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ABOUT PERIODIC STATES OF SYSTEMS OF MASSSERVICE WITH PERIODIC PARAMETERS

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ABSTRACT

In this article periodic solution from the theory of quenes is considered For the M/M/n/m queue when the Poisson processes are time homogeneous is given and for arrivals the service time distribution exponential. Arrrivals follow a Poisson distribution with an arrival rate $\lambda\ (t)$ that varies with time t and service times are exponential with a departure rate $\mu\ (t)$. The queue in discrete points is considered. The distance between discrete points equally period of $\lambda\ (t)$ and $\mu\ (t)$. The queue in discrete points is formed a Markov chain. This Markov chain aperiodical. This aperiodical Markov chain have asymptotical stationary solution. If $\lambda\ (t)$ and $\mu\ (t)$ continuous density and $\lambda\ (t)= \lambda\ \ (t+T)$, $\mu\ (t) = \mu\ (t+T)$. The transition probabilities $P_{ij}\ (t)$ satisfy an equation of Kolmogorov.

Keywords: Poisson Distribution, Markov Chain, Asymptotical Stationary Solution, Probability, Exponential Distribution.

PERİYODİK PARAMETRELER İLE KİTLE HİZMET SİSTEMLERİNİN PERİYODİK DURUMLARI HAKKINDA.

ÖZET

Bu makalede, quenes teorisinden periyodik çözüm ele alınmaktadır. M/M/n/m kuyruğu için Poisson işlemleri zaman homojen olduğunda verilen ve gelişler için servis süresi dağılımı üstel. Varışlar, t süresine göre değişen bir varış oranı λ (t) ile bir Poisson dağılımını takip eder. ve hizmet süreleri μ (t) kalkış oranıyla üsteldir. Sıra ayrık noktalar dikkate alınır. Ayrık noktalar arasındaki mesafe eşit olarak λ (t) ve μ (t) periyodu. Ayrık noktalardaki sıra bir Markov oluşturur zincir. Bu Markov zinciri periyodik değil. Bu periyodik olmayan Markov zinciri, asimptotik durağan çözüm. λ(t) ve μ(t) sürekli yoğunluk ve λ (t)= λ (t+T) , μ (t) = μ (t+T). Geçiş olasılıkları Pij (t), bir Kolmogorov denklemini karşılar.

Anahtar Kelimeler: Poisson Dağılımı, Markov Zinciri, Asimptotik Durağan Çözüm, Olasılık, Üstel Dağılım.

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In [1], using the methods of the qualitative theory of ordinary differential equations, the existence of periodic solutions for the queuing system in the case of periodicity of intensities was proved. Further, more general problems on the spectral approach [2], [3], [4] are considered. A large range of tasks in this direction is set in [5].

Consider a lossy queuing system consisting of n identical servers and m waiting places.

Devices receive a non-stationary Poisson call flow with a periodic leading function

$L(t_1)-L(t_2)=L(t_1+T)-L(t_2+T)$ (1)

Requirements are served by each device according to a quasi-exponential law:

P(service started at t_1 ends after t_2)=$e^{M(t_2)-M(t_1)}$

with periodic characteristic

$M(t_1)-M(t_2)=M(t_1+T)-M(t_2+T)$ (2)

Let P_0 , P_1 ,... P_n , P_{n+1} , ... P_{n+m} - e the probabilities that at the initial moment there is 0,1, ... n, n+1, ... n+m requirements ,$P_0(t)$, $P_1(t)$,… $P_n(t)$,… . $P_{n+m}(t)$, - probabilities that at time t there is 0 in the system; is 0; 1; ... n; n+1, ... n+m requirements [6].

Denote by Pij (t) probability Pj (t) given that Pk $=\delta_{ki}$. Then Pj (t) will be expressed as follows

$P_j(t)=\sum_i P_i P_{ij}(t)$

We also put $P_{ij}=P_{ij}(T)$ (3)

Theorem : There are $\bar{P}_j(t)$ (j= 0, 1, ... n+m), that

$$\bar{P}_j(t) = \bar{P}_j(t+T)$$

And

$$\lim_{t\to\infty}\left|P_{ij}(t) - \bar{P}_{ij}t\right| = 0$$

Proof. Consider the position of the system at successive times 0, T, 2T, ... KT, ...

From (1) and (2) it follows

$P_j(kT) = \sum_i P_i\{(k-1)T\} P_{ij}$

So the positions of the system at successive times 0, T, 2T, ... KT, ...
form a Markov chain with a finite number of states. Let us show that this chain is irreducible and non-periodic.

Indeed, for any i, j we have

1) $j > i$

$P_{ij} = \sum_{k=0}^{\infty} P$ (during time T, with an initial number of requests i in the system, exactly k services occurred and exactly j=i+k requests were received)> 0

2) $i \geq j$

$P_{ij} = \sum_{k=0}^{\infty} P($ during time T with the initial number of customers i in the system, exactly k requests and there were exactly i-j+k services) $\geq$ P(during the time T with the initial number of customers i, no customers have arrived in the system and exactly j-i services have occurred)>0

An irreducible non-periodic Markov chain with a finite number of states has asymptotic stationary distribution $\bar{P}_0, \bar{P}_1, \ldots \bar{P}_n, \bar{P}_{n+1}, \ldots \bar{P}_{n+m}$, independent of the initial one, and $\bar{P}_0, \bar{P}_1, \ldots \bar{P}_n, \bar{P}_{n+1}, \ldots \bar{P}_{n+m}$, are uniquely determined from the system of equations

$$\bar{P}_j = \sum_i \bar{P}_i P_{ij} \qquad (4)$$

Let's put now

$$\bar{P}_j = \sum_i \bar{P}_i P_{ij}(t)$$

From what has been proved, it follows that

$$\bar{P}_j(kT) = \bar{P}_j$$

$$P_j(kT) \to \bar{P}_j .$$

And

$$\bar{P}_j(t) = \sum_i \bar{P}_i([t]_T) P_{ij}(t - [t]_T) =$$

$$\sum_i \bar{P}_i P_{ij}(t - [t]_T),$$

$$P_j(t) = \sum_i P_i([t]_T) P_{ij}(t - [t]_T).$$

We have

$$\bar{P}_j(t) = \bar{P}_j(t + T),$$

$$\lim_{t \to \infty} \left| P_j(t) - \bar{P}_j t \right| = 0$$

Let L (t) и M (t) have continuous densities $\lambda(t)$ и $\mu(t)$ accordingly.

Then the following relations hold for the densities

$$\lambda\ (t) = \lambda\ (t+T)$$

$$\mu\ (t) = \mu\ (t+T)$$

For $P_{ij}\ (t)$ the system of differential equations is valid [7]

$$\frac{dP_{io}}{dt} = -\lambda P_{i0} + \mu P_{i1},$$

$$\frac{dP_{i1}}{dt} = \lambda P_{i0} - (\lambda+\mu)P_{i1} + 2\mu P_{i2},$$

$$\frac{dP_{in-1}}{dt} = \lambda P_{in-2} - (\lambda+(n-1)\mu)P_{in-1} + n\mu P_{in},$$

$$\frac{dP_{in}}{dt} = \lambda P_{in-1} - (\lambda+n\mu)P_{in} + n\mu P_{in+1},$$

$$\frac{dP_{in+1}}{dt} = \lambda P_{in} - (\lambda+n\mu)P_{in+1} + n\mu P_{in+2},$$

$$\frac{dP_{in+m-1}}{dt} = \lambda P_{in+m-2} - (\lambda+n\mu)P_{in+m-1} + n\mu P_{in+m},$$

$$\frac{dP_{in+m}}{dt} = \lambda P_{in+m-1} - n\mu P_{in+m},$$

under initial conditions

$$P_{ij}(0) = \delta_{ij}$$

Consider the case n=1, m=0. The system of differential equations for this case takes the form

$$\frac{d\,P_{io}}{dt} = -\lambda P_{io} + vP_{i1}$$

$$\frac{d\,P_{i1}}{dt} = \lambda P_{io} - vP_{i1}$$

To solve, we use the fact that

$$P_{i0} + P_{i1} = 1$$

Then

$$\frac{d\,P_{io}}{dt} = (-\lambda + v)P_{i0} + v$$

solving this equation, we get

$$P_{00}(t) = \left\{1 + \int_0^t v\ exp\langle\int_0^x(\lambda+\mathrm{v})\,dy\rangle dx\right\}\exp\left\{-\int_0^t(\lambda+\mathrm{v})dx\right\},$$

$$P_{01}(t) = 1 - \left\{1 + \int_0^t v\ exp\langle\int_0^x(\lambda+\mathrm{v})\,dy\rangle dx\right\}\exp\left\{-\int_0^t(\lambda+\mathrm{v})dx\right\},$$

$$P_{10}(t) = \left\{\int_0^t v\ exp\langle\int_0^x(\lambda+\mathrm{v})\,dy\rangle dx\right\}\exp\left\{-\int_0^t(\lambda+\mathrm{v})dx\right\},$$

$$P_{11}(t) = 1 - \left\{\int_0^t v\ exp\langle\int_0^x(\lambda+\mathrm{v})\,dy\rangle dx\right\}\exp\left\{-\int_0^t(\lambda+\mathrm{v})dx\right\}.$$

We introduce the notation

$$P_{00}(T) = q_0$$
$$P_{11}(T) = q_1$$

Then the transition matrix of the corresponding Markov chain will have the form

$$(P_{ij}) = \begin{pmatrix} q_0 & 1 - q_1 \\ 1 - q_0 & q_1 \end{pmatrix}$$

Let us write down the system of equations for determining $\overline{\mathrm{P}_J}$

$$\overline{\mathrm{P}_0} = q_0\overline{\mathrm{P}_0} + (1 - q_0)\overline{\mathrm{P}_0}\ ,$$
$$\overline{\mathrm{P}_1} = (1 - q_0)\overline{\mathrm{P}_0} + q_1\overline{\mathrm{P}_1}$$

Solving this system, we get

$$\overline{\mathrm{P}_0} = \frac{1 - q_1}{2 - q_0 - q_1}$$
$$\overline{\mathrm{P}_1} = \frac{1 - q_1}{2 - q_0 - q_1}$$

The probabilities of a periodic stationary state are given by the formulas

$$\overline{\mathrm{P}_0}(t) = \overline{\mathrm{P}_0}P_{00}(t) + \overline{\mathrm{P}_1}P_{10}(t)$$
$$\overline{\mathrm{P}_1}(t) = \overline{\mathrm{P}_0}P_{01}(t) + \overline{\mathrm{P}_1}P_{11}(t)$$

Consider a queuing system consisting of n identical service devices. The devices receive a non-stationary Poisson flow of demands with a periodic leading function (1). The demands are served by each device according to a quasi-exponential law with a periodic characteristic (2).
Let $\mathrm{P}_0(t), \mathrm{P}_1(t), \ldots, \mathrm{P}_k(t), \ldots$ the probabilities that at time t there are Q, 1, …, k, … requirements in the system, respectively.

We introduce the notation

$$L \quad = L_1(T) - L_1(O)$$
$$V = n[V(T) - V(O)]$$

Theorem

If $L \quad < V$, then there are $\overline{\mathrm{P}_\kappa}(t)$ (к=0,1,...) ,
$\sum_\kappa \overline{\mathrm{P}_\kappa}(t) = 1$; $\overline{\mathrm{P}_\kappa}(t) = \overline{\mathrm{P}_\kappa}(t + T)$

$$\lim_{t\to\infty} |P_k(t) - \overline{\mathrm{P}_\kappa}(t)| = 0$$

The proof requires a lemma. Lemma

Let an irreducible non-periodic Markov chain Z_n, accepting values $0, 1, \dots, к, \dots$ satisfies the following conditions:

1. 1. For all k, there is an expectation Z_n under condition $Z_{n-1} = к$, т. е.

$$\mathrm{M}\left({}^{Z_n}/_{Z_{n-1}} = k\right) = \sum_j j\, P_{kj} < \infty$$

2. For $k > K$ this mathematical expectation satisfies the requirement

$$\mathrm{M}\left({}^{Z_n}/_{Z_{n-1}} = к\right) \leq K - \varepsilon,$$

where $\varepsilon > 0$- arbitrary constant.

Then the Markov chain has a stationary distribution.

Proof of the lemma

It is known that for an irreducible non-periodic Markov chain with $n \to \infty$ there can be only two cases:

a. $Z_n \xrightarrow{P} \infty$, т.е. $P_k^{(n)} \to \infty$, for all k:
b. There is a unique stationary distribution and the distribution Z_n сходится к нconverges to it regardless of the initial distribution Z_0

Let us show that if $\mathrm{M}(Z_0) < \infty$, then for any n$\mathrm{M}(Z_0) < \infty$.

It is enough to show this for Z_1.

Denote

$$\mathrm{M} = \sup_{0 \leq k \leq K} M\left({}^{Z_n}/_{Z_0} = к\right)$$

then

$$M(Z_1) = \sum_j j\, P_j^{(1)} = \sum_j j \sum_k P_j^{(0)} P_{kj} = \sum_k P_k^{(0)} \sum_j j\, P_{kj} = \sum_k P_k^{(0)} P_{kj}\, \mathrm{M}\left({}^{Z_1}/_{Z_0} = к\right)$$

$$= \sum_{k \leq K} P_k^{(0)} \mathrm{M}\left({}^{Z_1}/_{Z_0} = к\right) + \sum_{k > K} P_k^{(0)} \mathrm{M}\left({}^{Z_1}/_{Z_0} = к\right) \leq M + M(Z_0) < \infty$$

Suppose now that for all k $P_k^{(n)} {}_{n\to\infty} \to 0$ then

$$\sup_{0 \leq k \leq K} P_k^{(n)} \underset{n\to\infty}{\to} 0$$

Therefore, under this assumption, for any initial distribution

$\mathrm{M}(Z_n) \underset{n\to\infty}{\to} \infty$. Пусть $\mathrm{M}(Z_0) < \infty$.

We choose N that , which n$> N$

$$K \sup_{0\le k\le K} P_k^{(n)} < \frac{1}{2}$$

$$MK \sup_{0\le k\le K} P_k^{(n)} < \frac{\varepsilon}{2}$$

Then

$\mathrm{M}(Z_n) = \sum_k P_k^{(n-1)} M\left({Z_n}/{Z_{n-1}} = \text{к}\right) + \sum_{k>K} P_k^{(n-1)} \mathrm{M}\left({Z_n}/{Z_{n-1}} = \text{к}\right) \le \frac{\varepsilon}{2} + \sum_k P_k^{(n-1)}\ k - \frac{\varepsilon}{2} \le M(Z_{n-1})$

The lemma is proven.

Proof of Theorem 2.

We will consider the state of the system at times 0, T, 2T, ... By virtue of (1) and (2), the system at these moments of time forms a Markov chain. In the proof of Theorem 1, it was shown that this chain is irreducible and non-periodic. Let us show that the chain satisfies the conditions of the lemma.

1. (during the time T tz the state k passed into the state j) P (during the time T at least j-krequirements were received) = K+

2. We choose from the relations

Let's put

Let us choose the satisfying relations

Then at

We choose M so that

Then for

Let's put

Consider now with

a) Denote i=j-k,

b)

c) Denote i=k-j,

Let us now show that, for k, the conditional expectation is bounded.

Thus, the conditions of the lemma are satisfied for the Markov chain and, therefore, it has a stationary distribution that does not depend on the initial

Let be the probability that the original queuing system is in state at time t, provided that at time 0 it was in state k

Let's put

Then

Note that if and have continuous densities and , then they satisfy an infinite system of differential equations with initial conditions

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