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A Local Similarity Representation for Generalized Q-Holomorphic Functions

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ABSTRACT

In this work, we prove a local similarity representation for the solution of equation

$$\frac{\partial w}{\partial \bar{z}} - Q\frac{\partial w}{\partial z} = Aw + B\bar{w}$$

where A and B are $m \times m$ lower triangular matrices, Q is a $m \times m$ type lower triangular matrix having zeros the main diagonal and the unknown fuction w is a $m \times s$-type complex matris valued functions.

Keywords: Q-Holomorphic Functions, Generalized Beltrami Systems

ÖZ

Bu makalede,

$$\frac{\partial w}{\partial \bar{z}} - Q\frac{\partial w}{\partial z} = Aw + B\bar{w}$$

denkleminin çözümleri için local bir benzerlik temsili ispatlanacaktır. Burada A ve B $\boldsymbol{m} \times \boldsymbol{m}$ tipinde alt ücgensel matrisler, Q $\boldsymbol{m} \times \boldsymbol{m}$ tipinde esas köşegeni sıfır olan alt üçgensel matris ve bilinmeyen w fonksiyonu ise $\boldsymbol{m} \times \boldsymbol{s}$-tipinde kompleks matris değerli bir fonksiyondur.
Anahtar Kelimeler: Q-Holomorf Fonksiyonlar, Genelleştirlmiş Beltrami Sistemi

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1. GİRİŞ (INTRODUCTION)

Let A and B be Holder continuous in a domain Ω. Then any solution $w \in C^{1,\alpha}$ of complex equation

$$\frac{\partial w}{\partial \bar{z}} = aw + b\bar{w} \quad (1.1)$$

can be written in the following form

$$w(z) = f(z) exps(z), \quad (1.2)$$

where f is analytic in Ω, s is Holder continuous and bounded function $\bar{\Omega}$, the closure of Ω. This representation, at first proven by Bers [1] and Vekua [2] , was called "similarity representation". By this similarity representation was to reduce the boundary value problem with (1.1) to related boundary value problem for analytic functions.

Subsequently, similarity representation was extended to the class of generalized hyperanalytic functions . A good surveys of methods in hyperanalytic case may be found in [3], Later in [4], [5] using Vekua and Bers techniques a function theory as given for the equation

$$Dw := \frac{\partial w}{\partial \bar{z}} - Q\frac{\partial w}{\partial z} = Aw + B\bar{w},$$

where w is $m \times s$-matrix, Q is self commuting with $m \times m$, A and B are commuting with Q which means

$$Q(z_1)A(z_2) = A(z_2)Q(z_1)$$

$$Q(z_1)B(z_2) = B(z_2)Q(z_1).$$

Solutions of such equation were called generalized (or pseudo) Q-holomorphic functions. Similarity principle was given for the solution of such an equation which is commuting with Q, continuous and bounded in $\mathbb{C}$ [4, p. 442]. But all attempt to date to find an adequate generalization of this global result for solution to

$$Dw := \frac{\partial w}{\partial \bar{z}} - Q\frac{\partial w}{\partial z} = Aw + B\bar{w} \quad (1.3)$$

have failed. Where A and B are $m \times m$ lower triangular matrices, Q is $m \times m$ lower triangular matrix having zeros the main diagonal and D is

$$D = \frac{\partial}{\partial \bar{z}} - Q\frac{\partial}{\partial z}.$$

In this work, we proved a local similarity representation for eq. (1.3).

We employ the standard norm on a matrix $B = (b_{ij})$ given by

$$\|B\|^2 = iz(B^*B) = \sum_{i,j} |b_{ij}|^2.$$

For convenience in exposition, we define certain spaces of $m \times s$ complex matrix valued functions represented by

$$w = \sum_{i=1}^{m}\sum_{j=1}^{s} w_{ij}e^{ij} \quad (1.4)$$

where w_{ij} is complex function and e^{ij} denotes $m \times s$ constants matrix in which i.th row and j.th column term is 1 and the others terms are 0. In general a function w given by (1.4) is said to lie in a given space if each of complex function w_{ij} is in that space. For example, if the set of functions satisfying the conditions

$$L^p(w, \bar{\Omega}) = \left(\iint_{\Omega} \|w(z)\|^p\right)^{1/p} < \infty$$

It will be denoted by $L^p(\bar{\Omega})$, where **Ω** is a some domain (i.e. an open set) in the plane. It is easily seen that if $w \epsilon L^p(\Omega)$ then

$$L_p(w_{ij},\Omega) := \left(\iint_\Omega |w_{ij}|^p\right)^{\frac{1}{p}} < \left(\iint_\Omega \|w(z)\|^p\right)^{\frac{1}{p}} < \infty,$$

where the norm | | is the standart norm on complex valued functions defined by $|f| := \sqrt{u^2+v^2}, f = u + iv$. That is, this criterion obviously is equivalent to the statement each $w_{ij} \in L_p(\Omega)$.

The space $C^m(\Omega)$ consists of those functions whose derivatives to order *m* are continuous in **Ω**. We write $C^0(\Omega) = C(\Omega)$.

We say $C^{m,\alpha}(\Omega)$, where $0 < \alpha < 1$ if *w* and its derivatives to order *m* are Holder continuous in $\bar{\Omega}$, the closure of **Ω** with exponent α (i.e. there is a positive constant M such that for $z_1, z_2 \in \bar{\Omega}$, $\|w(z_1) - w(z_1)\| < M|z_1 - z_2|^\alpha$, and similar inequalities hold for the derivatives to order *m*). The space $C^{m,\alpha}(\Omega)$ consists of those functions contained in $C^{m,\alpha}(\Omega_1)$, for every bounded sub domain Ω_1 of Ω with $\Omega_1 \subset \Omega$.

2. BÖLÜM 2 BENZERLİK PRENSİBİ (SIMILARITY PRİNCİBLES)

Two function will be called "similar" if they have zeros of the same order at the same points. For example, any solution of (1.1) which is continuous and bounded in **Ω** can be written as

$$w(z) = f(z)exps(z),$$

where *f* is analytic in **Ω**, s is Holder continuous and bounded function [2]. If $w(z)$ has a zero $k.$ order at some point z_0 then $f(z)$ has zero of same order at the same point. This shows that $w(z)$ is similar to $f(z)$. In this section we show that there exists a local similarity principles for Eq. (1.3).

2.1 Theorem: Suppose that the components of *A* and *B* are Holder continuous. Let *w* be solution of (1.3) with Holder continuous first partial derivatives such that w_{1l} not identically zero for $l = 1, \dots, s$. If z_0 is a point such that $w_{kl}(z_0) = 0$ for $k = 1, \dots, m, l = 1, \dots, s$ then in some neighborhood of z_0 there is a $m \times s$ matrix $E = \{E_{kli}\}$ and integers $n_{1l}, \dots, n_{ml}$ such that

$$w(z) = \sum_{k=1}^{m}\sum_{l=1}^{s} e^{kl} \sum_{i=1}^{k} E_{kli}(z - z_0)^{n_{il}}.$$

Moreover, each E_{kli}has the representation

$$E_{kli} = v_{kli} + \sum_{\mu=0}^{k-1} \alpha_{kli\mu}\left(\frac{\bar{z} - z_0}{z - z_0}\right)^\mu,$$

with v_{kli} Holder continuous and the $\alpha_{kli\mu}$ are complex constants.

Proof. Without loss of generality, we will assume that $z_0 = 0$. The results is proved by induction. Our induction hypothesis is that there exists a neighborhood S of 0 and a complex-valued function s_{jl} such that

$$w_{tl} = \exp(s_{tl})\left(\sum_{i=1}^{t} z^{n_{jl}}\sigma_{tlj}\right).$$

here

i. s_{tl} has generalized derivative with respect to $\bar{z}$ such that

$$\frac{\partial s_{tl}}{\partial \bar{z}} \in L^q(\bar{S})$$

for every $q \geq 1$.

ii. n_{jl} is positive integer

iii. σ_{tlt} is analytic in S with $\sigma_{tlt}(0) \neq 0$

iv. For $j \leq l$

$$\sigma_{tlj} = v_{tlj} + \sum_{\mu=0}^{t-1} \alpha_{tli\mu}\left(\frac{\bar{z}}{z}\right)^{\mu},$$

where $\alpha_{tli\mu} \in C$ and

$$v_{tlj} = \iint_S \frac{R_{tlj}(z)}{\zeta - z} d\xi d\eta$$

with $R_{tlj} \in L^p(\bar{S})$ for some $p > 2$ (Note that this implies that $v_{tlj} \in C^{0,\alpha}(\bar{S})$ for $\alpha = \frac{p-1}{p}$).

Note that (1.3) may be write following form

$$0 = \sum_{l=1}^{s}\left(\frac{\partial w_{1l}}{\partial \bar{z}} - a_{11}w_{1l} + b_{11}\overline{w}_{1l}\right)e^{1l}$$

$$+\sum_{i=2}^{m}\sum_{l=2}^{s}\left\{\frac{\partial w_{il}}{\partial \bar{z}}\right.$$

$$\left.-\sum_{j=1}^{i-1} a_{ij}w_{jl} + b_{ij}\overline{w}_{jl} - a_{ii}w_{il} - b_{ii}\overline{w}_{il}\right\}e^{il}.$$

First, we prove the result for w_{1l}in the case

$$\frac{\partial w_{1l}}{\partial \bar{z}} = a_{11}w_{1l} + b_{11}\overline{w}_{1l},$$

i.e. w_{1l} is pseudo-analytic. Thus we can apply the complex case to obtain the result. The s_{1l} is obtained as follow: let

$$P_{1l} = a_{11} + \begin{cases} b_{11}\dfrac{\overline{w_{1l}}}{w_{1l}}, & if\ w_{1l} \neq 0 \\ 0 & otherwise \end{cases}.$$

We define

$$s_{1l} := -\frac{1}{\pi}\iint_S \frac{P_{1l}}{\zeta - z} d\xi d\eta.$$

Since $w_{1l} \in C^{0,\alpha}(S), P_{1l}$ is bounded in S and hence in $L^q(\bar{S})$. Also, $s_{1l} \in C^{0,\alpha}(S)$, $0 < \alpha < 1$ and possesses a generalized derivative with respect to $\bar{z}, \frac{\partial s_{1l}}{\partial \bar{z}}$ (weak). If the function f in the statement of complex case has a zero of order n_{1l} at 0, then we define

$$\sigma_{1l1} = \frac{f_{1l}}{z^{n_{1l}}}.$$

This completes the first step of our induction hypothesis.

Now, we assume that our induction hypothesis holds for w_{tl} such that $1 \leq t \leq k-1$. The proof for w_{kl} will depend upon several lemmas which will be demonstrated at conclusion of the proof. From (1.3),

$$\frac{\partial w_{kl}}{\partial \bar{z}} = a_{kk}w_{kl} + b_{kk}\overline{w_{kl}}$$

$$+\sum_{j=1}^{k-1}\left(q_{kj}\frac{\partial w_{jl}}{\partial z} + a_{kj}w_{jl} + b_{kj}\overline{w_{jl}}\right)$$

$$:= a_{kk}w_{kl} + b_{kk}\overline{w_{kl}} + c_{kl}.$$

Define

$$P_{kl} = a_{kk} + \begin{cases} b_{kk}\dfrac{\overline{w_{kl}}}{w_{kl}}, & if\ w_{kl} \neq 0 \\ 0 & otherwise \end{cases},$$

$$s_{kl} := -\frac{1}{\pi}\iint_S \frac{P_{kl}(\zeta)}{\zeta - z} d\xi d\eta.$$

As in $t = 1$ we have $s_{kl} \in C^{0,\alpha}(S)$, $0 < \alpha < 1$, and $\frac{\partial s_{kl}}{\partial \bar{z}} = P_{kl}$ (weak) and $P_{kl} \in L^q(\bar{S}), q \geq 1$ [2, p. 34]. If we define $\psi = w_{kl}\exp(-s_{kl})$, we obtain

$$\frac{\partial \psi}{\partial \bar{z}} = \frac{\partial w_{kl}}{\partial \bar{z}}\exp(-s_{kl}) - w_{kl}\frac{\partial s_{kl}}{\partial \bar{z}}\exp(-s_{kl})$$
$$= (a_{kk}w_{kl} + b_{kk}\overline{w_{kl}} + c_{kl})\exp(-s_{kl})$$
$$- w_{kl}P_{kl}\exp(-s_{kl})$$
$$= c_{kl}\exp(-s_{kl})$$

By the Cauchy integral representation there is an analytic function ϕ such that

$$\psi(z) = \phi(z) - \frac{1}{\pi}\iint_S \frac{c_{kl}\exp(-s_{kl})}{\zeta - z} d\xi d\eta$$

in S (see [2, p. 34]). Using the induction hypothesis, it can be shown that

$$-\frac{1}{\pi}c_{kl}\exp(-s_{kl})$$
$$= \sum_{i=1}^{k-1}\left\{\zeta^{n_{il}}R_{kli} + \zeta^{n_{il}-1}\sum_{\mu=0}^{k-2}\lambda_{kli\mu}\left(\frac{\bar{\zeta}}{\zeta}\right)^{\mu}\right\}$$

where $\lambda_{kli\mu} \in C(S), R_{kli} \in L^p(\bar{S}), p > 2$ (see 2.4 Lemma). So

$$\psi(z) = \phi(z)$$
$$+\sum_{i=1}^{k-1}\iint_S \frac{\zeta^{n_{il}-1}}{\zeta - z}\left\{\zeta R_{kli} + \sum_{\mu=0}^{k-2}\lambda_{kli\mu}\left(\frac{\bar{\zeta}}{\zeta}\right)^{\mu}\right\} d\xi d\eta.$$

Since

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} + \frac{z}{\zeta^2} + \cdots + \frac{z^{n_{il}-1}}{\zeta^{n_{il}-1}} + \frac{z^{n_{il}}}{\zeta^{n_{il}}(\zeta - z)},$$

it follows that

$$\psi(z) = \phi(z)$$
$$+\sum_{i=1}^{k-1}\left\{\sum_{\rho=0}^{n_{il}-1} z^{\rho}\iint_S \frac{\zeta^{n_{il}-1}}{\zeta^{\rho+1}}\left\{\zeta R_{kli} + \sum_{\mu=0}^{k-2}\lambda_{kli\mu}\left(\frac{\bar{\zeta}}{\zeta}\right)^{\mu}\right\} d\xi d\eta\right\}$$
$$+\sum_{i=1}^{k-1} z^{n_{il}}\iint_S \frac{z^{n_{il}}}{\zeta^{n_{il}}(\zeta - z)}\left(\zeta R_{kli} + \sum_{\mu=0}^{k-2}\lambda_{kli\mu}\left(\frac{\bar{\zeta}}{\zeta}\right)^{\mu}\right) d\xi d\eta$$
$$:= \phi(z) + \sum_{\rho=0}^{N_l} k_{\rho}z^{\rho}$$
$$+\sum_{i=1}^{k-1} z^{n_{il}}\iint_S \left(\frac{R_{kli}}{\zeta - z} + \sum_{\mu=0}^{k-2}\frac{\lambda_{kli\mu}}{\zeta - z}\left(\frac{\bar{\zeta}}{\zeta}\right)^{\mu}\right) d\xi d\eta,$$

where $N_l = \max_{0 \leq j \leq k-1}(n_{il} - 1)$ and where the constant complex coefficients k_{ρ} are the sum of convergent integrals. Employing a forthcoming result (2.2 Lemma**)**, we have

$$\sum_{\mu=0}^{k-2}\iint_S \frac{\lambda_{kli\mu}}{\zeta(\zeta - z)}\left(\frac{\bar{\zeta}}{\zeta}\right)^{\mu} d\xi d\eta = \sum_{\mu=0}^{k-2}\lambda_{kli\mu}\left(\frac{\bar{z}}{z}\right)^{\mu+1}.$$

Define

$$\sigma_{kli} := \iint_S \frac{R_{kli}}{\zeta - z} d\xi d\eta$$

$$\Gamma := \phi(z) + \sum_{\rho=0}^{N_l} k_\rho z^\rho$$

then from complex case and the definition of ψ

$$w_{kl}(z) = \psi(z) exp\left(s_{kl}(z)\right)$$

$$= \left(\Gamma(z) + \sum_{i=1}^{k-1} \sigma_{kli} z^{n_{il}}\right) exp\left(s_{kl}(z)\right).$$

Suppose $\Gamma(z) \neq 0$. From (2.1), we have $\Gamma(0) = 0$ and since Γ is analytic in *S*, there is a $n_{kl} \geq 0$ such that

$$\sigma_{klk}(z) := \frac{\Gamma(z)}{z^{n_{kl}}}$$

is analytic in *S* and $\sigma_{klk}(0) \neq 0$. Hence (2.1) becomes

$$w_{kl}(z) = \sum_{i=1}^{k-1} (\sigma_{kli} z^{n_{il}}) \, exp\left(s_{kl}(z)\right)$$

as desired. Hence by induction we have all the results of the theorem. Finally we give the following three Lemmas used above.

2.2 Lemma For $m = 0,1, \ldots$ and $z \in S, z \neq 0$

$$\frac{-\pi}{m+1}\left(\frac{\bar{z}}{z}\right)^{m+1} = \iint_S \frac{1}{\zeta(\zeta - z)} d\xi d\eta.$$

[6]

2.3 Lemma Let $f \in C^{0,\alpha}(\bar{S}), 0 < \alpha < 1$, and let $g(z)$ be a bounded function in S with $g \in L^1(S)$. Then

$$\frac{f(z) - f(0)}{z} g(z) \in L^P(\bar{S}) \quad for\ some\ p > 2$$

[3]

2.4 Lemma For c_{kl} as defined in the proof of 2.1 Theorem,

$c_{kl}(\mathrm{z}) \exp\left(s_{kl}(z)\right)$

$$= -\pi \sum_{i=1}^{k-1} \left[\zeta^{n_{il}} R_{kli} + \zeta^{n_{il}-1} \sum_{\mu=0}^{k-2} \lambda_{kli\mu} \left(\frac{\bar{\zeta}}{\zeta}\right)^{\mu}\right],$$

where $\lambda_{kli\mu} \in C$ and $R_{kli} \in L^p(\bar{S})$ for some $p > 2$.

Proof The proof of this Lemma is similar to given in [3, p. 115]. So we omit the proof of Lemma.

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