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Axial Vibration Analysis of a Nanorod Embedded in Elastic Medium Using Nonlocal Strain Gradient Theory

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Abstract

Size-dependent axial vibration of a nanorod embedded in elastic medium is studied for the first time in this paper within the framework of the nonlocal strain gradient theory. The governing equation of motion of the problem is derived using the equilibrium condition and it is solved analytically to obtain the exact expression of vibration frequency for a fixed-fixed nanorod. The influences of the nonlocal parameter, the material length scale parameter and the elastic medium coefficient on the free vibration frequencies are investigated in detail. The results show that free vibration frequencies are significantly dependent on the size effects, and the size effects gain more importance at higher modes. Therefore, the classical continuum theory is inappropriate to investigate the mechanical behavior of nanostructures.

Keywords: Nanotechnology, Vibration, Nonlocal elasticity theory, Strain gradient theory, Nanorod

Yerel Olmayan Şekil Değiştirme Gradyanı Teorisi Kullanılarak Elastik Zemine Gömülü Nano Çubuğun Eksenel Titreşim Analizi

Özet

Elastik zemine gömülü bir nano çubuğun boyut etkisine bağlı eksenel titreşimi yerel olmayan şekil değiştirme gradyanı teorisi çerçevesinde ilk olarak bu çalışmada incelenmiştir. Probleme ait yönetici hareket denklemi denge şartı kullanılarak çıkarılmış, iki ucu ankastre nano çubuğun serbest titreşim frekansına ait kesin ifadeyi elde etmek için yönetici denklem analitik olarak çözülmüştür. Yerel olmayan parametre, malzeme uzunluk ölçek parametresi ve elastik zemin parametresinin serbest titreşim frekansları üzerindeki etkisi detaylı olarak incelenmiştir. Elde edilen sayısal sonuçlar göstermiştir ki; serbest titreşim frekansları boyut etkisine önemli derecede bağlıdır ve boyut etkisi yüksek modlarda daha çok önem kazanmaktadır. Bu nedenlerden dolayı, klasik sürekli ortamlar mekaniği nano ölçekteki yapıların analizi için uygun değildir.

Anahtar Kelimeler: Nanoteknoloji, Titreşim, Yerel olmayan elastisite teorisi, Şekil değiştirme gradyanı teorisi, Nano çubuk

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1. INTRODUCTION

With the advance of the technology, due to their novel properties nanostructures such as carbon nanotubes, nanorods and nanowires has been extensively used in various engineering applications, especially in micro and nanoelectromechanical system (MEMs and NEMs). For this reason the determination of the physical and mechanical behavior of nanostructures has become very important issue nowadays. The research method of nanostructures can be categorized into two groups. The first method group is based on the atomistic methods, i.e., molecular dynamic (MD) simulation. However, since the atomistic methods take more time and then they are computationally expensive, these methods are not always useful for the analysis of nanostructures that consist of a large number of atoms and molecules. The second method group based on the continuum mechanics can be good alternative in order to overcome this problem. On the other hand, experimental studies show that the size effect plays very crucial role on the mechanical behavior of nanostructures. Thus, the classical continuum theory, which is independent of the size effect, is insufficient in order to predict the behavior of the nanoscale structures. At this point, several size-dependent continuum theories, such as the nonlocal elasticity theory [1], the strain gradient theory [2], the modified couple stress theory [3] and the micropolar theory [4] were proposed. In the nonlocal elasticity theory proposed by Eringen [1], to account for scale effect it is assumed that the stress at a point is a function of strains at all points in the continuum. A large number of studies related with static [5,6], buckling [7-12] and vibration [13-20] analysis of nanostructures using Eringen's nonlocal elasticity theory can be found in literature.

Although nonlocal elasticity theory is widely used in the analysis of nanostructures, the main drawback of this theory is that it accounts for only softening effect whereas the stiffness enhancement, which is observed in experimental analysis of micro and nanostructures, is not included. In this context, the strain gradient theory and the modified couple stress theory are capable of incorporating the stiffness enhancement [21-30]. As seen from the previous discussion, the nonlocal elasticity theory and the strain gradient theory handle the different aspects of size-dependent mechanical behavior of micro and nanostructures. Therefore, there is a strong need to combine both size-dependent theories to realize the real behavior of micro and nanostructures. By connecting the nonlocal elasticity and the strain gradient theory, Challamel [31] proposed a generalized hybrid nonlocal law to investigate static bending, buckling, and free vibration of beams using different beam theories. Recently, Lim et al. [32] have presented nonlocal strain gradient theory that combine the nonlocal elasticity and the strain gradient theory for the wave propagation in Euler-Bernoulli and Timoshenko beams. Li and Hu [33] have investigated the post-buckling behavior of Euler-Bernoulli beam based on the nonlocal strain gradient theory.

The above literature survey reveals that the number of studies related to mechanical behavior of micro and nanostructures using the nonlocal strain gradient theory are very limited. To the best of author's knowledge, size-dependent axial vibration of nanorods embedded in elastic medium seems to be nonexistent.

The primary purpose of the current work is to fill this gap. In the present paper, free vibration characteristics of a nanorod embedded in elastic medium are examined in the context of the nonlocal strain gradient theory. The exact frequency expression is derived for a fixed-fixed nanorod. The effects of the nonlocal parameter, the material length scale parameter and the elastic medium coefficient on the free vibration frequencies are investigated in detail.

2. NONLOCAL STRAIN GRADIENT ROD MODEL

Based on the nonlocal strain gradient theory proposed by Lim et al. [32], the generalized nonlocal strain gradient constitutive relation for a one-dimensional structure can be given as;

$$\left[1-\left(e_0a\right)^2\nabla^2\right]\left[1-\left(e_1a\right)^2\nabla^2\right]t_{xx}=$$
$$E\left[1-\left(e_1a\right)^2\nabla^2\right]\varepsilon_{xx}-El_m{}^2\left[1-\left(e_0a\right)^2\nabla^2\right]\nabla^2\varepsilon_{xx} \tag{1}$$

where $\nabla^2=\partial^2/\partial x^2$ is the one-dimensional differential operator, t_{xx} is the total axial stress, ε_{xx} is the axial strain, e_0a and e_1a are the nonlocal parameters, and l_m is the material length scale parameter to determine the significance of higher-order strain gradient field. It is seen that Eq. (1) accounts for not only the nonlocal elastic stress field but also the strain gradient stress field. According to the assumption made in Lim et al. [32], when one considers that $e_0=e_1=e$, the generalized nonlocal strain gradient constitutive relation in Eq. (1) can be simplified as

$$\left[1-\left(ea\right)^2\nabla^2\right]t_{xx}=E\left[1-l_m^2\nabla^2\right]\varepsilon_{xx} \tag{2}$$

where E is the elasticity modulus. It can be seen from Eq. (1) that when the material length scale is taken as $l_m=0$, the Eringen's nonlocal stress model [1] is achieved as follows

$$\left[1-\left(ea\right)^2\nabla^2\right]t_{xx}=E\varepsilon_{xx} \tag{3}$$

On the other hand, by assuming $ea=0$ the pure strain gradient model of Aifantis [2] is obtained as

$$t_{xx}=E\left[1-l_m^2\nabla^2\right]\varepsilon_{xx} \tag{4}$$

For a one-dimensional structure, the nonlocal behavior can be neglected in the thickness direction. Thus, for a homogeneous isotropic rod with the length of L, the nonlocal strain gradient constitutive relation takes the following form

$$t_{xx}-\left(e_0a\right)^2\frac{\partial^2 t_{xx}}{\partial x^2}=E\varepsilon_{xx}-El_m^2\frac{\partial^2\varepsilon_{xx}}{\partial x^2} \tag{5}$$

Using Hamilton's principle the equation of motion for the axially vibrating rod embedded in an elastic medium can be obtained as

$$\frac{\partial N}{\partial x}+f\left(x,t\right)-k_u\,u\left(x,t\right)=\rho A\frac{\partial^2u\left(x,t\right)}{\partial t^2} \tag{6}$$

where $f\left(x,t\right)$ is the distributed axial load along x axis, k_u is the coefficient of the elastic medium, ρ is the mass density of the rod, A is the area of the cross-section, $u\left(x,t\right)$ is the axial displacement of the rod, t denotes the time, and N is the axial normal force which is defined by,

$$N=\int_A t_{xx}\,\mathrm{d}A \tag{7}$$

The axial strain ε_{xx} for a rod is given by,

$$\varepsilon_{xx}=\frac{\partial u}{\partial x} \tag{8}$$

From the Eqs. (5), (7) and (8), the axial force for the nonlocal strain gradient model can be obtained as,

$$N-\left(ea\right)^2\frac{\partial^2N}{\partial x^2}=EA\frac{\partial u}{\partial x}-EAl_m^2\frac{\partial^3u}{\partial x^3} \tag{9}$$

The explicit expression of the nonlocal axial normal force can be obtained by substituting Eq. (6) into Eq. (9) as,

$$N=EA\frac{\partial u}{\partial x}-EAl_m^2\frac{\partial^3u}{\partial x^3}+\left(ea\right)^2\left(\rho A\frac{\partial^3u}{\partial t^3}-\frac{\partial f}{\partial x}+k_u\frac{\partial u}{\partial x}\right) \tag{10}$$

By using Eq. (6) and (10), the equation of the motion for the nonlocal strain gradient rod model in terms of the axial displacement can be obtained as follows,

$$EA\frac{\partial^2 u}{\partial x^2} - EAl_m^2\frac{\partial^4 u}{\partial x^4} + \rho A\frac{\partial^2}{\partial t^2}\left[(ea)^2\frac{\partial^2 u}{\partial x^2} - u\right] + k_u\left[(ea)^2\frac{\partial^2 u}{\partial x^2} - u\right] = (ea)^2\frac{\partial^2 f}{\partial x^2} - f \tag{11}$$

It is clear that the equation of motion in (11) is the fourth order partial differential equation, and it requires four boundary conditions, two of which are the classical and the others are the non-classical boundary conditions. The related boundary conditions are given as [31],

$$\left(EA\frac{\partial u}{\partial x} - EAl_m^2\frac{\partial^3 u}{\partial x^3}\right)\delta u\Big|_0^L = 0 \tag{12}$$

$$\left(EAl_m^2\frac{\partial^2 u}{\partial x^2}\right)\frac{\delta\,\partial u}{\partial x}\Big|_0^L = 0 \tag{13}$$

For a fixed-fixed nanorod the following boundary conditions should be satisfied at both edges [21],

$$u(0) = u(L) = 0 \tag{14}$$

$$\frac{\partial^2 u(0)}{\partial x^2} = \frac{\partial^2 u(L)}{\partial x^2} = 0 \tag{15}$$

In the light of the above boundary conditions, the axial displacement of the nanorod can be expanded in the modal form as,

$$u(x,\, t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x) \tag{16}$$

where $q_n(t)$ are the unknown time-dependent generalized coordinates and $\phi_n(x)$ are the mode shapes of a fixed-fixed rod which are expresses as,

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots \tag{17}$$

For free vibration analysis, if the external load is set to zero ($f = 0$), and the time-dependent generalized coordinates are assumed to be as $q_n(t) = \bar{q}_n \sin\omega t$ in which ω is the natural frequency of the nanorod, and Eq. (16) is substituted into Eq. (11), one obtains the following relation after some mathematical amendments,

$$\left[-EA\left(\frac{n\pi}{L}\right)^2 - EAl_m^2\left(\frac{n\pi}{L}\right)^4 + \rho A(ea)^2\omega^2\left(\frac{n\pi}{L}\right)^2 + \rho A\omega^2 - k_u(ea)^2\left(\frac{n\pi}{L}\right)^2 - k_u\right]\sin\left(\frac{n\pi x}{L}\right)\bar{q}_n = 0 \tag{18}$$

For the non-trivial solution the expression in the square bracket must be zero. This condition yields the free vibration frequencies of the nanorod. After some mathematical processes, the free vibration frequency of the fixed-fixed nanorod is found as,

$$\omega_n = \sqrt{\frac{EA\left(\frac{n\pi}{L}\right)^2\left[1 + n^2\pi^2\left(\frac{l_m}{L}\right)^2\right]}{\rho A\left[1 + n^2\pi^2\left(\frac{ea}{L}\right)^2\right]} + \frac{k_u}{\rho A}} \tag{19}$$

It should be noted that the lowest frequency (ω_1) is the fundamental vibration frequency of the nanorod.

3. NUMERICAL RESULTS

In the numerical results, free vibration of the fixed-fixed nanorod based on the nonlocal strain gradient theory is investigated. Some numerical examples are presented to examine the effects of the nonlocal parameter and the material length scale parameter on the free vibration frequencies. In order to obtain the more general results, the following dimensionless quantities can be defined

Dimensionless nonlocal parameter:

$$\alpha = \frac{ea}{L} \tag{20}$$

Dimensionless material length scale parameter:

$$\beta = \frac{l_m}{L} \tag{21}$$

Dimensionless elastic medium parameter:

$$K_U = \frac{k_u L^2}{EA} \tag{22}$$

Dimensionless frequency parameter:

$$\Omega_n = \omega L \sqrt{\frac{\rho A}{EA}} \Rightarrow \Omega_n = \sqrt{\frac{n^2\pi^2\left(1+n^2\pi^2\beta^2\right)}{1+n^2\pi^2\alpha^2} + K_U} \tag{23}$$

In Tables 1-3, the first three dimensionless free vibration frequencies of the nanorod are presented for various values of the dimensionless nonlocal parameter (α) and the dimensionless elastic medium parameter (K_U).

Table 1. The first vibration frequency for various values of the nonlocal and the elastic medium parameters and $\beta = 0$

K_U	α			
	0	0,25	0,5	1
0	3,1415	2,4706	1,6871	0,9528
1	3,2969	2,6653	1,9612	1,3813
5	3,8561	3,3322	2,8011	2,4306
10	4,4575	4,0130	3,5841	3,3027

Table 2. The second vibration frequency for various values of the nonlocal and the elastic medium parameters and $\beta = 0$

K_U	α			
	0	0.25	0.5	1
0	6,2831	3,3742	1,9057	0,9875
1	6,3622	3,5193	2,1522	1,4054
5	6,6692	4,0479	2,9380	2,4444
10	7,0340	4,6244	3,6921	3,3128

In Tables 1-3, the dimensionless material length scale parameter is taken as $\beta = 0$. It can be seen from Tables 1-3 that an increase in the dimensionless nonlocal parameter (α) leads to a decrement in the dimensionless vibration frequencies. This is due to the softening effect caused by the inclusion of nonlocal parameter.

Table 3. The third vibration frequency for various values of the nonlocal and the elastic medium parameters and $\beta = 0$

K_U	α			
	0	0,25	0,5	1
0	9,4247	3,6821	1,9564	0,9944
1	9,4776	3,8154	2,1971	1,4102
5	9,6864	4,3078	2,9711	2,4472
10	9,9411	4,8536	3,7185	3,3149

The most important observation from Tables 1-3 is that the effect of the dimensionless nonlocal parameter on the frequencies increases as the mode number increases. In other words, the third frequency (Ω_3) is the most affected by the nonlocal parameter. For instance, regardless of K_U values the first and the third frequencies are $\Omega_1 = 3.1415$ and $\Omega_3 = 9.4247$, respectively when the nonlocal parameter is zero ($\alpha = 0$). However, if the nonlocal parameter is unity ($\alpha = 1$), the frequencies are $\Omega_1 = 0.9528$ and $\Omega_3 = 0.9944$, respectively.

Table 4 The first vibration frequency for various values of the material length scale and the elastic medium parameters and $\alpha = 0$

K_U	β			
	0	0,25	0,5	1
0	3,1415	3,9947	5,8499	10,357
1	3,2969	4,1179	5,9348	10,405
5	3,8561	4,5779	6,2627	10,596
10	4,4575	5,0948	6,6499	10,829

Tables 4-6 present the first three dimensionless frequency parameters for different values of the dimensionless material length scale parameter (β) and elastic medium parameter (K_U). Tables 4-6 reveal that in contrast to the nonlocal parameter, the material length scale parameter has an increasing effect on the frequencies. This behavior is the typical characteristics of the strain gradient theory (also couple stress theory). The inclusion of the material length scale parameter makes the rod stiffer, and then the stiffer rod yields the larger frequencies.

Table 5. The second vibration frequency for various values of the material length scale and the elastic medium parameters and $\alpha = 0$

K_U	β			
	0	0,25	0,5	1
0	6,2831	11,699	20,715	39,975
1	6,3622	11,742	20,739	39,987
5	6,6692	11,911	20,835	40,037
10	7,0340	12,119	20,955	40,100

Table 6. The third vibration frequency for various values of the material length scale and the elastic medium parameters and $\alpha = 0$

K_U	β			
	0	0,25	0,5	1
0	9,4247	24,123	45,402	89,325
1	9,4776	24,144	45,413	89,330
5	9,6864	24,227	45,457	89,353
10	9,9411	24,330	45,512	89,380

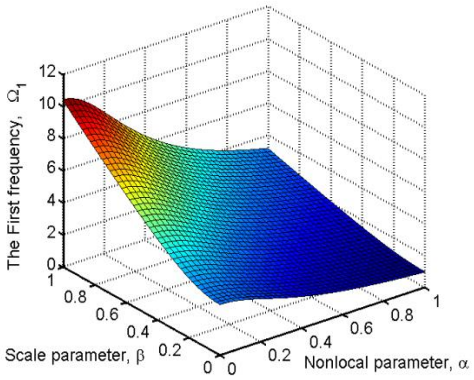

Figure 1. The variation of the first vibration frequency with the nonlocal and the material length scale parameters

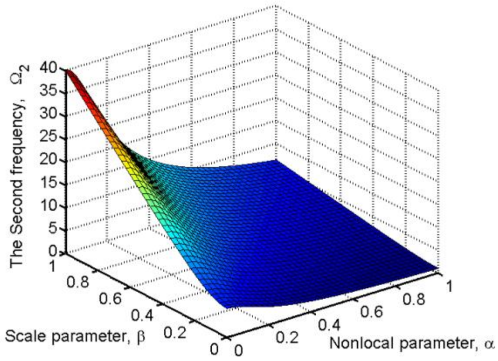

Figure 2. The variation of the second vibration frequency with the nonlocal and the material length scale parameters

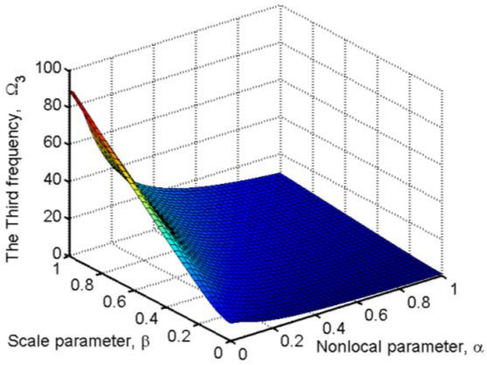

Figure 3. The variation of the third vibration frequency with the nonlocal and the material length scale parameters

Figures 1-3 display the variation of the first three frequencies with the nonlocal parameter (α) and the material length scale parameter (β). Here, the elastic medium parameter is taken as $K_U = 0$ since the similar figures are obtained for the other values of $K_U = 1, 5, 10$. As reported in the previous section, the frequency values are very sensitive to the nonlocal parameter and the material scale parameter.

It can be observed from Figures 1-3 that the material scale parameter is more effective on the higher modes than on the lower modes as also deduced for the nonlocal parameter. It should be noted here that when both nonlocal and the scale parameters are zero ($\alpha = \beta = 0$), the frequency values correspond to the classical counterparts.

Figure 4 shows the effect of the elastic medium parameter on the first vibration frequency for various values of the nonlocal and scale parameters. It is clear from the figure that an increase of the elastic medium parameter gives a rise in the first frequency. The reason of this behavior is due to the fact the elastic medium increase the stiffness of the system. Also, it should be noted here that the similar trend is observed for the other modes.

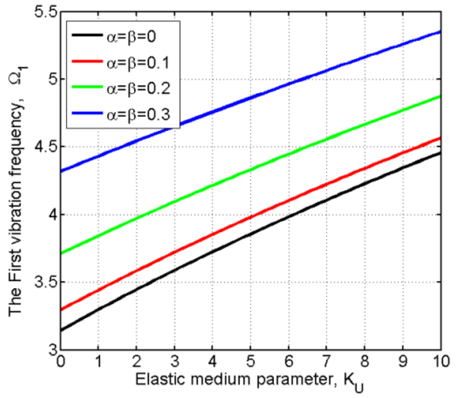

Figure 4. The variation of the first vibration frequency with the dimensionless elastic medium parameter

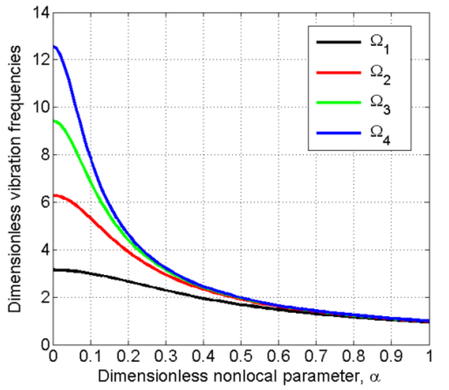

Figure 5. The variation of the first four vibration frequencies with the dimensionless elastic medium parameter

Figure 5 plots the variation of the first four vibration frequencies as a function of the dimensionless nonlocal parameter (α). In this figure, the scale and the elastic medium parameters are taken as $\beta = 0$ and $K_U = 0$. It is very interesting that the difference between the frequency values decreases significantly while the nonlocal parameter increases, and all frequency values go to the same value with the increase of the nonlocal parameter.

4. CONCLUSIONS

In this work, free vibration of an embedded nanorod is performed using a newly proposed nonlocal strain gradient theory. A frequency formula is presented for a fixed-fixed nanorod. The effects of the nonlocal parameter, material length scale parameter and the elastic medium on free vibration characteristics of the nanorod are investigated. From the results analyzed above, the most important observations are summarized as follows:

- The frequency values obtained by the classical (or local) theory are very different from those obtained by the nonlocal strain gradient theory. Thus, the classical continuum theory is not suitable to analyze the mechanical behavior of nanostructures;
- The nonlocal parameter decreases the frequency values whereas the material length scale parameter increases the frequencies;
- The effect of the nonlocal and the material scale parameters on the free vibration frequencies rises as the mode number increases;
- The vibration frequencies at the different modes approach to the same value with the increase of the nonlocal parameter;
- The elastic medium parameter plays an important role on free vibration frequencies;
- The tabulated results are presented for the first time in this paper, thus they can be a reference for the prospective researchers to validate their results.

5. REFERENCES

1. Eringen, A.C., 1972. Nonlocal Polar Elastic Continua; International Journal of Engineering Science, 10, pp. 1-16.
2. Aifantis, E.C., 1999. Strain Gradient Interpretation of Size Effects; International Journal of Fracture, 95, pp. 1-4.
3. Yang, F., Chong, A.C.M., Lam, D.C.C., Tong, P., 2002. Couple Stress Based Strain Gradient Theory for Elasticity; International Journal of Solids and Structures, 39, pp. 2731–2743.
4. Eringen, A.C., 1967. Theory of micropolar plates; Zeitschrift fur Angewandte Mathematik und Physik, 18, pp. 12–30.
5. Peddieson, J., Buchanan, G.R., McNitt R.P., 2003. Application of Nonlocal Continuum Models to Nanotechnology; International Journal of Engineering Science, 41, pp. 305-312.
6. Reddy, J.N., 2007. Nonlocal Theories for Bending, Buckling and Vibration of Beams; International Journal of Engineering Science, 45, pp. 288-307.
7. Adali S, 2008. Variational Principles for Multi-Walled Carbon Nanotubes Undergoing Buckling Based on Nonlocal Elasticity Theory; Physics Letters A, 372, pp. 5701-5705.
8. Murmu, T., Pradhan, S.C., 2009. Buckling Analysis of a Single-Walled Carbon Nanotube Embedded in an Elastic Medium Based on Nonlocal Elasticity and Timoshenko Beam Theory and Using DQM; Physica E, 41, pp. 1232-1239.
9. Murmu, T., Adhikari, S., 2011. Axial Instability of Double-Nanobeam-Systems; Physics Letters A, 375 pp. 601-608.
10. Pradhan, S.C., Reddy, G.K., 2011. Buckling Analysis of Single Walled Carbon Nanotube on Winkler Foundation Using Nonlocal Elasticity Theory and DTM; Computational Materials Science, 50, pp. 1052-1056.
11. Ansari, R., Sahmani, S., Rouhi, H., 2011. Axial Buckling Analysis of Single-Walled Carbon Nanotubes in Thermal Environments Via the Rayleigh–Ritz Technique; Computational Materials Science, 50 pp. 3050–3055.
12. Şimşek, M., Yurtcu, H.H., 2013. Analytical Solutions for Bending and Buckling of Functionally Graded Nanobeams Based on the Nonlocal Timoshenko Beam Theory; Composite Structures, 97, pp. 378-386.
13. Aydogdu, M., 2009. A General Nonlocal Beam Theory: Its Application to Nanobeam Bending, Buckling and Vibration; Physica E, 41, pp. 1651-1655.
14. Murmu, T., Pradhan, S.C., 2009. Thermo-Mechanical Vibration of a Single-Walled Carbon Nanotube Embedded in an Elastic Medium Based on Nonlocal Elasticity Theory; Computational Materials Science, 46, pp. 854-859.
15. Şimşek, M., 2010. Vibration Analysis of a Single-Walled Carbon Nanotube Under Action of a Moving Harmonic Load Based on Nonlocal Elasticity Theory; Physica E, 43, pp. 182-191.
16. Şimşek, M., 2011. Nonlocal Effects in the Forced Vibration of an Elastically Connected Double-Carbon Nanotube System Under a Moving Nanoparticle; Computational Materials Science, 50, pp. 2112-2123,
17. Şimşek, M., 2011. Forced Vibration of an Embedded Single-Walled Carbon Nanotube Traversed by a Moving Load Using Nonlocal Timoshenko Beam Theory; Steel Composite Structures, 11, pp. 59-76.
18. Eltaher, M.A., Emam, S.A., Mahmoud, F.F., 2012. Free Vibration Analysis of Functionally Graded Size-Dependent Nanobeams; Applied Mathematics and Computation, 218, pp. 7406–7420.
19. Thai, H.T., A., 2012.Nonlocal Beam Theory for Bending, Buckling, and Vibration of Nanobeams; International Journal of Engineering Science, 52, pp. 56–64.
20. Şimşek M., 2012. Nonlocal Effects in the Free Longitudinal Vibration of Axially Functionally Graded Tapered Nanorods; Computational Materials Science, 61, pp. 257–265.
21. Tsepoura, K.G., 2002. Papargyri-Beskou, S., Polyzos, D., Beskos, D.E., Static and Dynamic Analysis of a Gradient-Elastic Bar in Tension; Archive of Applied Mechanics, 72, pp. 483-497.
22. Park, S.K., Gao, X.L, 2006. Bernoulli–Euler Beam Model Based on a Modified Couple

Stress Theory; Journal of Micromechanics and Microengineering, 16, pp. 2355–2359.

23. Ma, H.M., Gao, X.L., Reddy, J.N., 2008. A Microstructure-Dependent Timoshenko Beam Model Based on a Modified Couple Stress Theory; Journal of the Mechanics and Physics of Solids, 56, pp. 3379-3391.

24. Şimşek M., 2010. Dynamic Analysis of an Embedded Microbeam Carrying A Moving Microparticle Based on the Modified Couple Stress Theory, International Journal of Engineering Science, 48, pp. 1721–1732.

25. Reddy, J.N., 2011. Microstructure-Dependent Couple Stress Theories of Functionally Graded Beams; Jo.urnal of the Mechanics and Physics of Solids, 59, pp. 2382-2399.

26. Şimşek, M., Kocatürk, T., Akbaş, Ş. D., 2013. Static Bending of a Functionally Graded Microscale Timoshenko Beam Based on the Modified Couple Stress Theory; Composite Structures, 95, pp. 740-747.

27. Şimşek, M., Reddy, J.N, 2013. Bending and Vibration of Functionally Graded Microbeams Using a New Higher Order Beam Theory and the Modified Couple Stress Theory; International Journal of Engineering Science, 64, pp. 37–53.

28. Şimşek, M., Reddy, J.N, 2013. A Unified Higher Order Beam Theory for Buckling of a Functionally Graded Microbeam Embedded in Elastic Medium Using Modified Couple Stress theory, Composite Structures, 101, pp. 47-58.

29. Akgöz, B., Civalek, Ö., 2013. Longitudinal Vibration Analysis of Strain Gradient Bars Made of Functionally Graded Materials (FGM); Composites: Part B, 55, pp. 263-268.

30. Şimşek, M., 2014. Nonlinear Static and Free Vibration Analysis of Microbeams Based on the Nonlinear Elastic Foundation Using Modified Couple Stress Theory and He's Variational Method; Composite Structures, 112, pp. 264-272.

31. Challamel N., 2013. Variational Formulation of Gradient or/and Nonlocal Higher-Order Shear Elasticity Beams, Composite Structures, 105, pp. 351–368.

32. Lim C.W., Zhang G., Reddy J.N., 2015. A Higher-Order Nonlocal Elasticity and Strain Gradient Theory and its Applications in Wave Propagation, Journal of the Mechanics and Physics of Solids, 78, pp. 298–313,

33. Li L., Hu Y, 2015. Buckling Analysis of Size-Dependent Nonlinear Beams Based on a Nonlocal Strain Gradient Theory; International Journal of Engineering Science, 97, pp. 84–94.