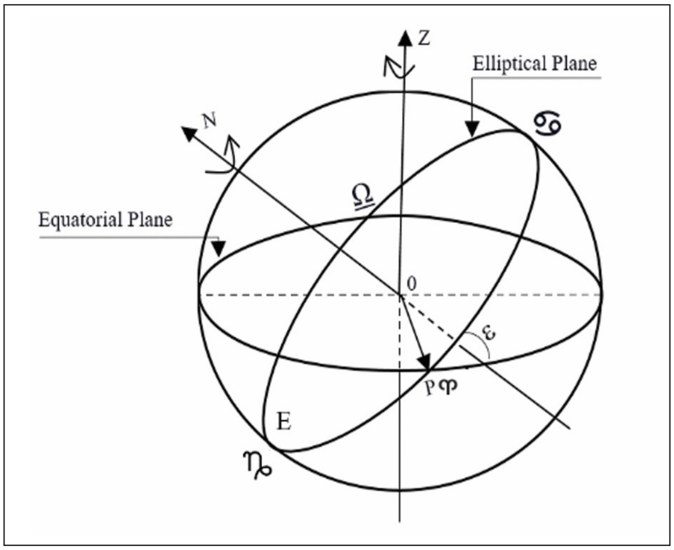

Figure 3. The system in which the apparent motion of the Sun occurs.

$$L(w_1) = Q \times i \times Q^* \tag{18}$$

So the calculations are as such:

$$Q = Q_2^* \times Q_1 = (cos(c\theta)/2 - k\, sin(c\theta)/2) \times (cos\, \theta/2 - j\, a sin\, \theta/2 + k\, b sin\, \theta/2) \tag{19}$$

$$Q=(cos(c\theta)/2\, cos\, \theta/2 + b\, sin(c\theta)/2\, sin\, \theta/2) - i\, a\, sin(c\theta)/2\, sin\, \theta/2 - j\, a\, cos(c\theta)/2\, sin\, \theta/2 \tag{20}$$

Accordingly, when the rotation operator produced by the Q quaternion is applied to $w = (i,0,0)$,, the pure quaternion $W = (W_1, W_2, W_2)$ is obtained as shown below.

$$W = Q \times i \times Q^* \tag{21}$$

When the calculations are made

$$W_1 = (cos(c\theta)cos\, \theta + b sin(c\theta)\, sin\theta)i,\ W_2 = (b cos(c\theta) sin\, \theta - sin(c\theta)\, cos\theta)j,\ W_3 = (a sin\theta)k \tag{22}$$

is found. As a result of transferring pure quaternion W to the vector space, $V = (V_1, V_2, V_3)$ is obtained.

For $0 \leq \theta \leq 2\pi$, $c = 365.25$ (365.25 the number of days in a year), $a = sin\ 23^0\ 27'$ and $b = cos\ 23^0\ 27'$

$$\begin{cases} V_1 = X = \cos\theta\cos(c\theta) + b\sin\theta\sin(c\theta) \\ V_2 = X = b\sin\theta\cos(c\theta) - \cos\theta\sin(c\theta) \\ V_3 = Z = a\sin\theta \end{cases} \tag{23}$$

If the graphic of the equation (23) we obtained above was drawn, this curve would cover the entirety of the sphere found between the planes $z = -\, sin\ 23^0 27'$ and $z = sin\ 23^0 27'$ because the constant c is $c = 365.25$. For this reason, to be able to visually show the shape of the curve, $c = 12$ is chosen instead of $c = 365.25$ and this way the graphic shown in Figure 4 is obtained. According to Güçler et al. (2022).

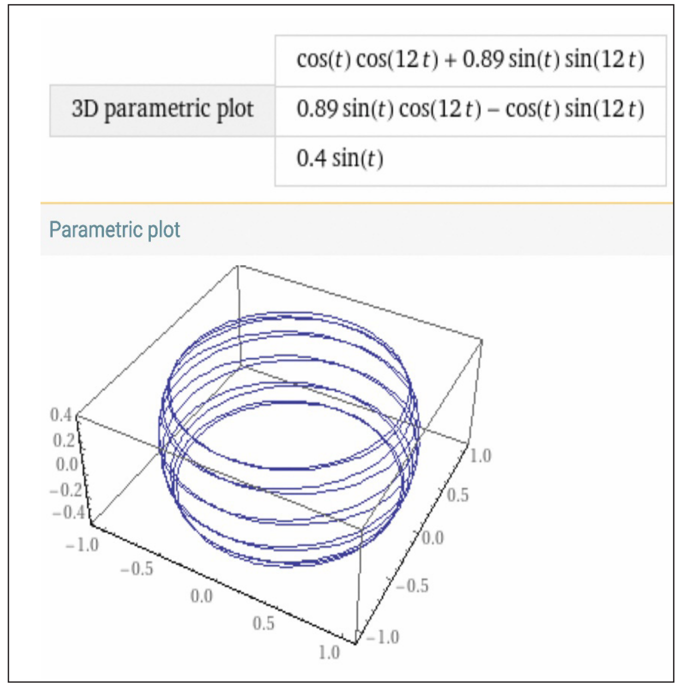

Figure 4. The curve of the apparent motion of the Sun for c = 12.

If in equation (23) $\varepsilon = -90^0$ and it is accepted that the second rotation occurs in the positive direction, then the parametric equation of the spherical spiral is procured. This means that the answer to the question "Can the curve that is formed still be considered a spherical spiral?" that was posited above is an affirmative " Yes, it can still be considered a spherical spiral." However, the question arises, "Are there other examples that can be provided to clarify this subject?". The answer to this question is also an affirmative "Yes, there are".

Another example of this motion is the apparent motion of the Sun as seen by an observer found on another planet in the solar system, for example, on planet Venus (Güçler 2023) or Mercury (Güçler 2023). Because the speed with which these planets move around themselves is much slower than the speed with which Earth rotates around itself, the advantages which the use of the method brings are more clearly apparent.

As it is shown in Figure 4, the curve that is obtained is a spherical spiral confined to a particular region. So what does the quaternion used to obtain this curve tell us? To answer this, we will examine the quaternion Q. For $Q = (q_0, q_1,) q_2, q_3)$:

$$\begin{cases} q_0 = (\cos(c\theta)/2\cos\theta/2 + b\sin(c\theta)/2\sin\theta/2) \\ q_1 = (-a\sin(c\theta)/2\sin\theta/2) \\ q_2 = (-a\cos(c\theta)/2\sin\theta/2) \\ q_3 = (b\cos(c\theta)/2\sin\theta/2 - \sin(c\theta)/2\cos\theta/2) \end{cases} \quad (24)$$

By showing $"Q" = \cos\psi/2 + \alpha_q \sin\psi/2$, $\alpha_q = (q_1, q_2, q_3)$, we have,

$$\cos\psi/2 = \cos(c\theta)/2\cos\theta/2 + b\sin(c\theta)/2\sin\theta/2 \quad (25)$$

From this equation, ψ angle of rotation can be found.

The Sun completes its motion in the ecliptic orbit in approximately 365.25 days. Therefore, in one day $\Delta\theta \approx 1^0$, where $\Delta\theta$ is the difference between angle θ_2 taken by the sun in x+1 days, and angle θ_1 taken by the Sun in x days. Due to this, if we analyse the apparent daily motion of the Sun, the angular distance between the two points that the Sun appears in, in the celestial sphere will be $\Delta\theta < 1^0$. This means that $\Delta\theta/2 < (1/2)^0$. If we calculate the value of $\sin\Delta\theta/2$ for such a small angle the result can be accepted as equal to zero and the value of $\cos\Delta\theta/2$ can be accepted as equal to one. In this case:

$$\cos\psi/2 = \cos(c\Delta\theta/2)\cos\Delta\theta/2 + b\sin(c\Delta\theta/2)\sin\Delta\theta/2 = \cos(c\Delta\theta/2) \quad (26)$$

$$\cos\psi/2 = \cos 365{,}25\,(\Delta\theta/2) \rightarrow \psi = 365{,}25\,\Delta\theta \quad (27)$$

Additionally:

$$\begin{cases} q_1 = -a\sin(c\Delta\theta/2)\sin\Delta\theta/2 = 0 \\ q_2 = -a\cos(c\Delta\theta/2)\sin\Delta\theta/2 = 0 \\ q_3 = b\cos(c\Delta\theta/2)\sin\Delta\theta/2 - \sin(c\Delta\theta)/2\cos\Delta\theta/2 = \\ -\sin(c\Delta\theta)/2 \end{cases} \quad (28)$$

The direction $\alpha_q = (q_1, q_2, q_3)$ of quaternion Q found as a result of the calculation above, is the same as the direction of $Q_2^* = \cos(c\theta)/2 - k\sin(c\theta)/2$. So α_q is the same as the axis of the apparent daily motion of the Sun. This motion occurs in parallel to the equatorial plane. This is an expected result because when astronomers study the apparent daily motion of the Sun, they accept that this motion takes place parallel to the equatorial plane.

When the angle of rotation $\psi = 365.25\,\Delta\theta$ is converted from degrees to units of time, the time taken for the Sun to travel from one point to the other is obtained.

4. Conclusion

First conclusion: Let an object move with the velocity ω of the sphere with a radius r along any great circle of the sphere and let this object move with the velocity cω, where $c > 2$, parallel to XY plane. In this case, this object traces a spiral confined in a section such as $2r\sin\vartheta$, where ϑ is the small angle between the plane determined by the great circle and XY plane. This section lies between two parallel planes that cut through the sphere. Each plane is located at a distance equal to the $r\sin\vartheta$ of the XY plane.

Second conclusion: The apparent motion of the Sun can be shown as an example of the graph described in the first result. In addition, the quaternion used in obtaining this curve provides important data for interpreting the apparent motion of the Sun.

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