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STABILIZATION OF DISCRETE SYSTEMS WITH BOUNDED PARAMETERS

Taner BÜYÜKKÖROĞLU*, Vakif DZHAFAROV

Department of Mathematics, Faculty of Science, Anadolu University, Eskişehir, Turkey

ABSTRACT

In this paper stabilization problem of linear discrete single input, single output plant by affine stabilizator is considered. It is assumed that stabilizing vector is bounded and its values are changed in a given box. We use the Schur-Szegö parameters (reflection coefficients) and obtain conditions for nonexistence and existence of a stabilizing vector.

Keywords: Schur stability, Reflection coefficients, Stabilizing vector

1. INTRODUCTION

Recall that a single polynomial

$$p(s) = a_1 + a_2 s + \cdots + a_n s^{n-1} + a_{n+1} s^n$$

is called Schur stable if all its roots lie in the unit open disc. In order to use the Schur-Szegö parameters we assume that $a_{n+1} = 1$. In this case the above polynomial becomes

$$p(s) = a_1 + a_2 s + \cdots + a_n s^{n-1} + s^n, \tag{1}$$

which corresponds to an n-dimensional vector $p = (a_1, a_2, \dots, a_n)^T \in \mathbb{R}^n$, where the symbol "T" stands for the transpose. The vector p is called stable if the corresponding polynomial $p(s)$ (1) is stable. From hence, we will understand "stable" to mean "Schur stability".

For the definition of Schur-Szegö parameters (reflection coefficients) $k_1, k_2, \dots, k_n$ of the polynomial (1) we refer to [1,2,3]. A map $f: \mathbb{R}^n \to \mathbb{R}^n$, $(k_1, k_2, \dots, k_n)^T \to (f_1, f_2, \dots, f_n)^T$ which is defined recursively by the formula

$$(f_1, f_2, \dots, f_n)^T = M_n(k_n) \begin{bmatrix} 0^T \\ M_{n-1}(k_{n-1}) \end{bmatrix} \cdots \begin{bmatrix} 0^T \\ M_1(k_1) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is called the reflection map. Here

$$M_j(k_j) = I_{j+1} + k_j E_{j+1}, \quad E_j = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix},$$

I_j is a $j \times j$ unit matrix, E_j is a $j \times j$ dimensional unit Henkel matrix.

The reflection map f is multilinear, that is, affine linear with respect to each variable k_i.

The polynomial (1) is stable if and only if there exist $k_1, k_2, \dots, k_n$ such that $|k_i| < 1$ and $a_i = f_i(k_1, k_2, \dots, k_n)$ for all $i = 1,2, \dots, n$. In this case, the numbers $k_1, k_2, \dots, k_n$ are called the reflection coefficients or Schur-Szegö parameters of the polynomial (1) (see [2]).

*Corresponding Author: tbuyukkoroglu@anadolu.edu.tr

Consider a linear plant $G(s) = \frac{a(s)}{b(s)}$ and regulator $R(s,c) = \frac{n(s,c)}{d(s,c)}$, where $c = (c_1, c_2, \dots, c_l)^T \in \mathcal{C}$ is control vector and $\mathcal{C}$ is a box:

$$\mathcal{C} = \{(c_1, c_2, \dots, c_l)^T : c_i^- \le c_i \le c_i^+, i = 1,2,\dots,l\}.$$

It is assumed that a, b, n and d are polynomials in s and both $n(s,c)$ and $d(s,c)$ are affine linear with respect to c. The closed-loop system has characteristic polynomial

$$\begin{aligned} p(s,c) &= a(s)\,n(s,c) + b(s)\,d(s,c) \\ &= p_0(s) + c_1 p_1(s) + c_2 p_2(s) + \cdots + c_l p_l(s). \end{aligned} \tag{2}$$

Without loss of generality assume that degree$(p_0) = n$, degree$(p_i) < n$ $(i = 1,2,\dots,l)$ and $p_0(s)$ is monic polynomial.

The stabilization problem is the determination of a parameter $c \in \mathcal{C}$ such that the characteristic polynomial (2) becomes stable. The relation (2) can be written in a vector equation in $\mathbb{R}^n$ as follows (see [2,3]). Define column vectors $p^0, p^1, \dots, p^l$, where the vector p^i correspond to the polynomial $p_i(s)$ $(i = 0,1,\dots,l)$. Since degree$(p_i) < n$ $(i = 1,2,\dots,l)$ we add zero components to the vector representation of $p_i(s)$ $(i = 0,1,\dots,l)$ in order to have dimension n. For example, assume that

$$p_0(s) = s^4 + 2s^3 + s^2 + 3s + 1, \quad p_1(s) = 2s^3 + s^2 + 1, \quad p_2(s) = s^2 + s.$$

Then

$$p^0 = (1,3,1,2)^T, \quad p^1 = (1,0,1,2)^T, \quad p^2 = (0,1,1,0)^T.$$

The relation (2) can be written as the following vector equation

$$p(c) = Ac + p^0 \tag{3}$$

where A is $n \times l$ dimensional matrix with column vectors p^i $(i = 1,2,\dots,l)$.

Let the monic polynomial (1) be given and p is the corresponding n-dimensional vector. The vector p is said to be stable if the corresponding polynomial (1) is stable. Define the following open subset of $\mathbb{R}^n$:

$$\mathcal{D} = \{(a_1, a_2, \dots, a_n)^T \in \mathbb{R}^n : \text{(1) is stable}\}.$$

It is shown in [4] that the closed convex hull of $\mathcal{D}$ is a polytope with $(n+1)$ vertices:

$$\overline{\text{co}}\,\mathcal{D} = \text{co}\,\{V^1, V^2, \dots, V^{n+1}\}, \tag{4}$$

where $\overline{\text{co}}$ stands for the closure of the convex hull and n-dimensional column vectors V^i correspond to the polynomial $(s-1)^i (s+1)^{n-i}$ $(0 \le i \le n)$.

Define

$$\mathcal{P} = \{Ac + p^0 : c \in \mathcal{C}\}. \tag{5}$$

The following proposition is obvious.

Proposition 1 There exists a stabilizing parameter $c \in \mathcal{C}$ if and only if $\mathcal{P} \cap \mathcal{D} \neq \emptyset$. If $\mathcal{P} \cap \text{co}\,\{V^1, V^2, \dots, V^{n+1}\} = \emptyset$ then there is no stabilizing parameter c.

Stabilization problems for discrete systems with unbounded stabilizing vector c have been considered in [3,5-9]. In [5,7] stabilizing vector from $\mathbb{R}^l$ is sought by randomly generating a great number of stable points or stable segments in the parameter space. In [6], D-decomposition method for the case $l = 2$ is described, stabilization of matrix families and various solution methods are discussed. In [3] the Bernstein expansion method is applied for stabilization. In [8] the locations of vertices V^i with respect to the affine set $\mathcal{P}$ is used for stabilization. The outer and inner approximation of the stabilizing set by using the interlacing property of Schur stable polynomials is considered in [9, p. 367].

In this paper, we consider stabilization problem with bounded stabilizing vector. In Section 2, we reduce the necessary condition $\mathcal{P} \cap \mathrm{co}\,\{V^1, V^2, \dots, V^{n+1}\} \neq \emptyset$ to a linear programming (LP) problem. The unfeasibility of this LP proves the nonexistence of a stabilizing vector. In Section 3, we apply least square minimization for stabilization.

2. LINEAR PROGRAMMING

In this section, we investigate the condition

$$\mathcal{P} \cap \mathrm{co}\,\{V^1, V^2, \dots, V^{n+1}\} \neq \emptyset. \tag{6}$$

This problem is reduced to feasibility problem of a suitable standard LP problem. If the feasibility set is empty then there is no stabilizing parameter c.

Since

$$\mathrm{co}\,\{V^1, V^2, \dots, V^{n+1}\} = \{x_1 V^1 + x_2 V^2 + \cdots + x_{n+1} V^{n+1} \colon x_1 + x_2 + \cdots + x_{n+1} = 1,$$
$$x_1 \geq 0, x_2 \geq 0, \dots, x_{n+1} \geq 0\}$$

the condition (6) is equivalent to the following: there exist $(c_1, c_2, \dots, c_l) \in [c_1^-, c_1^+] \times [c_2^-, c_2^+] \times \cdots \times [c_l^-, c_l^+]$ and $x_1 \geq 0$, $x_2 \geq 0$, … , $x_{n+1} \geq 0$, $x_1 + x_2 + \cdots + x_{n+1} = 1$ such that

$$x_1 V^1 + x_2 V^2 + \cdots + x_{n+1} V^{n+1} = c_1 p^1 + c_2 p^2 + \cdots + c_l p^l + p^0. \tag{7}$$

By introducing new variables, without loss of generality, we can assume that $c_1^- = c_2^- = \cdots = c_l^- = 0$ that is $\mathcal{C} = [0, c_1^+] \times [0, c_2^+] \times \cdots \times [0, c_l^+]$.

Define the following variables:

$$\begin{array}{llll} x_{n+2} = c_1, & x_{n+3} = c_2, & \cdots, & x_{n+l+1} = c_l, \\ x_{n+l+2} = c_1^+ - c_1, & x_{n+l+3} = c_2^+ - c_2, & \cdots, & x_{n+2l+1} = c_l^+ - c_l. \end{array}$$

The equality (7) is equivalent to the feasibility of the following LP problem:

$$\begin{bmatrix} V^1 & V^2 & \cdots & V^{n+1} & -p^1 & -p^2 & \cdots & -p^l & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+l+1} \\ x_{n+l+2} \\ \vdots \\ x_{n+2l+1} \end{bmatrix} = \begin{bmatrix} p^0 \\ 1 \\ c_1^+ \\ c_2^+ \\ \vdots \\ c_l^+ \end{bmatrix} \tag{8}$$

$$x_i \geq 0 \quad (i = 1,2, \dots, n + 2l + 1).$$

In the above LP feasibility problem $Ax = b$, $x \geq 0$, the matrix A is $(n + l + 1) \times (n + 2l + 1)$ dimensional and the variable x is $(n + 2l + 1)$ dimensional.

If the feasibility set of (8) is empty then there is no stabilizing vector (see Example 1). If the feasibility set is nonempty the solution set gives several inequalities from which a stabilizing vector can be determined easily by inspection (see Example 2).

Example 1 Consider the plant

$$G(s)=\frac{-0.7s^2-0.2s-2.5}{s^5-1.4s^4+s^3+2.4s^2+1.2s-1.9}$$

and controller $R(s,c)=\frac{c_1+c_2s+c_3s^2}{s}$ with $c_1\in[0{,}5]$, $c_2\in[0{,}5]$, $c_3\in[0{,}5]$.

The closed loop characteristic polynomial is

$$\begin{aligned}p(s,c)=s^6-1.4s^5+2.4s^4+5.25s^3+5.99s^2+6.59s-3.25\\ +c_1(-0.7s^2-0.2s-2.5)+c_2(-0.7s^3-0.2s^2-2.5s)\\ +c_3(-0.7s^4-0.2s^3-2.5s^2)\end{aligned}$$

Then $p^0=(-3.25,6.59,5.99,5.25,2.4,-1.4)^T$, $p^1=(-2.50,-0.20,-0.70,0,0,0)^T$, $p^2=(0,-2.50,-0.20,-0.70,0,0)^T$, $p^3=(0,0,-2.50,-0.20,-0.7,0)^T$. For $n=6$, the vertices $V^1,V^2,\dots,V^7$ are

$V^1=(1,6,15,20,15,6)^T$, $V^2=(-1,-4,-5,0,5,4)^T$, $V^3=(1,2,-1,-4,-1,2)^T$,
$V^4=(-1,0,3,0,-3,0)^T$, $V^5=(1,-2,-1,4,-1,-2)^T$, $V^6=(-1,4,-5,0,5,-4)^T$,
$V^7=(1,-6,15,-20,15,-6)^T$

The corresponding LP problem (8) is

$$\begin{bmatrix}1&-1&1&-1&1&-1&1&2.5&0&0&0&0&0\\6&-4&2&0&-2&4&-6&0.2&2.5&0&0&0&0\\15&-5&-1&3&-1&-5&15&0.6&0.2&2.5&0&0&0\\20&0&-4&0&4&0&-20&0&0.6&0.2&0&0&0\\15&5&-1&-3&-1&5&15&0&0&0.6&0&0&0\\6&4&2&0&-2&-4&-6&0&0&0&0&0&0\\1&1&1&1&1&1&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0&0&1&0\\0&0&0&0&0&0&0&0&0&1&0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\\vdots\\x_{13}\end{bmatrix}=\begin{bmatrix}-3.25\\6.59\\5.99\\5.25\\2.4\\-1.4\\1\\5.0\\5.0\\5\end{bmatrix}$$

$$x_i\geq 0\quad (i=1,2,\dots,13).$$

The feasible set of this LP is empty, that is, $\mathcal{P}\cap\operatorname{co}\{V^1,V^2,\dots,V^7\}=\emptyset$ and there is no stabilizing parameter.

Example 2 Let the family (2) be given as

$$\begin{aligned}p(s,c)=s^5-2.9s^4+5.84s^3-3.064s^2+0.8783s-2.28035+c_1(-s^3+1)\\ +c_2(0.5s^2+s)\end{aligned}$$

and $c_1\in[0{,}2]$, $c_2\in[0{,}1]$.

For $n=5$, the vertices $V^1,V^2,\dots,V^6$ are

$V^1=(1,5,10,10,5)^T$, $V^2=(-1,-3,-2,2,3)^T$, $V^3=(1,1,-2,-2,1)^T$,
$V^4=(-1,1,2,-2,-1)^T$, $V^5=(1,-3,2,2,-3)^T$, $V^6=(-1,-5,-10,10,-5)^T$.

The corresponding LP problem (8) is

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 0 & 0 \\ 5 & -3 & 1 & 1 & -3 & 5 & 0 & 0 & -1 & 1 \\ 10 & -2 & -2 & 2 & 2 & -10 & 0 & 0 & -0.5 & 0.5 \\ 10 & 2 & -2 & -2 & 2 & 10 & 1 & -1 & 0 & 0 \\ 5 & 3 & 1 & -1 & -3 & -5 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} = \begin{bmatrix} -2.28035 \\ 0.8783 \\ -3.064 \\ 5.84 \\ -2.9 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$x_i \geq 0 \quad (i = 1,2,\ldots,10).$$

The solution set of this LP consists of the union of 11 polyhedral sets. For example, the first solution set has the following form:

$$\begin{array}{ll} 0 \leq x_1 \leq 0.009, & 0 \leq x_2 \leq 0.033, \\ x_3 = 0.107 - 3.999\,x_1 - 1.999\,x_2, & x_4 = 0.206 + 5.111\,x_1 + 1.333\,x_2, \\ x_5 = 0.314 - 3.222\,x_1 - 0.666\,x_2, & x_6 = 0.371 + 1.111\,x_1 + 0.333\,x_2, \\ x_7 = 2.123 - 12.444\,x_1 - 5.333\,x_2, & x_8 = 0.3507 + 21.3333\,x_1, \\ x_9 = -0.123 + 12.444\,x_1 + 5.333\,x_2, & x_{10} = 0.649 - 21.333\,x_1. \end{array}$$

Take the following point from this set

$$(x_1, x_2, \ldots, x_{10})^T = (0.002, 0.01, 0.079012, 0.229552, 0.300896, 0.376552, 2.044782, 0.3933666, -0.044782, 0.606334)^T.$$

Recall that $c_1 = x_7$, $c_2 = x_8$. The values $c_1^0 = x_7 = 2.044782$, $c_2^0 = x_8 = 0.3933666$ are stabilizing, since the polynomial

$$p(s, c^0) = s^5 - 2.9s^4 + 3.795218s^3 - 2.86731670s^2 + 1.2716666s - 0.235568$$

is stable.

3. LEAST SQUARE MINIMIZATION

The condition (6) is only a necessary condition for the existence of a stabilizing parameter. On the other hand we can give corresponding example where (6) is satisfied but there is no stabilizing parameter (see Example 3). This shows that the condition (6) is only a necessary condition for the existence. In this section, we consider the Euclidean distance between the sets $\mathcal{P}$ and $\mathcal{D}$. Define

$$\alpha = \min_{c\in\mathcal{C},\, k\in[-1,1]^n} \|Ac + p^0 - f(k)\|,$$

where $c = (c_1, c_2, \ldots, c_l)^T$, $\mathcal{C} = [0, c_1^+] \times [0, c_2^+] \times \cdots \times [0, c_l^+]$, $k = (k_1, k_2, \ldots, k_n)^T$, $[-1,1]^n = [-1,1] \times [-1,1] \times \cdots \times [-1,1]$, $f(k) = (f_1(k), f_2(k), \ldots, f_n(k))^T$ and f is the reflection map, k is the reflection vector.

The condition $\alpha = 0$ is "almost" a necessary and sufficient condition for the existence of a stabilizing parameter.

Proposition 2 If $\alpha > 0$ then there is no a stabilizing parameter. If $\alpha = 0$ then there exists a stabilizing parameter or there exists $c \in \mathcal{C}$ such that the corresponding polynomial $p(s, c)$ has all roots belonging to the closed unit disc.

In what follows the minimizations have been made by using Maple minimization commands.

Example 3 Consider the family

$$p(s,c) = s^5 - 1.4s^4 + 0.4s^3 + 4.12s^2 - 1.3s - 1.5 + c_1(0.7s^3 - 0.3s^2 + 0.6) + c_2(0.5s^2 + s + 0.1) + c_3(s^3 - 0.3s + 0.2),$$

$c_1 \in [0,5]$, $c_2 \in [0,1]$, $c_3 \in [0,1]$.

For this example, the feasibility set of LP problem (8) is nonempty. Consider the following minimization problem

$$F(c,k) = \|Ac + p^0 - f(k)\|^2 \quad \rightarrow \quad \min \tag{9}$$

where $c = (c_1, c_2, c_3)^T \in [0,5] \times [0,1] \times [0,1]$, $k = (k_1, k_2, \dots, k_5) \in [-1,1]^5$.
The optimal value of (9) is positive (≈ 4.999). Therefore, there is no stabilizing parameter.

Example 4 Consider the family

$$p(s,c) = s^5 - 3.2s^4 + 4.24s^3 - 7.758s^2 - 2.3699s + 1.1872 + c_1(0.3s^2 + 1.1s + 0.7) + c_2(1.5s^2 + 2s - 0.6) + c_3(0.5s^3 - 2.4s - 2.2),$$

$c_1 \in [0,3]$, $c_2 \in [0,3]$, $c_3 \in [0,4]$.
The corresponding LP problem (8) has nonempty feasibility set. The minimization problem

$$F(c,k) = \|Ac + p^0 - f(k)\|^2 \quad \rightarrow \quad \min$$

has optimal value zero ($\approx 2.3\ 10^{-8}$). The optimal point is the following

$c_1 = 1.94598672835119$, $c_2 = 2.12919162384265$, $c_3 = 0.816115998634540$,
$k_1 = 0.954034132402279$, $k_2 = -0.891166832620269$, $k_3 = 0.773998874801903$,
$k_4 = -0.544090555311313$, $k_5 = 0.523579479105046$.

For $c_1 = 1.94598672835119$, $c_2 = 2.12919162384265$ the corresponding polynomial is

$$p(s,c) = s^5 - 3.2s^4 + 4.648057999s^3 - 3.980416546s^2 + 2.070390252s - 0.5235794617$$

and its roots are

$$\begin{aligned}
z_1 &= 0.9309235937, & |z_1| &= 0.9309235937,\\
z_2 &= 0.8235418792 + 0.4120655366\,i, & |z_2| &= 0.9208795976,\\
z_3 &= 0.8235418792 - 0.4120655366\,i, & |z_3| &= 0.9208795976,\\
z_4 &= 0.3109963240 + 0.7526681217\,i, & |z_4| &= 0.814388123,\\
z_5 &= 0.3109963240 - 0.7526681217\,i, & |z_5| &= 0.814388123.
\end{aligned}$$

Therefore the polynomial $p(s,c)$ is stable.

4. CONCLUSION

In this paper, we consider Schur stabilization problem for discrete single input, single output systems. It is assumed that the control parameter is bounded and control function is affine linear with respect to the control parameter. Two approaches are considered: Linear programming and least square minimization.

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