Weierstrass Representation, Degree and Classes of the Surfaces in the Four Dimensional Euclidean Space

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Abstract
We study two parameters families of Bour-type and Enneper-type minimal surfaces using the Weierstrass representation in the four dimensional Euclidean space. We obtain implicit algebraic equations, degree and classes of the surfaces.

Keywords — 4-space, surface, Weierstrass representation, degree, class.
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1 Surfaces in $\mathbb{R}^4$

In Moore [4], we find a general definition of rotation surfaces in $\mathbb{R}^4$:

$$\begin{aligned} X(u,t) = (&x_1(u)\cos(at) - x_2(u)\sin(at),\\ &x_1(u)\cos(at) + x_2(u)\sin(at),\\ &x_3(u)\cos(bt) - x_4(u)\sin(bt),\\ &x_3(u)\cos(bt) + x_4(u)\sin(bt)). \end{aligned}$$

We propose that we look at a restricted case of this, found in Ganchev-Milousheva [2] :

$$W(u,t) = (x_1(u), x_2(u), r(u)\cos(t), r(u)\sin(t)).$$

The first we think is a bit too general since the curve is not located in any subspace before rotation.

At any rate this has:

$$g(\partial u, \partial u) = r'^2 + (x_1)'^2 + (x_2)'^2 = 1$$

if we use arc length parametrization, $g(\partial u, \partial t) = 0$ and $g(\partial t, \partial t) = r^2$.

Using the Weierstrass representation in Section 2, we give two parameters familes of Bour's-type (in Section 3) and Enneper's-type (in Section 4) minimal surfaces in the four dimensional Euclidean space. We also calculate implicit algebraic equations of the surfaces, degrees and classes of the surfaces.

2 Weierstrass equations for a minimal surface in $\mathbb{R}^4$

In Hoffman and Osserman [3] , p.45, they give the Weierstrass equations for a minimal surface in $\mathbb{R}^4$:

$$\Phi(z) = \frac{\psi}{2}(1 + fg, i(1 - fg), f - g, -i(f + g)).$$

Here ψ is analytic and the order of the zeros of ψ must be greater than the order of the poles of f, g at each point. If $\psi = 2z$ and $f = f_1 + if_2$, $g = g_1 + ig_2$ then

$$\begin{aligned} X_x - iX_y &= \Phi(z) \\ &= z(1 + fg, i(1 - fg), f - g, -i(f + g)) \\ &= ((1 + f_1g_1 - f_2g_2)x - (f_2g_1 + f_1g_2)y, \\ &\quad (f_2g_1 + f_1g_2)x - y + f_1g_1y - f_2g_2y, \\ &\quad (f_1 - g_1)x + (-f_2 + g_2)y, (f_2 + g_2)x + (f_1 + g_1)y) \\ &\quad - i(-y - f_1(g_2x + g_1y) + f_2(-g_1x + g_2y), \\ &\quad (-1 + f_1g_1 - f_2g_2)x - (f_2g_1 + f_1g_2)y, \\ &\quad + (-f_2 + g_2)x + (-f_1 + g_1)y, \\ &\quad (f_1 + g_1)x - (f_2 + g_2)y) \end{aligned}$$

We set

$$\begin{aligned} w_1 = (&-(f_2g_1x + f_1g_2x - y + f_1g_1y - f_2g_2y), \\ &(1 + f_1g_1 - f_2g_2)x - (f_2g_1 + f_1g_2)y, \\ &-((f_2 + g_2)x + (f_1 + g_1)y), \\ &(f_1 - g_1)x + (-f_2 + g_2)y) \end{aligned}$$

which is perpendicular to X_x, and

$$\begin{aligned} w_2 = (&-((-1 + f_1g_1 - f_2g_2)x - (f_2g_1 + f_1g_2)y), \\ &-y - f_1(g_2x + g_1y) + f_2(-g_1x + g_2y), \\ &-(f_1x + g_1x - (f_2 + g_2)y), \\ &-f_2x + g_2x + (-f_1 + g_1)y) \end{aligned}$$

which is perpendicular to X_y.

So far we see that:

$$\begin{aligned} b &= \langle X_x, w_2 \rangle \\ &= -(-1 + f_1^2 + f_2^2)(1 + g_1^2 + g_2^2)(x^2 + y^2) \\ &= -\langle X_y, w_1 \rangle, \end{aligned}$$

while

$$\begin{aligned} a &= \langle X_x, X_x \rangle \\ &= \langle X_y, X_y \rangle \\ &= (1 + f_1^2 + f_2^2)(1 + g_1^2 + g_2^2)(x^2 + y^2) \\ &= \langle w_j, w_j \rangle. \end{aligned}$$

Now we can use Gram-Schmidt to find an orthonormal basis for the normal space. We let $e_1 = X_x / \sqrt{a}$ and $e_2 = X_y / \sqrt{a}$. Then we get

$$n_1 = \sqrt{\frac{a}{a^2 - b^2}}\left(w_1 + \frac{b}{a}X_y\right),$$

$$n_2 = \sqrt{\frac{a}{a^2 - b^2}}\left(w_2 - \frac{b}{a}X_x\right),$$

where

$$a^2 - b^2 = 4(f_1^2 + f_2^2)(x^2 + y^2)^2(g_1^2 + g_2^2 + 1)^2,$$

$$\sqrt{\frac{a}{a^2 - b^2}} = \sqrt{\frac{1 + f_1^2 + f_2^2}{4(f_1^2 + f_2^2)(x^2 + y^2)(g_1^2 + g_2^2 + 1)}},$$

$$\frac{b}{a} = -\frac{-1 + f_1^2 + f_2^2}{1 + f_1^2 + f_2^2}.$$

Then

$$n_1=\sqrt{\frac{a}{a^2-b^2}}\left[\begin{pmatrix}-(f_1g_2+f_2g_1)x-(-1+f_1g_1-f_2g_2)y\\(1+f_1g_1-f_2g_2)x-(f_2g_1+f_1g_2)y\\-(f_2+g_2)x-(f_1+g_1)y\\(f_1-g_1)x+(-f_2+g_2)y\end{pmatrix}+\frac{b}{a}\begin{pmatrix}-(f_1g_2+f_2g_1)x+(-1-f_1g_1-f_2g_2)y\\(-1+f_1g_1-f_2g_2)x-(f_2g_1+f_1g_2)y\\(-f_2+g_2)x+(-f_1+g_1)y\\(f_1+g_1)x-(f_2+g_2)y\end{pmatrix}\right].$$

With $x=r\cos(\theta)$, $y=r\sin(\theta)$, $f_1=r^m\cos(m\theta)$, $f_2=r^m\sin(m\theta)$, $g_1=r^n\cos(n\theta)$, $g_2=r^n\sin(n\theta)$ we have: normals $n_1(r,\theta)$:

$$\frac{1}{\sqrt{(r^{2m}+1)(r^{2n}+1)}}\begin{pmatrix}r^m\sin(\theta)-r^n\sin((m+n+1)\theta)\\r^m\cos(\theta)+r^n\cos((m+n+1)\theta)\\-r^{m+n}\sin((n+1)\theta)-\sin((m+1)\theta)\\-r^{m+n}\cos((n+1)\theta)+\cos((m+1)\theta)\end{pmatrix}$$

and $n_2(r,\theta)$:

$$\frac{1}{\sqrt{(r^{2m}+1)(r^{2n}+1)}}\begin{pmatrix}r^m\cos(\theta)-r^n\cos((m+n+1)\theta)\\-r^m\sin(\theta)-r^n\sin((m+n+1)\theta)\\-r^{m+n}\cos((n+1)\theta)-\cos((m+1)\theta)\\r^{m+n}\sin((n+1)\theta)-\sin((m+1)\theta)\end{pmatrix}$$

3 Bour's family of surfaces

We now choose, in analogy with the surface case, $\psi=2z$, $f=z^m$ and $g=z^n$, with $m\neq n$. This gives:

$$\Phi(z)=z(1+z^{m+n},i(1-z^{m+n}),z^m-z^n,-i(z^m+z^n)).$$

We integrate to get:

$$\int\Phi(z)dz=\begin{pmatrix}\frac{z^2}{2}+\frac{z^{m+n+2}}{m+n+2}\\i\left(\frac{z^2}{2}-\frac{z^{m+n+2}}{m+n+2}\right)\\\frac{z^{m+2}}{m+2}-\frac{z^{n+2}}{n+2}\\-i\left(\frac{z^{m+2}}{m+2}+\frac{z^{n+2}}{n+2}\right)\end{pmatrix}.$$

We let $z=re^{i\theta}$ and take the real part

$$\mathsf{B}_{m,n}(r,\theta)=\begin{pmatrix}\frac{r^2\cos(2\theta)}{2}+\frac{r^{m+n+2}\cos((m+n+2)\theta)}{m+n+2}\\-\frac{r^2\sin(2\theta)}{2}+\frac{r^{m+n+2}\sin((m+n+2)\theta)}{m+n+2}\\\frac{r^{m+2}\cos((m+2)\theta)}{m+2}-\frac{r^{n+2}\cos((n+2)\theta)}{n+2}\\\frac{r^{m+2}\sin((m+2)\theta)}{m+2}+\frac{r^{n+2}\sin((n+2)\theta)}{n+2}\end{pmatrix}.$$

Example: For $m=2,\ n=0$, we have $\mathsf{B}_{2,0}(r,\theta)$:

$$\begin{pmatrix}\frac{r^2\cos(2\theta)}{2}+\frac{r^4\cos(4\theta)}{4}\\-\frac{r^2\sin(2\theta)}{2}+\frac{r^4\sin(4\theta)}{4}\\-\frac{r^2\cos(2\theta)}{2}+\frac{r^4\cos(4\theta)}{4}\\\frac{r^2\sin(2\theta)}{2}+\frac{r^4\sin(4\theta)}{4}\end{pmatrix}=\begin{pmatrix}x(r,\theta)\\y(r,\theta)\\z(r,\theta)\\w(r,\theta)\end{pmatrix},$$

and $\mathsf{B}_{2,0}(u,v)$:

$$\begin{pmatrix}\frac{1}{2}(u^2-v^2)+\frac{1}{4}u^4-\frac{3}{2}u^2v^2+\frac{1}{4}v^4\\-uv+u^3v-uv^3\\-\frac{1}{2}(u^2-v^2)+\frac{1}{4}u^4-\frac{3}{2}u^2v^2+\frac{1}{4}v^4\\uv+u^3v-uv^3\end{pmatrix}=\begin{pmatrix}x(u,v)\\y(u,v)\\z(u,v)\\w(u,v)\end{pmatrix}.$$

We want to find normals n_1 and n_2 of the Bour's minimal surface

$$\mathsf{B}_{2,0}(u,v)=(x(u,v),y(u,v),z(u,v),w(u,v)),$$

and degree of the algebraic Bour minimal surface.

Hence, we find the implicit equations $Q(x,y,z,w)=0$ of $\mathsf{B}_{2,0}(u,v)$ using elimination techniques in the cartesian coordinates x, y, z, w as follow:

$$y^2 + 4xy^2 + y^4 - 2yw - 8xyw - 4y^3w - 3w^2 + 4xw^2 + 6y^2w^2 - 4yw^3 + w^4,$$

and

$$-3y^2 + y^4 + 4y^2z - 2yw - 4y^3w - 8yzw + w^2 + 6y^2w^2 + 4zw^2 - 4yw^3 + w^4.$$

without z and x, respectively. But we should get with x, y, z, w. On the other hand, we use the Sylvester elimination technique and find the implicit eq. as follows:

$$\det\begin{pmatrix} 1 & 0 & A & 0 \\ 0 & 1 & 0 & A \\ 1 & 0 & B & 0 \\ 0 & 1 & 0 & B \end{pmatrix} = (B-A)^2$$
$$= (2x + 2z - 2wy + 2xz + w^2 - x^2 + y^2 - z^2)^2,$$

where

$$A = -2(x-z)^2 + 2(x+z),$$
$$B = -(x-z)^2 - (w-y)^2.$$

For short, taking $r^4 = t^2 = k,$ then we get

$$\det\begin{pmatrix} 1 & A \\ 1 & B \end{pmatrix} = B - A$$
$$= 2x + 2z - 2wy + 2xz + w^2 - x^2 + y^2 - z^2.$$

Hence,the irreducible implicit equation is

$$Q(x,y,z,w) = 2x + 2z - 2wy + 2xz + w^2 - x^2 + y^2 - z^2$$

with $\deg(\mathfrak{B}_{2,0}) = 2.$ So, $\mathfrak{B}_{2,0}$ is an algebraic minimal surface in 4-space. Then find P_1 using

$$xX_1 + yY_1 + zZ_1 + wW_1 + P_1 = 0,$$

where

$$n_1 = (X_1(u,v), Y_1(u,v), Z_1(u,v), W_1(u,v)),$$

and $P_1 = P_1(u,v).$ Similarly, find P_2 using

$$xX_2 + yY_2 + zZ_2 + wW_2 + P_2 = 0,$$

where

$$n_2 = (X_2(u,v), Y_2(u,v), Z_2(u,v), W_2(u,v)),$$

and $P_2 = P_2(u,v).$ Therefore, inhomogeneous tangential coordinates of the Bour surface, using n_1 (resp. using n_2), are $a_1 = X_1 / P_1,$ $b_1 = Y_1 / P_1,$ $c_1 = Z_1 / P_1,$ $d_1 = W_1 / P_1$ (resp. $a_2 = X_2 / P_2,$ $b_2 = Y_2 / P_2,$ $c_2 = Z_2/P_2,$ $d_2 = W_2/P_2$).

Hence, we can find the implicit eq.

$$\widetilde{Q_1}(a_1, b_1, c_1, d_1) = 0$$

(resp. $\widetilde{Q_2}(a_2, b_2, c_2, d_2) = 0$)

of

$$\mathfrak{B}_{2,0}(u,v)$$

using elimination techniques in the inhomogeneous tangential coordinates a_1, b_1, c_1, d_1 (resp. a_2, b_2, c_2, d_2) and can find the classes of the algebraic Bour minimal surface (we have 2 normals and then have 2 classes).

$$(n_1)_{2,0}(r,\theta) = \frac{1}{\sqrt{2(r^4+1)}} \begin{pmatrix} r^2\sin(\theta) - \sin(3\theta) \\ r^2\cos(\theta) + \cos(3\theta) \\ -r^2\sin(\theta) - \sin(3\theta) \\ -r^2\cos(\theta) + \cos(3\theta) \end{pmatrix}$$
$$= \begin{pmatrix} X_1(r,\theta) \\ Y_1(r,\theta) \\ Z_1(r,\theta) \\ W_1(r,\theta) \end{pmatrix},$$

and then

$$(n_1)_{2,0}(u,v) = \begin{pmatrix} \frac{v\left(u^4+2u^2v^2-3u^2+v^4+v^2\right)}{\left(u^2+v^2\right)^{\frac{3}{2}}\sqrt{2\left(\left(u^2+v^2\right)^2+1\right)}} \\ \frac{u\left(u^4+2u^2v^2+u^2+v^4-3v^2\right)}{\left(u^2+v^2\right)^{\frac{3}{2}}\sqrt{2\left(\left(u^2+v^2\right)^2+1\right)}} \\ -\frac{v\left(u^4+2u^2v^2+3u^2+v^4-v^2\right)}{\left(u^2+v^2\right)^{\frac{3}{2}}\sqrt{2\left(\left(u^2+v^2\right)^2+1\right)}} \\ -\frac{u\left(u^4+2u^2v^2-u^2+v^4+3v^2\right)}{\left(u^2+v^2\right)^{\frac{3}{2}}\sqrt{2\left(\left(u^2+v^2\right)^2+1\right)}} \end{pmatrix} = \begin{pmatrix} X_1(u,v) \\ Y_1(u,v) \\ Z_1(u,v) \\ W_1(u,v) \end{pmatrix}.$$

Using $xX_1 + yY_1 + zZ_1 + wW_1 + P_1 = 0$, we get

$$P_1 = \frac{v\sqrt{2}\left(\sqrt{u^2+v^2}\right)^3}{4\sqrt{(u^2+v^2)^2+1}},$$

and then

$$a_1 = X_1/P_1 = \frac{2(u^4+2u^2v^2-3u^2+v^4+v^2)}{(u^2+v^2)^3},$$

$$b_1 = Y_1/P_1 = \frac{2u(u^4+2u^2v^2+u^2+v^4-3v^2)}{v(u^2+v^2)^3},$$

$$c_1 = Z_1/P_1 = \frac{-2(u^4+2u^2v^2+3u^2+v^4-v^2)}{(u^2+v^2)^3},$$

$$d_1 = W_1/P_1 = \frac{-2u(u^4+2u^2v^2-u^2+v^4+3v^2)}{v(u^2+v^2)^3}.$$

Hence, in the inhomogeneous tangential coordinates a_1, b_1, c_1, d_1, parametric eq. of Bour surface is

$$\mathfrak{B}_{2,0}(u,v) = \frac{2}{v(u^2+v^2)^3}\begin{pmatrix} v(u^4+2u^2v^2-3u^2+v^4+v^2) \\ u(u^4+2u^2v^2+u^2+v^4-3v^2) \\ -v(u^4+2u^2v^2+3u^2+v^4-v^2) \\ -u(u^4+2u^2v^2-u^2+v^4+3v^2) \end{pmatrix} = \begin{pmatrix} a_1(u,v) \\ b_1(u,v) \\ c_1(u,v) \\ d_1(u,v) \end{pmatrix}.$$

So, we have 6 implicit eqs.

$$\widetilde{Q}_1(a_1,b_1,c_1,d_1) = 0$$

of

$$\mathfrak{B}_{2,0}(u,v)$$

using elimination techniques in the inhomogeneous tangential coordinates a_1, b_1, c_1, d_1, as follow:

$$\begin{aligned}\widetilde{Q}_1(a_1,b_1,c_1,d_1) = {} & -a_1^2b_1 + a_1^2d_1 + 2a_1b_1c_1 \\ & - 2a_1c_1d_1 - b_1c_1^2 + c_1^2d_1 \\ & - 4a_1b_1 + 4c_1d_1,\end{aligned}$$

or

$$\begin{aligned}\widetilde{Q}_1(a_1,b_1,c_1,d_1) = {} & -a_1^3 + 2a_1^2c_1 + 2a_1b_1^2 \\ & - 2a_1b_1d_1 - a_1c_1^2 - b_1^2c_1 \\ & + c_1d_1^2 + 4a_1^2 + 4a_1c_1 \\ & + 4b_1d_1 + 4d_1^2,\end{aligned}$$

or

$$\begin{aligned}\widetilde{Q}_1(a_1,b_1,c_1,d_1) = {} & -2a_1^3 + 5a_1^2c_1 + 3a_1b_1^2 - 4a_1b_1d_1 \\ & - 4a_1c_1^2 + a_1d_1^2 - 2b_1^2c_1 + 2b_1c_1d_1 \\ & + c_1^3 + 8a_1^2 + 4a_1c_1 - 4b_1^2 + 4b_1d_1 \\ & - 4c_1^2 + 8d_1^2,\end{aligned}$$

or

$$\begin{aligned}\widetilde{Q}_1(a_1,b_1,c_1,d_1) = {} & a_1^4 - 4a_1^3c_1 + 6a_1^2c_1^2 - 4a_1c_1^3 \\ & + c_1^4 - 2a_1^2c_1 - 2a_1b_1^2 + 4a_1b_1d_1 \\ & + 4a_1c_1^2 - 2a_1d_1^2 - 2c_1^3 - 16a_1^2 \\ & - 24a_1c_1 - 8b_1^2 - 24b_1d_1 - 8c_1^2 \\ & - 16d_1^2,\end{aligned}$$

or

$$\begin{aligned}\widetilde{Q}_1(a_1,b_1,c_1,d_1) = {} & -2a_1^3b_1 + 6a_1^2b_1c_1 - a_1b_1^3 + 3a_1b_1^2d_1 \\ & - 6a_1b_1c_1^2 - 3a_1b_1d_1^2 + a_1d_1^3 + 2b_1c_1^3 \\ & - 16a_1^2b_1 + 16a_1^2d_1 + 16a_1b_1c_1 - 4a_1c_1d_1 \\ & - 4b_1^3 - 8b_1^2d_1 - 12b_1c_1^2 + 4b_1d_1^2 + 8d_1^3 \\ & - 32a_1b_1 + 32c_1d_1,\end{aligned}$$

or

$$\begin{aligned}\widetilde{Q}_1(a_1,b_1,c_1,d_1) = {} & a_1b_1^4 - 4a_1b_1^3d_1 + 6a_1b_1^2d_1^2 - 4a_1b_1d_1^3 + a_1d_1^4 \\ & - 16a_1^4 + 38a_1^3c_1 + 14a_1^2b_1^2 - 44a_1^2b_1d_1 \\ & - 28a_1^2c_1^2 + 22a_1^2d_1^2 + 16a_1b_1^2c_1 + 6a_1c_1^3 \\ & + 4b_1^4 + 4b_1^3d_1 - 8b_1^2c_1^2 - 12b_1^2d_1^2 - 4b_1d_1^3 \\ & + 8d_1^4 + 144a_1^3 - 144a_1^2c_1 - 208a_1b_1^2 \\ & + 168a_1b_1d_1 + 104a_1c_1^2 + 40a_1d_1^2 + 80b_1^2c_1 \\ & - 24c_1^3 - 320a_1^2 - 224a_1c_1 + 96b_1^2 - 224b_1d_1 \\ & + 96c_1^2 - 320d_1^2.\end{aligned}$$

So,

$$classes(\mathfrak{B}_{2,0}) = 3,4,5.$$

We can use the same techniques for $(n_2)_{2,0}$:

$$(n_2)_{2,0}(r,\theta) = \frac{1}{\sqrt{2(r^4+1)}}\begin{pmatrix} r^2\cos(\theta)-\cos(3\theta) \\ -r^2\sin(\theta)-\sin(3\theta) \\ -r^2\cos(\theta)-\cos(3\theta) \\ r^2\sin(\theta)-\sin(3\theta)\end{pmatrix} = \begin{pmatrix} X_2(r,\theta) \\ Y_2(r,\theta) \\ Z_2(r,\theta) \\ W_2(r,\theta)\end{pmatrix},$$

$$(n_2)_{2,0}(u,v) = \begin{pmatrix} X_2(u,v) \\ Y_2(u,v) \\ Z_2(u,v) \\ W_2(u,v)\end{pmatrix}.$$

4 Enneper's family of surfaces

We now choose, in analogy with the surface case, $\psi = 2$, $f = z^m$ and $g = z^n$, with $m \neq n$. This gives:

$$\Phi(z) = (1+z^{m+n}, i(1-z^{m+n}), z^m - z^n, -i(z^m + z^n)).$$

We integrate to get:

$$\int \Phi(z)dz = \begin{pmatrix} z + \frac{z^{m+n+1}}{m+n+1} \\ i\left(z - \frac{z^{m+n+1}}{m+n+1}\right) \\ \frac{z^{m+1}}{m+1} - \frac{z^{n+1}}{n+1} \\ -i\left(\frac{z^{m+1}}{m+1} + \frac{z^{n+1}}{n+1}\right)\end{pmatrix}.$$

We let $z = re^{i\theta}$ and take the real part

$$\mathsf{E}_{m,n}(r,\theta) = \begin{pmatrix} r\cos(\theta) + \frac{r^{m+n+1}\cos((m+n+1)\theta)}{m+n+1} \\ -r\sin(\theta) + \frac{r^{m+n+1}\sin((m+n+1)\theta)}{m+n+1} \\ \frac{r^{m+1}\cos((m+1)\theta)}{m+1} - \frac{r^{n+1}\cos((n+1)\theta)}{n+1} \\ \frac{r^{m+1}\sin((m+1)\theta)}{m+1} + \frac{r^{n+1}\sin((n+1)\theta)}{n+1}\end{pmatrix}.$$

Example: For $m = 2, \quad n = 0$, we have $\mathsf{E}_{2,0}(r,\theta)$:

$$\begin{pmatrix} \frac{r^3\cos(3\theta)}{3} + r\cos(\theta) \\ \frac{r^3\sin(3\theta)}{3} - r\sin(\theta) \\ \frac{r^3\cos(3\theta)}{3} - r\cos(\theta) \\ \frac{r^3\sin(3\theta)}{3} + r\sin(\theta)\end{pmatrix} = \begin{pmatrix} x(r,\theta) \\ y(r,\theta) \\ z(r,\theta) \\ w(r,\theta)\end{pmatrix},$$

and $\mathsf{E}_{2,0}(u,v)$:

$$\begin{pmatrix} \frac{1}{3}u^3 - uv^2 + u \\ u^2 v - \frac{1}{3}v^3 - v \\ \frac{1}{3}u^3 - uv^2 - u \\ u^2 v - \frac{1}{3}v^3 + v \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \\ w(u,v) \end{pmatrix},$$

where $u = r\cos\theta, \quad v = r\sin\theta.$

We want to find normals n_1 and n_2 of the Enneper's minimal surface

$$\mathsf{E}_{2,0}(u,v) = (x(u,v), y(u,v), z(u,v), w(u,v)),$$

and degree of the algebraic Enneper minimal surface.

We have $r + A = 0$, $r^3 + B = 0$ and

$$Syl(A,B,r) = \det\begin{pmatrix} 1 & A & 0 & 0 \\ 0 & 1 & A & 0 \\ 0 & 0 & 1 & A \\ 1 & 0 & 0 & B \end{pmatrix} = B - A^3,$$

where

$$A = -\tfrac{1}{4}((x-z)^2 + (w-y)^2),$$
$$B = -\tfrac{9}{4}((x+z)^2 + (w+y)^2).$$

Hence, we find the irreducible implicit equation $Q(x,y,z,w) = 0$ of $\mathsf{E}_{2,0}(u,v)$ using elimination techniques in the cartesian coordinates x, y, z, w as follows:

$$\begin{aligned}
&w^6 - 6w^5y + 3w^4x^2 - 6w^4xz + 15w^4y^2 + 3w^4z^2 \\
&- 12w^3x^2y + 24w^3xyz - 20w^3y^3 - 12w^3yz^2 + 3w^2x^4 \\
&- 12w^2x^3z + 18w^2x^2y^2 + 18w^2x^2z^2 - 36w^2xy^2z \\
&- 12w^2xz^3 + 15w^2y^4 + 18w^2y^2z^2 + 3w^2z^4 - 144w^2 \\
&- 6wx^4y + 24wx^3yz - 12wx^2y^3 - 36wx^2yz^2 + 24wxy^3z \\
&+ 24wxyz^3 - 6wy^5 - 12wy^3z^2 - 6wyz^4 - 288wy + x^6 \\
&- 6x^5z + 3x^4y^2 + 15x^4z^2 - 12x^3y^2z - 20x^3z^3 + 3x^2y^4 \\
&+ 18x^2y^2z^2 + 15x^2z^4 - 144x^2 - 6xy^4z - 12xy^2z^3 - 6xz^5 \\
&- 288xz + y^6 + 3y^4z^2 + 3y^2z^4 - 144y^2 + z^6 - 144z^2.
\end{aligned}$$

Its degree is $\deg(\mathsf{E}_{2,0}) = 6$. So, $Q(x,y,z,w) = 0$ is an implicit algebraic Enneper type minimal surface in 4-space. Then find P_1 using

$$xX_1 + yY_1 + zZ_1 + wW_1 + P_1 = 0,$$

where

$$n_1 = (X_1(u,v), Y_1(u,v), Z_1(u,v), W_1(u,v)),$$

and $P_1 = P_1(u,v)$. Similarly, find P_2 using

$$xX_2 + yY_2 + zZ_2 + wW_2 + P_2 = 0,$$

where

$$n_2 = (X_2(u,v), Y_2(u,v), Z_2(u,v), W_2(u,v)),$$

and $P_2 = P_2(u,v)$. Therefore, inhomogeneous tangential coordinates of the Enneper surface, using n_1 (resp. using n_2), are $a_1 = X_1 / P_1$, $b_1 = Y_1 / P_1$, $c_1 = Z_1 / P_1$, $d_1 = W_1 / P_1$ (resp. $a_2 = X_2 / P_2$, $b_2 = Y_2 / P_2$, $c_2 = Z_2 / P_2$, $d_2 = W_2 / P_2$).

Hence, we can find the implicit eq.

$$\widetilde{Q_1}(a_1, b_1, c_1, d_1) = 0$$

(resp. $\widetilde{Q_2}(a_2, b_2, c_2, d_2) = 0$)

of

$$\widetilde{\mathfrak{E}}_{2,0}(u,v)$$

using elimination techniques in the inhomogeneous tangential coordinates a_1, b_1, c_1, d_1 (resp. a_2, b_2, c_2, d_2) and can find the classes of the algebraic Enneper minimal surface (we have 2 normals and then have 2 classes). $(n_1)_{2,0}(r,\theta)$ is as follows:

$$(n_1)_{2,0}(r,\theta) = \frac{1}{\sqrt{2(r^4+1)}}\begin{pmatrix} -\sin(2\theta) \\ r^2+\cos(2\theta) \\ -\sin(2\theta) \\ -r^2+\cos(2\theta) \end{pmatrix} = \begin{pmatrix} X_1(r,\theta) \\ Y_1(r,\theta) \\ Z_1(r,\theta) \\ W_1(r,\theta) \end{pmatrix},$$

and $(n_1)_{2,0}(u,v)$ is as follows:

$$\frac{1}{(u^2+v^2)\sqrt{2\left((u^2+v^2)^2+1\right)}}\begin{pmatrix} -2uv \\ (u^2+v^2)^2+(u^2-v^2) \\ -2uv \\ (u^2-v^2)-(u^2+v^2)^2 \end{pmatrix} = \begin{pmatrix} X_1(u,v) \\ Y_1(u,v) \\ Z_1(u,v) \\ W_1(u,v) \end{pmatrix}.$$

Using $xX_1 + yY_1 + zZ_1 + wW_1 + P_1 = 0,$ we get

$$P_1 = \frac{2\sqrt{2}\,v(u^2+v^2)}{3\sqrt{(u^2+v^2)^2+1}},$$

and then

$$a_1 = X_1/P_1 = -\frac{3u}{2(u^2+v^2)^2},$$

$$b_1 = Y_1/P_1 = \frac{3\left((u^2+v^2)^2+u^2-v^2\right)}{4v(u^2+v^2)^2},$$

$$c_1 = Z_1/P_1 = -\frac{3u}{2(u^2+v^2)^2},$$

$$d_1 = W_1/P_1 = -\frac{3\left((u^2+v^2)^2+v^2-u^2\right)}{4v(u^2+v^2)^2}.$$

Hence, using $\mathsf{E}_{2,0}(u,v)$ and $(n_1)_{2,0}(u,v),$ we get the first parametric eq. of Enneper type surface

$$\widetilde{\mathfrak{E}}_{2,0}(u,v)$$

in the inhomogeneous tangential coordinates a_1, b_1, c_1, d_1 as follows:

$$\widetilde{\mathfrak{E}}_{2,0}(u,v) = \frac{3}{4v(u^2+v^2)^2}\begin{pmatrix} -2uv \\ (u^2+v^2)^2+u^2-v^2 \\ -2uv \\ (u^2+v^2)^2+v^2-u^2 \end{pmatrix} = \begin{pmatrix} a_1(u,v) \\ b_1(u,v) \\ c_1(u,v) \\ d_1(u,v) \end{pmatrix}.$$

Then we have implicit eq.

$$\widetilde{Q}_1(a_1,b_1,c_1,d_1) = 0$$

of the first surface

$$\widetilde{\mathfrak{E}}_{2,0}(u,v)$$

using elimination techniques in the inhomogeneous tangential coordinates a_1, b_1, c_1, d_1 , as follows:

$$\begin{aligned}\widetilde{Q}_1(a_1,b_1,c_1,d_1) &= 16a_1^2b_1^6 - 96a_1^2b_1^5d_1 + 240a_1^2b_1^4d_1^2 \\ &- 320a_1^2b_1^3c_1^3 + 240a_1^2b_1^2d_1^4 - 96a_1^2b_1d_1^5 + 16a_1^2d_1^6 \\ &- 144a_1^2b_1^4 + 288a_1^2b_1^3d_1 - 288a_1^2b_1d_1^3 + 144a_1^2d_1^4 - 36b_1^6 \\ &+ 108b_1^4d_1^2 - 108b_1^2d_1^4 + 36d_1^6 - 1296a_1^4 - 648a_1^2b_1^2 \\ &- 1296a_1^2b_1d_1 - 648a_1^2d_1^2 - 81b_1^4 - 324b_1^3d_1 - 486b_1^2d_1^2 \\ &- 324b_1d_1^3 - 81d_1^4.\end{aligned}$$

So,

$$class(\widetilde{\mathfrak{E}}_{2,0}) = 8.$$

We can use the same techniques for

$$(n_2)_{2,0}(r,\theta) = \frac{1}{\sqrt{2r^4(r^4+1)}} \begin{pmatrix} r^4 - r^2\cos(2\theta) \\ -r^2\sin(2\theta) \\ -r^4 - r^2\cos(2\theta) \\ -r^2\sin(2\theta) \end{pmatrix} = \begin{pmatrix} X_2(r,\theta) \\ Y_2(r,\theta) \\ Z_2(r,\theta) \\ W_2(r,\theta) \end{pmatrix},$$

and $(n_2)_{2,0}(u,v)$:

$$\frac{1}{(u^2+v^2)\sqrt{2\left((u^2+v^2)^2+1\right)}} \begin{pmatrix} (u^2+v^2)^2 + v^2 - u^2 \\ -2uv \\ -(u^2+v^2)^2 + v^2 - u^2 \\ -2uv \end{pmatrix} = \begin{pmatrix} X_2(u,v) \\ Y_2(u,v) \\ Z_2(u,v) \\ W_2(u,v) \end{pmatrix}.$$

Using $\mathsf{E}_{2,0}(u,v)$ and $(n_2)_{2,0}(u,v)$, we get

$$P_2 = -\frac{2\sqrt{2}\,u(u^2+v^2)}{3\sqrt{(u^2+v^2)^2+1}}.$$

Hence, we obtain the second surface:

$$\tilde{\mathfrak{E}}_{2,0}(u,v) = \frac{3}{4v(u^2+v^2)^2} \begin{pmatrix} -\left[(u^2+v^2)^2 + v^2 - u^2\right] \\ 2uv \\ -\left[(u^2+v^2)^2 + u^2 - v^2\right] \\ 2uv \end{pmatrix} = \begin{pmatrix} a_2(u,v) \\ b_2(u,v) \\ c_2(u,v) \\ d_2(u,v) \end{pmatrix}.$$

So, we have implicit eq.

$$\tilde{Q}_2(a_2,b_2,c_2,d_2) = 0$$

of the second surface

$$\tilde{\mathfrak{E}}_{2,0}(u,v)$$

using elimination techniques in the inhomogeneous tangential coordinates a_2, b_2, c_2, d_2 as follows:

$$\begin{aligned} \tilde{Q}_2(a_2,b_2,c_2,d_2) &= 16a_2^6b_2^2 - 96a_2^5b_2^2c_2 + 240a_2^4b_2^2c_2^2 \\ &- 320a_2^3b_2^2c_2^3 + 240a_2^2b_2^2c_2^4 - 96a_2b_2^2c_2^5 + 16b_2^2c_2^6 \\ &- 36a_2^6 - 144a_2^4b_2^2 + 108a_2^4c_2^2 + 288a_2^3b_2^2c_2 - 108a_2^2c_2^4 \\ &- 288a_2b_2^2c_2^3 + 144b_2^2c_2^4 + 36c_2^6 - 81a_2^4 - 324a_2^3c_2 \\ &- 648a_2^2b_2^2 - 486a_2^2c_2^2 - 1296a_2b_2^2c_2 - 324a_2c_2^3 \\ &- 1296b_2^4 - 648b_2^2c_2^2 - 81c_2^4. \end{aligned}$$

Then we have

$$class(\tilde{\mathfrak{E}}_{2,0}) = 8.$$

5 References

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