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Some sums related to the terms of generalized Fibonacci autocorrelation sequences $\left\{a_{k,n}(\tau)\right\}_{\tau}^{\infty}$

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ABSTRACT

In this paper, we give the terms of the generalized Fibonacci autocorrelation sequences $\left\{a_{k,n}(\tau)\right\}_{\tau}^{\infty}$ defined as

$$a_{k,n}(\tau) := a_n(U_{ki}, \tau)$$

and some interesting sums involving terms of these sequences for an odd integer number k and nonnegative integers τ, n.

Keywords: Fibonacci numbers, generalized Fibonacci autocorrelation sequences, sums

$\left\{a_{k,n}(\tau)\right\}_{\tau}^{\infty}$ geneleştirilmiş Fibonacci otokorelasyon dizilerinin terimlerini içeren bazı bağıntılar

ÖZ

Bu makalede, k tek tamsayı ve τ, n negatif olmayan tamsayı olmak üzere

$$a_{k,n}(\tau) := a_n(U_{ki}, \tau)$$

terimlerine sahip $\left\{a_{k,n}(\tau)\right\}_{\tau}^{\infty}$ genelleştirilmiş Fibonacci otokorelasyon diziler ve bu dizilerin terimlerini içeren bazı toplamlar verildi.

AnahtarKelimeler: Fibonacci sayıları, genelleştirilmiş Fibonacci otokorelasyon dizileri, toplamlar

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1. GİRİŞ (INTRODUCTION)

For $a,b,p,q \in \mathbb{R}$, the second order sequence $\{W_n(a,b;p,q)\}$ is defined for $n>0$ by

$$W_{n+1}(a,b;p,q) = pW_n(a,b;p,q) - qW_{n-1}(a,b;p,q)$$

in which $W_0(a,b;p,q)=a,\ W_1(a,b;p,q)=b$. When $q=-1$, $W_n(0,1;p,-1)=U_n$ and $W_n(2,p;p,-1)=V_n$. When $p=1$, $U_n=F_n$ (n th Fibonacci number) and $V_n=L_n$ (n th Lucas number).

If α and β are the roots of equation $x^2-px-1=0$ the Binet formulas of the sequences $\{U_n\}$ and $\{V_n\}$ have the forms

$$U_n=\frac{\alpha^n-\beta^n}{\alpha-\beta} \text{ and } V_n=\alpha^n+\beta^n,$$

respectively.

E. Kılıç and P. Stanica [1], derived the following recurrence relations for the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ for $k\geq 0$, $n>0$,

$$U_{k(n+1)}=V_kU_{kn}+(-1)^{k+1}U_{k(n-1)}$$

and

$$V_{k(n+1)}=V_kV_{kn}+(-1)^{k+1}V_{k(n-1)},$$

where the initial conditions of the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ are 0, U_k and 2, V_k respectively. The Binet formulas of the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ are given by

$$U_{kn}=\frac{\alpha^{kn}-\beta^{kn}}{\alpha-\beta} \text{ and } V_{kn}=\alpha^{kn}+\beta^{kn},$$

respectively.

P. Filipponi and H.T. Freitag [2] defined the terms $a_n(S_i,\tau)$ of the autocorelation sequences of any sequence $\{S_i\}_0^\infty$ as

$$a_n(S_i,\tau):=\sum_{i=0}^{n}S_iS_{i+\tau},\ (0\leq\tau\leq n), \tag{1}$$

where the subscript $i+\tau$ must be considered as reduced modulo $n+1$ and τ, n are nonnegative integers. It is clearly that autocorrelation sequences differ from the definition of cyclic autocorrelation function for periodic sequences with period n+1 [3].

For positive integer number τ, the authors gave

$$a_n(S_i,\tau)=a_n(S_i,n-\tau+1)$$

and

$$a_n(S_i,\tau)=\sum_{i=0}^{n-\tau}S_iS_{i+\tau}+\sum_{i=0}^{\tau-1}S_{i+n-\tau+1}S_i.$$

The terms of the Fibonacci autocorrelation sequences $\{a_{k,n}(\tau)\}_\tau^\infty$ were defined as

$$a_n(\tau):=a_n(F_i,\tau)$$

and they obtained some sums involving the terms $a_n(\tau)$ as follows:

$$\sum_{i=0}^{n}a_n(i)=(F_{n+2}-1)^2,$$

$$10\sum_{i=0}^{n}\binom{n}{i}a_n(i) = \begin{cases} 2L_{3n+2}-5F_{2n+2}+L_{n+1}, & \text{if } n \text{ is even} \\ 2L_{3n+2}-L_{n+1}(5F_{n+1}-1), & \text{if } n \text{ is odd} \end{cases}.$$

Inspiring by studies in [2], we consider subsequence $\{S_i\}_0^\infty$ of the autocorrelation sequences of subsequence $\{S_{ki}\}_0^\infty$ defined as

$$a_n(S_{ki},\tau):=\sum_{i=0}^{n}S_{ki}S_{k(i+\tau)}\ (0\leq\tau\leq n), \tag{2}$$

where the subscript $i+\tau$ must be considered as reduced modulo $n+1$. It can clearly be seen that

$$a_n\left(S_{ki},\tau\right)=a_n\left(S_{ki},n-\tau+1\right) \quad (3)$$

and

$$\sum_{i=0}^{n}S_{ki}S_{k(i+\tau)}=\sum_{i=0}^{n-\tau}S_{ki}S_{k(i+\tau)}+\sum_{i=0}^{\tau-1}S_{k(i+n-\tau+1)}S_{ki},$$

where τ is positive integer number.

For example, for $n=6$, $k=5$ and $\tau=3$ in (3),

$$\begin{aligned}&a_6\left(S_{5i},3\right)\\&=S_0S_{15}+S_5S_{20}+S_{10}S_{25}+S_{15}S_{30}+S_{20}S_0\\&+S_{25}S_5+S_{30}S_{10}\\&=a_6\left(S_{5i},4\right).\end{aligned}$$

In this paper, taking generalized Fibonacci subsequence $\left\{U_{ki}\right\}_0^{\infty}$ instead of subsequence $\left\{S_{ki}\right\}_0^{\infty}$ in (2), we write the terms of the generalized Fibonacci autocorrelation sequences $\left\{a_{k,n}\left(\tau\right)\right\}_{\tau}^{\infty}$ as

$$a_{k,n}\left(\tau\right)=\sum_{i=0}^{n}U_{ki}U_{k(i+\tau)}$$

and obtain some sums involving the numbers $a_{k,n}\left(\tau\right)$, where an odd integer k and nonnegative integers τ, n. Throughout this paper, we will take $\left\{W_n\right\}$ instead of $\left\{W_n\left(a,b;p,q\right)\right\}$.

The following Fibonacci identities and sums in [4] will be used widely throughout the proofs of Theorems:

$$V_{k(m+n)}+V_{k(m-n)}=\begin{cases}V_{km}V_{kn}, & \text{if } n \text{ is even}\\ \Delta U_{km}U_{kn}, & \text{if } n \text{ is odd}\end{cases}, \quad (4)$$

$$V_{k(m+n)}-V_{k(m-n)}=\begin{cases}\Delta U_{km}U_{kn}, & \text{if } n \text{ is even}\\ V_{km}V_{kn}, & \text{if } n \text{ is odd}\end{cases}, \quad (5)$$

$$U_{k(m+n)}+U_{k(m-n)}=\begin{cases}U_{km}V_{kn}, & \text{if } n \text{ is even}\\ V_{km}U_{kn}, & \text{if } n \text{ is odd}\end{cases}, \quad (6)$$

$$U_{k(m+n)}-U_{k(m-n)}=\begin{cases}V_{km}U_{kn}, & \text{if } n \text{ is even}\\ U_{km}V_{kn}, & \text{if } n \text{ is odd}\end{cases},$$

(7)

$$\begin{aligned}&\sum_{i=r}^{n}W_{k(ci+d)}\\&=\Big[W_{k(cr+d)}-W_{k(c(n+1)+d)}-(-1)^cW_{k(c(r-1)+d)}\\&\quad+(-1)^cW_{k(cn+d)}\Big]\Big/\left(1-V_{kc}+(-1)^c\right)\end{aligned} \quad (8)$$

$$\begin{aligned}&\sum_{i=r}^{n}(-1)^iW_{k(ci+d)}=\Big[(-1)^rW_{k(cr+d)}\\&\quad+(-1)^nW_{k(c(n+1)+d)}+(-1)^{c+r}W_{k(c(r-1)+d)}\\&\quad+(-1)^{c+n}W_{k(cn+d)}\Big]\Big/\left(1+V_{kc}+(-1)^c\right),\end{aligned} \quad (9)$$

$$\begin{aligned}&\sum_{i=r}^{n}iW_{k(ci+d)}=\Big[\left(r+2(r-1)(-1)^c\right)W_{k(cr+d)}\\&+\left(n+2(n+1)(-1)^c\right)W_{k(cn+d)}-(r-1)W_{k(c(r+1)+d)}\\&-\left(r+1+2r(-1)^c\right)W_{k(c(r-1)+d)}-(n+1)W_{k(c(n-1)+d)}\\&-\left(n+1+2n(-1)^c\right)W_{k(c(n+1)+d)}+nW_{k(c(n+2)+d)}\\&+rW_{k(c(r-2)+d)}\Big]\Big/\left(1-V_{kc}+(-1)^c\right)\end{aligned} \quad (10)$$

and

$$\begin{aligned}&\sum_{i=r}^{n}(-1)^{i-1}iW_{k(ci+d)}=\Big[n(-1)^{n+1}W_{k(c(n+2)+d)}\\&+r(-1)^{r-1}W_{k(c(n-2)+d)}-(n+1)(-1)^nW_{k(c(n-1)+d)}\\&+\left(2n(-1)^{c+n+1}-(n+1)(-1)^n\right)W_{k(c(n-1)+d)}\\&-\left((r-1)(-1)^r-2r(-1)^{c+r-1}\right)W_{k(c(r-1)+d)}\\&+\left(r(-1)^{r-1}-2(r-1)(-1)^{c+r}\right)W_{k(cr+d)}\\&+\left(n(-1)^{n+1}-2(n+1)(-1)^{c+n}\right)W_{k(cn+d)}\\&-(r-1)(-1)^rW_{k(c(r+1)+d)}\Big]\Big/\left(1+V_{kc}+(-1)^c\right),\end{aligned} \quad (11)$$

where $\Delta=\left(V_k^2+4\right)/U_k^2$.

2. SOME IDENTITIES INVOLVING THE TERMS $a_{k,n}\left(\tau\right)$ ($a_{k,n}\left(\tau\right)$ TERİMLERİNİ İÇEREN BAZI ÖZELLİKLER)

In this section, we will give closed-form expressions for terms of the generalized Fibonacci autocorrelation sequences $\{a_{k,n}(\tau)\}_{\tau}^{\infty}$. Now, we give auxiliary Lemma before the proof of main Theorems.

Lemma 2.1. Let k be an odd integer number.

For even τ,

$$V_k a_{k,n}(\tau) = \begin{cases} U_{k(n+1)}U_{k(n-\tau)} + U_{kn}U_{k\tau}, & \text{if } n \text{ is even} \\ U_{kn}\left(U_{k(n-\tau+1)} + U_{k\tau}\right), & \text{if } n \text{ is odd} \end{cases},$$

and for odd τ,

$$V_k a_{k,n}(\tau) = \begin{cases} U_{kn}U_{k(n-\tau+1)} + U_{k(n+1)}U_{k(\tau-1)}, & \text{if } n \text{ is even} \\ U_{k(n+1)}\left(U_{k(n-\tau)} + U_{k(\tau-1)}\right), & \text{if } n \text{ is odd} \end{cases}.$$

Proof. Let n and τ be even integers. Using Binet formula of generalized Fibonacci sequence $\{U_{kn}\}$, we write

$$\begin{aligned} a_{k,n}(\tau) &= \sum_{i=0}^{n-\tau} U_{ki}U_{k(i+\tau)} + \sum_{i=0}^{\tau-1} U_{ki}U_{k(i+n-\tau+1)} \\ &= \frac{1}{\Delta}\left\{\sum_{i=0}^{n-\tau}\left(\alpha^{k(2i+\tau)} + \beta^{k(2i+\tau)} - \beta^{ki}\alpha^{k(i+\tau)} - \alpha^{ki}\beta^{k(i+\tau)}\right)\right. \\ &\quad \left. + \sum_{i=0}^{\tau-1}\left(\alpha^{k(2i+n-\tau+1)} + \beta^{k(2i+n-\tau+1)} - \beta^{ki}\alpha^{k(i+n-\tau+1)} - \alpha^{ki}\beta^{k(i+n-\tau+1)}\right)\right\} \\ &= \frac{1}{\Delta}\left\{\sum_{i=0}^{n-\tau}\left(V_{k(2i+\tau)} - (-1)^{ki}V_{k\tau}\right) + \sum_{i=0}^{\tau-1}\left(V_{k(2i+n-\tau+1)} - (-1)^{ki}V_{k(n-\tau+1)}\right)\right\}. \end{aligned}$$

From (8) and the sums

$$\sum_{i=0}^{n-\tau}(-1)^i = \begin{cases} 1, & \text{if } n,\tau \text{ are same parities} \\ 0, & \text{if } n,\tau \text{ are different parities} \end{cases},$$

$$\sum_{i=0}^{\tau-1}(-1)^i = \begin{cases} 1, & \text{if } \tau \text{ is odd} \\ 0, & \text{if } \tau \text{ is even} \end{cases},$$

we write

$$\Delta a_{k,n}(\tau) = \left(V_{k(2n-\tau+1)} - V_{k(\tau+1)} + V_{k(n+\tau)} - V_{k(n-\tau)}\right)\Big/ V_k.$$

By (5), we have

$$V_k a_{k,n}(\tau) = U_{k(n+1)}U_{k(n-\tau)} + U_{kn}U_{k\tau}.$$

The other equalities are obtained similar to the proof. Thus we have the conclusion.

For example, for $a=\tau=0$ and $b=k=p=1$ in Lemma 2.1, it is clearly seen that $a_{1,n}(0) = F_{n+1}F_n$ [1].

Now, we will investigate some sums involving the terms $a_{k,n}(\tau)$.

Theorem 2.1. Let k be an odd integer number. We have

$$V_k\sum_{i=0}^{n}(-1)^i a_{k,n}(i) = \begin{cases} \left(U_{k(n+1)} - U_{kn} + U_k\right)^2 \Big/ V_k, & \text{if } n \text{ is odd} \\ U_{k(n+1)}U_{kn}, & \text{if } n \text{ is even} \end{cases}$$

and

$$V_kV_{3k}\sum_{i=0}^{n}(-1)^i a_{k,n-i}(i) = \begin{cases} U_{k(n-1)} + \left(V_{k(2n+4)} - V_{k(2n+1)}\right)\Big/\Delta \\ \quad -V_{k(n-1)} + V_{3k}\left(V_k + V_{2k}\right)\Big/\Delta V_k, & \text{if } n \text{ is odd} \\ U_{k(n+2)} + \left(V_{k(2n+4)} - V_{k(2n+1)}\right)\Big/\Delta \\ \quad -V_{k(n+2)} - V_{3k}\left(V_k - 2\right)\Big/\Delta V_k, & \text{if } n \text{ is even} \end{cases}.$$

Proof. For even number n, observed that

$$\sum_{i=0}^{n}(-1)^i a_{k,n}(i) = a_{k,n}(0) - a_{k,n}(1) + \ldots + a_{k,n}(n).$$

From the equality $a_{k,n}(i)=a_{k,n}(n-i+1)$ and Lemma 2.1, we get

$$\sum_{i=0}^{n}(-1)^i a_{k,n}(i)$$
$$=a_{k,n}(0)-a_{k,n}(1)+...-a_{k,n}(n-1)+a_{k,n}(n)$$
$$=a_{k,n}(0)=U_{k(n+1)}U_{kn}\big/V_k .$$

For odd number n, we write

$$\sum_{i=0}^{n}(-1)^i a_{k,n}(i)=a_{k,n}(0)-a_{k,n}(1)+...-a_{k,n}(n)$$
$$=a_{k,n}(0)+\sum_{\tau=1}^{(n-1)/2}a_{k,n}(2\tau)-\sum_{\tau=1}^{(n+1)/2}a_{k,n}(2\tau-1).$$

Using the equality $U^2{}_{kn}-U_{k(n-1)}U_{k(n+1)}=U_k{}^2$ in [5], (7), (8) and Lemma 2.1, we have the claimed result. The remaining formulas are similarly proven.

Theorem 2.2. Let k be an odd integer number. We have

$$V_k{}^2\sum_{i=0}^{n} i\, a_{k,n}(i)=(n+1)\times$$
$$\begin{cases}\left(U_{k(n+1)}+U_{kn}\right)\left(U_{kn}-U_k\right), & \text{if } n \text{ is odd}\\ U_{k(n+1)}\left(U_{kn}+U_{k(n-1)}-U_k\right)-U_{kn}U_k, & \text{if } n \text{ is even}\end{cases}$$

and

$$V_k^2\sum_{i=0}^{n}(-1)^i i a_{k,n}(i) \tag{12}$$
$$=\begin{cases}(n+1)\left(U_{kn}-U_{k(n+1)}\right)\left(U_{kn}-U_k\right), & \text{if } n \text{ is odd}\\ (n+1)\left(U_{kn}U_k-U_{k(n+1)}U_{k(n-1)}\right) & \\ \quad +(n-1)U_{k(n+1)}\left(U_{kn}-U_k\right) & \\ \quad +4U_{kn}\left(U_{k(n+1)}-U_{kn}\right)\big/V_k, & \text{if } n \text{ is even}\end{cases}.$$

Proof. For odd number n, we write

$$\sum_{i=0}^{n} i\, a_{k,n}(i)=a_{k,n}(1)+2a_{k,n}(2)+...+na_{k,n}(n)$$
$$=\sum_{i=0}^{(n-1)/2}2ia_{k,n}(2i)+\sum_{i=0}^{(n-1)/2}(2i+1)a_{k,n}(2i+1).$$

By Lemma 2.1, we have

$$\sum_{i=0}^{n} i\, a_{k,n}(i)=\frac{1}{V_k}\left\{U_{kn}\sum_{i=0}^{(n-1)/2}2i\left(U_{k(n-2i+1)}+U_{2ki}\right)\right.$$
$$\left.+U_{k(n+1)}\sum_{i=0}^{(n-1)/2}(2i+1)\left(U_{k(n-2i-1)}+U_{2ki}\right)\right\}.$$

From (5), (7), (8) and (10), we get

$$V_k^2\sum_{i=0}^{n} i\, a_{k,n}(i)=(n+1)\left(U_{k(n+1)}+U_{kn}\right)\left(U_{kn}-U_k\right)$$

as claimed. Similarly, for even n, the proof is clearly obtained. With the help of (11), the proof of the other result is given. Thus the proof is completed.

For example, taking $a=0$ and $b=k=1$ in (12), it is clearly seen that

$$p^2\sum_{i=0}^{n}(-1)^i i\, a_{1,n}(i)$$
$$=\begin{cases}(n+1)\left(U_{n+1}-U_n\right)\left(1-U_n\right), & \text{if } n \text{ is odd}\\ (n+1)\left(U_n-U_{n+1}U_{n-1}\right) & \\ \quad +(n-1)U_{n+1}\left(U_n-1\right) & \\ \quad +4U_n\left(U_{n+1}-U_n\right)\big/p, & \text{if } n \text{ is even}\end{cases}.$$

Theorem 2.3. Let k be an odd integer number. We have

$$\Delta U_{2k}\sum_{i=0}^{n}(-1)^{\binom{i+1}{2}}a_{k,n}(i)$$
$$=\begin{cases}\Delta U_k U_{kn}U_{k(n+1)}, & \text{if } n\equiv 0 \pmod 4\\ U_{k(n+1)}\times & \\ \left(V_{k(n+1)}+V_{k(n-1)}+2\left(V_k-V_{kn}\right)\right), & \text{if } n\equiv 1 \pmod 4\\ \left(V_k-2\right)U_{k(2n+1)}+\left(V_k+2\right)U_k & \\ +2V_k\left(U_{k(n+1)}-U_{kn}\right), & \text{if } n\equiv 2 \pmod 4\\ V_kU_{kn}\left(V_{k(n+1)}-2\right), & \text{if } n\equiv 3 \pmod 4\end{cases}$$

and

$$\Delta V_k U_{3k}\sum_{i=0}^{n}(-1)^{\binom{i+1}{2}}a_{k,n-i}(i) = \begin{cases} U_{k(2n+4)}-U_{k(2n+1)}+U_{3k}(V_k+1) \\ -(-1)^{\binom{n}{2}}(V_k+V_{2k})U_{k(n-1)}, & \text{if } n \text{ is odd} \\ U_{k(2n+4)}-U_{k(2n+1)}+U_{3k} \\ -(-1)^{\binom{n+1}{2}}(V_k+V_{2k})U_{k(n+2)}, & \text{if } n \text{ is even} \end{cases}.$$

Proof. For the second sum, the proof can be given. Let $n\equiv 0(mod4)$. Observed that

$$\sum_{i=0}^{n}(-1)^{\binom{i+1}{2}}a_{k,n-i}(i)$$
$$=(-1)^{\binom{1}{2}}a_{k,n}(0)+(-1)^{\binom{2}{2}}a_{k,n-1}+\ldots+(-1)^{\binom{n+1}{2}}a_{k,0}(n)$$
$$=a_{k,n}(0)-a_{k,n-1}(1)+\ldots+a_{k,1}(n-1)+a_{k,0}(n)$$
$$=a_{k,n}(0)+\sum_{i=1}^{n/4}a_{k,n-4i}(4i)+\sum_{i=1}^{n/4}a_{k,n-4i+1}(4i-1)$$
$$-\sum_{i=1}^{n/4}a_{k,n-4i+2}(4i-2)-\sum_{i=1}^{n/4}a_{k,n-4i+3}(4i-3).$$

By (5) and Lemma 2.1, we get

$$V_k\sum_{i=0}^{n}(-1)^{\binom{i+1}{2}}a_{k,n-i}(i)=U_{k(n+1)}U_{kn}$$
$$+\frac{1}{\Delta}\sum_{i=1}^{n/4}\Big\{-V_{3k}\left(V_{k(2n-12i+4)}+V_{k(2n-12i+7)}\right)$$
$$-V_k\left(V_{4ki}+V_{k(4i-1)}\right)+\Delta U_{k(n-8i+4)}U_{4k}\Big\}.$$

From (4)-(6) and (8), we write

$$V_k\sum_{i=0}^{n}(-1)^{\binom{i+1}{2}}a_{k,n-i}(i) \tag{13}$$
$$=U_{k(n+1)}U_{kn}+\frac{1}{\Delta U_{3k}}\left(U_{k(n-1)}-U_{k(2n+1)}\right)$$
$$-U_{k(2n-2)}+U_{k(-n-2)}+\frac{1}{\Delta U_k}\left(U_{2k}-U_{k(n+2)}\right)$$
$$+U_k-U_{k(n+1)}$$
$$=\frac{U_{k(2n+4)}-U_{k(2n+1)}-(V_k+V_{2k})U_{k(n+2)}+U_{3k}}{\Delta U_{3k}}$$

as claimed. For $n=2(\bmod 4)$,

$$\Delta V_k U_{3k}\sum_{i=0}^{n}(-1)^{\binom{i+1}{2}}a_{k,n-i}(i) \tag{14}$$
$$=U_{k(2n+4)}-U_{k(2n+1)}+(V_k+V_{2k})U_{k(n+2)}+U_{3k}.$$

By (13) and (14), for even number n, the desired results are obtained. Similarly, for $n\equiv 1,3(\bmod 4)$, the remaining results are proven. The proof of the other result is hold. Thus, the proof is completed.

Theorem 2.4. Let k be an odd integer number. We have

$$\Delta U_{2k}\sum_{i=0}^{n}(-1)^{\binom{i+2}{2}}a_{k,n}(i) = \begin{cases} V_k\left(U_{k(n+1)}+U_{kn}\right)(2-V_{kn}) \\ \quad -\Delta U_k U_{kn}^2, & \text{if } n\equiv 0(\bmod 4) \\ \Delta U_k\left(U_k^2-U_{k(n+1)}U_{kn}-U_{kn}^2\right) \\ \quad +V_k U_{k(n+1)}\left(2-V_{k(n-1)}\right), & \text{if } n\equiv 1(\bmod 4) \\ -\Delta U_k U_{kn}U_{k(n+1)}, & \text{if } n\equiv 2(\bmod 4) \\ -V_k U_{kn}\left(V_{k(n+1)}-2\right), & \text{if } n\equiv 3(\bmod 4) \end{cases}$$

and

$$\Delta V_k V_{2k} U_{3k} \sum_{i=0}^{n} (-1)^{\binom{i+2}{2}} a_{k,n-i}(i)$$

$$= \begin{cases} V_{2k}\left(U_{3k} - U_{k(2n+4)} - U_{k(2n+1)}\right) \\ +U_{2k}\left(V_{k(n-3)} + \Delta U_{kn} U_{3k}\right) & \text{if } n \equiv 0 \pmod 4 \\ +V_k^2 U_{k(n+1)} + V_{2k}^2 U_{k(n+2)}, \\ -V_{2k}\left(U_{k(2n+4)} + U_{k(2n+1)}\right. \\ \left. \qquad + U_{3k}\left(V_k - 1\right)\right) & \text{if } n \equiv 1 \pmod 4 \\ -U_{k(n-1)}\left(\Delta U_{2k}^2 + V_{2k}\left(V_k + 2\right)\right), \\ V_{2k}\left(-U_{k(2n+4)} - U_{k(2n+1)}\right. \\ \left. +\left(V_k - V_{2k}\right) U_{k(n+2)} + U_{3k}\right), & \text{if } n \equiv 2 \pmod 4 \\ -V_{2k}\left(U_{k(2n+4)} + U_{k(2n+1)}\right. \\ \left. +U_{3k}\left(V_k - 1\right)\right) + V_k V_{2k} U_{k(n-1)} \\ +2U_{k(n+1)} V_{4k} + U_{4k} V_{k(n-1)}, & \text{if } n \equiv 3 \pmod 4 \end{cases}.$$

Proof. Let $n \equiv 0 \pmod 4$. Consider that

$$\sum_{i=0}^{n} (-1)^{\binom{i+2}{2}} a_{k,n}(i)$$
$$= -a_{k,n}(0) - a_{k,n}(1) + a_{k,n}(2) + \ldots + a_{k,n}(n-1) - a_{k,n}(n)$$
$$= -\sum_{i=1}^{n/4} a_{k,n}(4i-4) - \sum_{i=1}^{n/4} a_{k,n}(4i-3) + \sum_{i=1}^{n/4} a_{k,n}(4i-2) + \sum_{i=1}^{n/4} a_{k,n}(4i-1) - a_{k,n}(n).$$

From (7) and Lemma 2.1, we write

$$\sum_{i=0}^{n} (-1)^{\binom{i+2}{2}} a_{k,n}(i)$$
$$= -\frac{U_{kn}^2}{V_k} - \left(U_{k(n+1)} + U_{kn}\right) \sum_{i=1}^{n/4} \left(U_{k(n-4i+3)} - U_{k(4i-3)}\right).$$

By (5), (7) and (8), we have

$$\sum_{i=0}^{n} (-1)^{\binom{i+2}{2}} a_{k,n}(i)$$
$$= -\frac{1}{V_k} U_{kn}^2 - \frac{V_k}{\Delta U_{2k}} \left(U_{k(n+1)} + U_{kn}\right)\left(V_{kn} - 2\right)$$
$$= -\frac{1}{V_k} U_{kn}^2 - \frac{1}{\Delta U_k} \left(U_{k(n+1)} + U_{kn}\right)\left(V_{kn} - 2\right)$$

as claimed. For $n \equiv 1, 2, 3 \pmod 4$, the proofs are clearly given. Similarly, the other result is given. Thus, we have the conclusion.

REFERENCES (KAYNAKÇA)

[1] E. Kılıç and P. Stanica, "Factorizations and Representations of Second Order Linear Recurrences with Indices in Arithmetic Progressions", *Bol. Soc. Mat. Mex. III. Ser.*, vol. 15, no. 1, pp. 23-25, 2009.

[2] P. Filipponi and H.T. Freitag, "Autocorrelation Sequences", *The Fibonacci Quarterly*, vol. 32, no. 4, pp. 356-368, 1994.

[3] S. Golomb, Shift Register Sequences, Laguna Hills: CA: Aegean Park Press, 1982.

[4] Y. Türker, N. Ömür and E.Kılıç, "Sums of Products of the Terms of the Generalized Lucas Sequence $\{V_{kn}\}$", *Hacettepe Journal of Mathematics and Statistics*, vol. 4, no. 2, pp. 147-161, 2011.

[5] T. Koshy, Fibonacci and Lucas Numbers with Applications, New York: Pure and Applied Mathematics, Wiley-Interscience, 2001.

[6] S. Vajda, Fibonacci&Lucas Numbers and the Golden Section, New York: John Wiley&Sons, Inc., 1989.