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Variants of the New Caristi Theorem

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Abstract

The well-known Caristi fixed point theorem has numerous generalizations and modifications. Recently there have appeared its equivalent dual forms and generalizations based on new concept of lower semicontinuity from above by several authors. In the present article, we give new proofs of such new versions and their equivalent formulations by applying our Metatheorem in the ordered fixed point theory.

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1. Introduction

The Banach contraction principle has numerous generalizations or modifications for various types of spaces with new metrics or topologies and new contractive conditions. Similarly, since the appearances of the Ekeland variational principle [8-10] and the Caristi fixed point theorem [4], nearly one thousand works followed on their modifications, equivalents, generalizations, applications, and related topics. Many of them are concerned with new spaces extending complete metric spaces, new metrics or topologies on them, and new order relations extending the so-called Caristi order.

While the author was working on such subjects in 1985-2000, in order to give some equivalents of the Ekeland principle, we obtained a Metatheorem in [18-23] on fixed point theorems related to the order theory. It claims that certain order theoretic maximal element statements are equivalent to theorems on fixed points, stationary points, common fixed points, common stationary points of families of maps or multimaps. As usual in the mathematical community, our Metatheorem was not attracted for a long period.

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Recently in 2022, we came back to our Metatheorem after 22 years have passed and obtained its extended versions in [24-27] with a large number of new results. The most recent version of them is a modification of the 2023 Metatheorem in [28]. This article is based on the modification given in [32]. It is applied to the traditional order theoretic results and, consequently, to the so-called Ordered Fixed Point Theory in [28]. This can be comparable to traditional several fields in the fixed point theory, that is, Analytical, Metric, and Topological fixed point theories.

Recently there have appeared two types of generalizations of the original Caristi fixed point theorem on complete metric spaces.

The first kind is to replace the concept of lower semicontinuity of functions in the Caristi theorem by a little general concept of the lower semicontinuity *from above*. This concept was derived by Kirk-Saliga [15] in 2001 and Chen-Cho-Yang [5] in 2002 and followed by Lin-Du [16] in 2007 and Ansari [1] in 2014.

The second kind is the dual versions of the Caristi theorem adopting upper semicontinuity instead of lower semicontinuity of the real-valued function defining the Caristi order. Such results were appeared in [1, 5, 16].

In the present article, our modified 2023 Metatheorem in [32] is applied to obtain some equivalent formulations of the results in [5] and [16] for more useful applications.

Section 2 is to introduce the so-called Extreme (Maximal or Minimal) Element Principle based on our new 2023 Metatheorem and its particular forms. In Section 3, we introduce our strengthened versions of the Caristi theorem as a basis of our study in this article. Section 4 devotes to improve and strengthen the results of Chen-Cho-Yang [5] with new proofs. In Section 5. we obtain equivalent formulations of the result of Lin-Du [16]. Section 6 is to recall some history related to the equivalences of metric completeness. Finally, Section 7 devotes to epilogue.

2. Extreme Element Principles

In order to deduce some equivalents of the well-known central result of Ekeland [8-10] on the variational principle for approximate solutions of problems, we obtained a metatheorem in [17–23]. Later we found more additional conditions and, consequently, we obtained extended versions of the metatheorem in 2022 [24–27]. Finally we obtained the 2023 Metatheorem in [28] with a large number of applications. Such Metatheorem consists of several logically equivalent statements and guarantees the truth of all items when so is one of them. Since 1985, we have shown nearly one hundred cases of such situation. In the present article, we assume a certain modified form of the 2023 Metatheorem in [32].

Let $(X, \preceq)$ be a preordered set; that is, X is a nonempty set and the order $\preceq$ is reflexive and transitive. The partial order $\preceq$ is the one having additional anti-symmetry.

A maximal or minimal element will be called an extreme element. From our new 2023 Metatheorem in [32], we deduce the following prototype of Extreme Element Principles as in [29] for multimaps having *nonempty values*:

Theorem A. *Let $(X, \preceq)$ be a preordered set and A be a nonempty subset of X. Then the following statements are equivalent:*

(α) *There exists a maximal (resp. minimal) element $v \in A$, that is, $v \not\preceq w$ (resp. $w \not\preceq v$) for any $w \in X\backslash\{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ such that, for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$), then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $x \preceq f(x)$ (resp. $f(x) \preceq x$) for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(δ) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for any $x \in A\backslash F(x)$ there exists $y \in X\backslash\{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$). Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in F(v)$ for all $F \in \mathfrak{F}$.*

(ϵ) *If* $\mathfrak{F}$ *is a family of multimaps* $F : A \multimap X$ *such that* $x \preceq y$ *(resp.* $y \preceq x$*) holds for any* $x \in A$ *and any* $y \in F(x)\backslash\{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in A$*, that is,* $\{v\} = F(v)$ *for all* $F \in \mathfrak{F}$*.*

(η) *If* Y *is a subset of* X *such that for each* $x \in A\backslash Y$ *there exists a* $z \in X\backslash\{x\}$ *satisfying* $x \preceq z$ *(resp.* $z \preceq x$*), then there exists a* $v \in A \cap Y$*.*

Remark. (1) Note that we claimed that $(\alpha) - (\eta)$ are equivalent in Theorem A and did not say that they are true. For a counter-example, consider the real line $\mathbb{R}$ with the usual order. However, we gave many examples that they are true based on the original sources; see the articles mentioned in our [28].

(2) All the elements v's in Theorem A are same as we have seen in the proof of Metatheorem in [28], and (η) simply tells that Y is nonempty when $X = A$ and the location of the common point v.

(3) When $\mathfrak{F}$ is a singleton, each of $(\beta) - (\epsilon)$ is is denoted $(\beta 1) - (\epsilon 1)$, respectively. These are also equivalent to $(\alpha) - (\eta)$ in Theorem A.

Let $(X, \preceq)$ be a preordered set and $F : X \multimap X$ a multimap. For every $x \in X$, we denote

$$S_+F(x) := \{z \in X : u \preceq z \text{ for some } u \in F(x)\},$$

$$(\textit{resp. } (S_-F(x) := \{z \in X : z \preceq u \text{ for some } u \in F(x)\}).$$

From Theorem A, we have several variants in [28] as follows:

Theorem A1. *Let* $(X, \preceq)$ *be a partially ordered set,* $F : X \multimap X$ *be a multimap, and* $x_0 \in X$ *such that* $A = (S_+F(x_0), \preceq)$ *(resp.* $A = (S_-F(x_0), \preceq)$*) has an upper bound (resp. a lower bound)* $v \in A$*.*

Then the equivalent statements $(\alpha) - (\eta)$ *of Theorem A hold.*

For the identity map $F = 1_X$, let

$$S_+(x) := \{y \in X : x \preceq y\} \quad (\textit{resp. } S_-(x) := \{y \in X : y \preceq x\}).$$

Then Theorem A1 reduces to the following:

Theorem A2. *Let* $(X, \preceq)$ *be a partially ordered set, and* $x_0 \in X$ *such that* $A = (S_+(x_0), \preceq)$ *(resp.* $A = (S_-(x_0), \preceq)$ *has an upper bound (resp. a lower)* $v \in A$*. Then the equivalent statements* $(\alpha) - (\eta)$ *of Theorem A1 hold.*

Let $(X, \preceq)$ be a partially ordered set and $G(x, y)$ mean $y \preccurlyeq x$ (resp. $x \preccurlyeq y$) in Metatheorem* in [32]. Then we have the following:

Theorem A.2.* *Let* $(X, \preceq)$ *be a partially ordered set,* $x_0 \in X$*, and* $A = S_+(x_0)$ *(resp.* $A = S_-(x_0)$*) have an upper bound (resp. a lower bound)* $v \in A$*. Then the following equivalent statements hold:*

(α) $v \in A$ *is a maximal (resp. minimal) element, that is,* $v \npreceq w$ *(resp.* $w \npreceq v$*) for any* $w \in X\backslash\{v\}$*.*

($\theta 1$) $v \in A$ *satisfies that, for each chain* C *in* $S_+(v)$ *(resp.* $S_-(v)$*) , we have* $\bigcap_{x\in C} S_+(x) \neq \emptyset$ *(resp.* $\bigcap_{x\in C} S_-(x) \neq \emptyset$*).*

($\theta 2$) $v \in A$ *satisfies that, for a maximal chain* C^* *in* $S_+(v)$ *(resp.* $S_-(v)$*), we have* $\bigcap_{x\in C^*} S_+(x) \neq \emptyset$ *(resp.* $\bigcap_{x\in C^*} S_-(x) \neq \emptyset$*).*

For the motivation of this theorem and its proof, we have a long story as shown in [33]. The conditions $(\theta 1)$ and $(\theta 2)$ are originated from [11] and Theorem A2* extends a part of ([2], Theorem 5.1). See also ([33], Theorem 5.1*).

For multimaps permitting *empty values*, we derive the following Empty Element Principle from the old 2023 Metatheorem:

Theorem A*. *Let* $(X, \preceq)$ *be a preordered set and* A *be a nonempty subset of* X*. Then the following statements are equivalent:*

(α) *There exists a maximal (resp. minimal) element* $v \in A$*, that is,* $v \not\preceq w$ *(resp.* $w \not\preceq v$*) for any* $w \in X \backslash \{v\}$.

($\zeta 1$) *If a multimap* $F : A \multimap X$ *such that, for all* $x \in A$ *with* $F(x) \neq \emptyset$*, there exists* $y \in X \backslash \{x\}$ *satisfying* $x \preceq y$ *(resp.* $y \preceq x$*), then there exists* $v \in A$ *such that* $F(v) = \emptyset$.

($\zeta 2$) *Let* $\mathfrak{F}$ *be a family of multimaps* $F : A \multimap X$ *such that, for all* $x \in A$ *with* $F(x) \neq \emptyset$*, there exists* $y \in X \backslash \{x\}$ *satisfying* $x \preceq y$ *(resp.* $y \preceq x$*). Then there exists* $v \in A$ *such that* $F(v) = \emptyset$ *for all* $F \in \mathfrak{F}$.

Proof. Note that $(\zeta 2) \Longrightarrow (\zeta 1)$ is clear.

$(\alpha) \Longrightarrow (\zeta 2)$: By (α) there exists $v \in A$ such that $v \not\preceq w$ (resp. $w \not\preceq v$) holds for all $w \in X \backslash \{v\}$. Suppose to the contrary, there exists $F \in \mathfrak{F}$ such that $F(v) \neq \emptyset$. By hypothesis, there exists $w \in X$ with $w \neq v$ and $v \preceq w$ (resp. $w \preceq v$) holds. Therefore it leads a contradiction and $F(v) = \emptyset$ for all $F \in \mathfrak{F}$.

$(\zeta 1) \Longrightarrow (\alpha)$: Suppose that, for each $x \in A$, there exists $y \in X \backslash \{x\}$ such that $x \preceq y$ (resp. $y \preceq x$) holds. Let us define a multimap $F : A \multimap X$ by

$$F(x) = \{y \in X : x \preceq y\} \neq \emptyset \quad (\text{resp. } F(x) = \{y \in X : y \preceq x\} \neq \emptyset.)$$

for all $x \in A$. Then, by ($\zeta 1$), there exists $v \in A$ such that $F(v) = \emptyset$. This is a contradiction. □

A function $\varphi : X \to \mathbb{R}$ is said to be lower semicontinuous at $x \in X$ if for any sequence $\{x_n\} \subset X$,

$$\lim_{n\to\infty} x_n = x \in X \implies \varphi(x) \leq \lim_{n\to\infty} \inf \varphi(x_n).$$

Similarly, upper semicontinuity can be defined.

From now on, our key results in this article will be denoted Theorems B, C, D, $\cdots$.

3. Dual Forms of the Caristi Theorem

In our previous article [30], we obtained various forms of Minimal Element Principles from Theorem A and applied them to known or new works related to minimality. We were based on our 2023 Metatheorem in [28], a prototype of Minimal Element Principles, and the Brøndsted-Jachymski Principle for the minimality. These are dual versions of the corresponding ones for the maximality.

From now on Max($\preceq$) (resp. Min($\preceq$)) denotes the set of maximal (resp. minimal) elements of the order $\preceq$, and Fix(f) (resp. Per(f)) denotes the set of all fixed points (resp. periodic) points of a map $f : X \to X$, respectively.

The following is our strengthened version of the Caristi fixed point theorem given in [28]:

Theorem B. *If* (X, d) *is a complete metric space and* $\varphi : X \to \mathbb{R}^+$ *lower semicontinuous, then in the Brøndsted order* ($x \preccurlyeq y$ *iff* $d(x, y) \leq \varphi(x) - \varphi(y)$)*, every progressive map* $f : X \to X$ (*that is,* $x \preccurlyeq f(x)$ *for all* $x \in X$) *satisfies*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preccurlyeq) \neq \emptyset.$$

In [30], we showed that Theorem A(γ) implies the following dual to the Caristi fixed point theorem:

Theorem C. *Let* $(X, \preccurlyeq)$ *be a partially ordered complete metric space, and a function* $\varphi : X \to \mathbb{R}^+$ *be upper semicontinuous and bounded from above in the Brøndsted order. Then every anti-progressive map* $f : X \to X$ (*that is,* $f(x) \preccurlyeq x$ *for all* $x \in X$) *satisfies*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preccurlyeq) \neq \emptyset.$$

Motivated by such results, in [29,30], we obtained various forms of Minimal Element Principles and applied them to known or new works related to minimality. We were based on our 2023 Metatheorem, a prototype of Minimal Element Principles, and the Brøndsted-Jachymski Principle for the minimality. These are duals of the corresponding related results on the maximality.

In the following, Theorems B and C can be improved by adopting new extended concepts of the lower (or upper) semi-continuity.

4. Variants due to Chen-Cho-Yang [5]

Kirk-Saliga [15] in 2001 and Chen-Cho-Yang [5] in 2002 introduced the following concept of lower semicontinuity from above:

Definition 4.1. [5] Let X be a metric space. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower semicontinuous from above* if, for any point $x \in X$, $x_n \to x$ as $n \to \infty$ and $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq \cdots$ imply $\lim_{n\to\infty} f(x_n) \geq f(x)$.

Obviously, the usual lower semicontinuity implies lower semicontinuity from above, but the converse does not hold. In fact, Chen-Cho-Yang [5] gave an example of a function which is lower semicontinuous from above at a point, but not lower semicontinuous at that point.

Similarly, Lin-Du [16] defined the following motivated by [5]:

Definition 4.1.* Let X be a topological space. A function $f : X \to \overline{\mathbb{R}}$ is said to be *upper semicontinuous from below* if, for any point $x \in X$, $x_n \to x$ as $n \to \infty$ and $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_n) \leq \cdots$ imply $\lim_{n\to\infty} f(x_n) \leq f(x)$.

Chen-Cho-Yang [5] showed that the Weierstrass theorem, Ekeland's variational principle, and Caristi's fixed point theorem hold for lower semicontinuity from above.

Proposition 4.2. [5] *Let D be a compact subset of a metric space X and a function $\phi : D \to \overline{\mathbb{R}}$ be lower semicontinuous from above and bounded from below. Then there exists $x_0 \in D$ such that $\phi(x_0) = \inf_{x\in D} \phi(x)$.*

Similarly, we can obtain the following:

Proposition 4.2.* *Let D be a compact subset of a metric space X and a function $\phi : D \to \overline{\mathbb{R}}$ be upper semicontinuous from below and bounded from above. Then there exists $x_0 \in D$ such that $\phi(x_0) = \sup_{x\in D} \phi(x)$.*

Recall the following in [5]:

Theorem 4.3. (Caristi's Fixed Point Theorem) *Let $(D,\ d)$ be a complete metric space and a function $\phi : D \to \mathbb{R}^+$ be lower semi-continuous from above. Suppose that a mapping $f : D \to D$ satisfies the following:*

$$d(x, f(x)) \leq \phi(x) - \phi(f(x)) \quad \text{for all } x \in D.$$

Then there exists $x_0 \in D$ such that $f(x_0) = x_0$.

Note that (D, d) can be made into a partially ordered set by defining

$$x \preceq y \iff d(x, y) \leq \phi(x) - \phi(y)$$

for $x,\ y \in D$.

Recently, we gave an elementary proof of Theorem 4.3 in [33] independent from Zorn's Lemma or its equivalents. For the earlier proofs of the Caristi theorem, see Kirk [14].

Then we can apply Theorem A as follows:

Theorem D. *Let (D, d) be a metric space and $\phi : D \to \overline{\mathbb{R}}$ be upper semi-continuous from below and bounded from above (resp. lower semi-continuous from above and bounded from below).*

Then the following statements are equivalent:

(0) *(D, d) is complete.*

(α) *There exists a maximal (resp. minimal) element $v \in D$; that is, $v \not\preceq w$ (resp. $w \not\preceq v$) for any $w \in D\backslash\{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : D \to D$ such that, for any $x \in D$ with $x \neq f(x)$, there exists a $y \in D\backslash\{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$), then $\mathfrak{F}$ has a common fixed element $v \in D$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(γ) *If* $\mathfrak{F}$ *is a family of maps* $f : D \to D$ *satisfying* $x \preceq f(x)$ *(resp.* $f(x) \preceq x$*) for all* $x \in D$ *with* $x \neq f(x)$*, then* $\mathfrak{F}$ *has a common fixed element* $v \in D$*, that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$*.*

(δ) *Let* $\mathfrak{F}$ *be a family of multimaps* $F : D \multimap D$ *such that, for any* $x \in D \backslash F(x)$ *there exists* $y \in D\backslash\{x\}$ *satisfying* $x \preceq y$ *(resp.* $y \preceq x$*). Then* $\mathfrak{F}$ *has a common fixed element* $v \in D$*, that is,* $v \in F(v)$ *for all* $F \in \mathfrak{F}$*.*

(ϵ) *If* $\mathfrak{F}$ *is a family of multimaps* $F : D \multimap D$ *such that* $x \preceq y$ *(resp.* $y \preceq x$*) holds for any* $x \in D$ *and any* $y \in F(x)\backslash\{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in D$*, that is,* $\{v\} = F(v)$ *for all* $F \in \mathfrak{F}$*.*

(η) *If* Y *is a subset of* D *such that, for each* $x \in D\backslash Y$*, there exists a* $z \in D\backslash\{x\}$ *satisfying* $x \preceq z$ *(resp.* $z \preceq x$*), then there exists a* $v \in D \cap Y = Y$*.*

Remark 4.4. Recall that Kirk [13] in 1976 showed the metric completeness (0) is characterized by the Caristi theorem (γ1). Hence (0) $\iff$ (γ1) in Theorem D gives a simple proof of the Caristi theorem. Moreover, Park [17] in 1984 showed that seven statements are equivalent to metric completeness. These are essential properties of complete metric spaces and not for to check only whether a metric space is complete or not.

By applying Theorem D, we can improve or strengthen well-known theorems as follows:

Remark 4.5. (1) Theorem D(α) shows that the Weierstrass Theorem in [5] can be extended to complete subsets instead of compact subsets.

(2) In view of Theorem D, we can replace the lower (resp. upper) semicontinuity in Theorems B and C by lower semicontinuity from above (resp. upper semicontinuity from below) with more strengthend conclusions.

(3) Consider the following particular but equivalent case of (γ):

(γ1) *If a map* $f : D \to D$ *satisfying* $x \preceq f(x)$ *(resp.* $f(x) \preceq x$*) for all* $x \in D$ *with* $x \neq f(x)$*, then* f *has a fixed element* $v \in D$*, that is,* $v = f(v)$*.*

Here $f(x) \preceq x$ means the Caristi condition

$$f(x) \preceq x \iff d(x, f(x)) \leq \phi(x) - \phi(f(x)).$$

Therefore (γ1) holds by the Caristi theorem, and so does Theorem C by Theorem A.

In our previous works, we gave several equivalent formulations of the Caristi theorem and their origins. See [24, 26–31].

The origin of Theorem D is the following due to Brunner [3] in 1987:

Corollary 4.6. [3] *If* (X, ρ) *is a complete metric space and* $\varphi : X \to \mathbb{R}$ *is bounded above and upper semi-continuous, then in the Brøndsted order* ($x \preccurlyeq y$ *iff* $\rho(x, y) \leq \varphi(y) - \varphi(x)$)*, there is a maximal element.*

5. Extensions due to Lin-Du [16]

In 2008, Lin and Du [16] introduced the τ-function which generalizes the w-distance due to Takahashi [35]. They established a generalized Ekeland's variational principle in the setting of lower semi-continuity from above and τ-functions. As applications of their Ekeland variational principle, they derived generalized Caristi's (common) fixed point theorems, a generalized Takahashi's nonconvex minimization theorem, a nonconvex minimax theorem, a nonconvex equilibrium theorem and a generalized flower petal theorem for l.s.c. from above functions or l.s.c. functions in the complete metric spaces. They also proved that these theorems also imply their Ekeland variational principle.

Throughout this section, unless specified otherwise, (X, d) is a metric space and $\varphi : (-\infty, \infty] \to (0, \infty)$ is a nondecreasing function. A function f is said to be *proper* if $f \not\equiv \infty$.

The following definition of τ-*function* is different from the definition of τ-distance, it is a generalization of w-distance in [12].

Definition 5.1. [16] A function $p : X \times X \to [0, \infty)$ is called a τ-*function* if the following conditions hold:

(τ1) for all $x,\ y,\ z \in X,\ p(x, z) \leq p(x, y) + p(y, z)$;

(τ2) if $x \in X$ and $\{x_n\}$ in X with $\lim_{n\to\infty} y_n = y$ and $p(x, y_n) \leq M$ for some $M = M(x) > 0$, then $p(x, y) \leq M$;

(τ3) for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, and if there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} p(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$;

(τ4) for $x,\ y,\ z \in X,\ p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$.

It is known [12] that if p is a w-distance on $X \times X$, then for all $x,\ y,\ z \in X,\ p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$.

Remark 5.2. Every w-distance, introduced and studied by Kada et al. [12], is a τ-function.

After such preparation, Lin and Du [16] in 2006 obtained the following generalization of Ekeland's variational principle for l.s.c. from above functions:

Theorem 5.3. [16] *Let X be a complete metric space, $g : X \to (-\infty, \infty]$ be a proper l.s.c. function from above and bounded from below, and p be a τ-function on $X \times X$. Then there exists $v \in X$ such that $p(v, x) > \varphi(g(v))(g(v) - g(x))$ for all $x \in X\backslash\{v\}$.*

Now we apply our Metatheorem to Theorem 5.3:

Theorem E. *Let (X, d) be a metric space, $g : X \to (-\infty, \infty]$ be a proper l.s.c. function from above and bounded from below and p be a τ-function on $X \times X$.*

Then the following equivalent statements hold:

(0) *(X, d) is complete.*

(α) *There exists an element $v \in X$ such that $p(v, w) > \varphi(g(v))(g(v) - g(w))$ for any $w \in X\backslash\{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ such that, for any $x \in X$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $p(x, y) \leq \varphi(g(x))(g(x) - g(y))$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ satisfying $p(x, f(x)) \leq \varphi(g(x))(g(x) - g(f(x)))$ for all $x \in X$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(δ) *Let $\mathfrak{F}$ be a family of multimaps $F : X \multimap X$ such that, for any $x \in X\backslash F(x)$, there exists $y \in X\backslash\{x\}$ satisfying $p(x, y) \leq \varphi(g(x))(g(x) - g(y))$. Then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v \in F(v)$ for all $F \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $F : X \multimap X$ such that $p(x, y) \leq \varphi(g(x))(g(x) - g(y))$ holds for any $x \in X$ and any $y \in F(x)\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in X$, that is, $\{v\} = F(v)$ for all $F \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that, for each $x \in X\backslash Y$, there exists a $z \in X\backslash\{x\}$ satisfying $p(x, y) \leq \varphi(g(x))(g(x) - g(y))$, then there exists a $v \in X \cap Y = Y$.*

Proof. Recall that (0) $\Longrightarrow$ (α) holds by Theorem 5.3. Note that (α) $-$ (η) are logically equivalent by Theorem A. Conversely, (γ) shows

$$d(x, f(x)) \leq g(x) - g(f(x)) \text{ for } p = d,$$

with the constant function $\varphi = 1$. Hence (γ1) implies the Caristi fixed point theorem. Now by Kirk's characterization [13] of metric completeness, we have (γ) $\Longrightarrow$ (0). This completes our proof. □

Note that $p(v, v) = 0$ throughout in Theorem E.

As a first application of their generalized Ekeland variational principle, Lin-Du [16] derived the following generalization of Caristi's theorem for a family of multimaps:

Theorem 5.4. [16] *Let (X, d) be complete, p and g be the same as in Theorem 5.3. Let I be any index set and for each $i \in I$, let $T_i : X \multimap X$ be a multimap such that for each $x \in X$, there exists $y = y(x, i) \in T_i(x)$ with*

$$p(x, y) \leq \varphi(g(x))(g(x) - g(y)).$$

Then there exists $v \in X$ such that $v \in \bigcap_{i\in I} T_i(v)$, that is, the family of multimaps $\{T_i\}_{i\in I}$ has a common fixed point in X, and $p(v, v) = 0$.

This is just $(0) \Longrightarrow (\gamma)$ in Theorem E.

In [16], Corollary 2.1 is an equivalent form of Theorem 2.2 for a family of single-valued maps. Therefore, we have Theorem E$(\epsilon) \Longrightarrow$ Theorem 2.2 [16] $\Longrightarrow$ Corollary 2.1 [16] $\Longrightarrow$ Theorem E(γ). Hence they are all equivalent.

In [30], we showed our equivalent formulations imply many new facts.

6. Extensions of Cobzaş [7]

Our Metatheorem was originated from the Ekeland Principle which has equivalent forms like the Caristi fixed point theorem, Takahashi's minimization theorem, and many others. Our recent applications of Metatheorem to those theorems were given in [23-28].

Recently, Cobzaş [7] in 2022 gave versions of Ekeland, Takahashi, Caristi Principles in preordered quasi-metric spaces, the equivalence to some completeness results for the underlying quasi-metric spaces.

For convenience, Cobzaş [7] formulated these three principles as follows:

Theorem 6.1. (Ekeland, Takahashi and Caristi principles) *Let (X, d) be a complete metric space and $\varphi : X \to \mathbb{R} \cup \{\infty\}$ a proper bounded below l.s.c. function. Then the following statements hold:*

[wEk] *There exists $z \in \operatorname{dom} \varphi$ such that $\varphi(z) < \varphi(x) + d(x, z)$ for all $x \in X \backslash \{z\}$.*

[Tak] *If for every $x \in \operatorname{dom} \varphi$ with $\varphi(x) > \inf \varphi(X)$ there exists an element $y \in \operatorname{dom} \varphi \backslash \{\mathrm{x}\}$ such that $\varphi(y) + d(x, y) \leq \varphi(x)$, then φ attains its minimum on X, i.e., there exists $z \in \operatorname{dom} \varphi$ such that $\varphi(z) = \inf \varphi(X)$.*

[Car] *If the mapping $f : X \to X$ satisfies $d(f(x), x) + \varphi(f(x)) \leq \varphi(x)$ for all $x \in \operatorname{dom} \varphi$, then f has a fixed point in $\operatorname{dom} \varphi$, i.e., there exists $z \in \operatorname{dom} \varphi$ such that $f(z) = z$.*

Here [wEk] means the weak Ekeland principle, [Tak] the Takahashi principle, and [Car] the Caristi fixed point theorem. Also we denote $\operatorname{dom} \varphi = \{\mathrm{x} \in \mathrm{X} : -\infty < \varphi(\mathrm{x}) < \infty\}$.

As we have seen in previous sections, the following holds:

Theorem 6.1.* *Let (X, d) be a complete metric space and a proper function $\phi : X \to \overline{\mathbb{R}}$ be lower semi-continuous from above and bounded from below (resp. upper semi-continuous from below and bounded from above).*

Then the three statements in Theorem 6.1 hold.

Here ϕ is proper means $\phi \not\equiv \infty$ and $\phi \not\equiv -\infty$.

Motivated by Example 4.1 of [6], we derive the following from Theorem D:

Theorem F. *Let (X, d) be a metric space and a proper function $\varphi : X \to \overline{\mathbb{R}}$ be l.s.c. from above and bounded from below (resp. u.s.c. from below and bounded from above). Let $A = \operatorname{dom} \varphi$.*

Then the following statements are equivalent:

(0) (X, d) *is complete.*

(α) *There exists a maximal (resp. minimal) element $v \in A$ such that*

$$d(v, w) > \varphi(v) - \varphi(w) \quad (resp.\ d(v, w) > \varphi(w) - \varphi(v))$$

for any $w \in X\backslash\{v\}$. [wEk]

(β) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying*

$$d(x, y) \leq \varphi(x) - \varphi(y) \quad (resp.\ d(x, y) \leq \varphi(y) - \varphi(x)),$$

then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)) \quad (resp.\ d(x, f(x)) \leq \varphi(f(x)) - \varphi(x))$$

for all $x \in A\backslash\{f(x)\}$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$. [Car]

(δ) *Let $\mathfrak{F}$ be a family of multimaps $T : A \multimap X$ such that, for any $x \in A\backslash T(x)$, there exists $y \in X\backslash\{x\}$ satisfying*

$$d(x, y) \leq \varphi(x) - \varphi(y) \quad (resp.\ d(x, y) \leq \varphi(y) - \varphi(x)),$$

then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in T(v)$ for all $T \in \mathfrak{F}$.

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $T : A \multimap X$ such that*

$$d(x, y) \leq \varphi(x) - \varphi(y) \quad (resp.\ d(x, y) \leq \varphi(y) - \varphi(x))$$

holds for any $x \in A$ and any $y \in T(x)\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = T(v)$ for all $T \in \mathfrak{F}$.

(η) *If Y is a subset of X such that for each $x \in A\backslash Y$ there exists a $z \in X\backslash\{x\}$ satisfying*

$$d(x, z) \leq \varphi(x) - \varphi(z) \quad (resp.\ d(x, z) \leq \varphi(z) - \varphi(x)),$$

then there exists a $v \in A \cap Y$.

In this theorem, note that "(α) $\Longleftrightarrow$ [wEk] with its dual form," and "(γ1) $\Longleftrightarrow$ [Car] with the first form" of the following particular form of γ:

(γ1) *If $f : X \to X$ is a map such that*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)) \quad (resp.\ d(x, f(x)) \leq \varphi(f(x)) - \varphi(x))$$

for any $x \in X$, then f has a fixed element $v \in X$, that is, $v = f(v)$.

Moreover, we show an application of Theorem A*(ζ1) as follows:

Theorem 6.2. *Theorem A*(ζ1) implies the Takahashi principle and its dual form.*

Proof. For any $x \in A$ with $\varphi(x) > \inf \varphi(X)$,

$$T(x) = \{y \in X : \varphi(x) > \varphi(y)\}.$$

Suppose that for each $x \in A$, there exists $y \in A\backslash\{x\} \subset X\backslash\{x\}$ such that $d(x, y) \leq \varphi(x) - \varphi(y)$. Then, by ($\zeta$1), we have $v \in A$ such that $T(v) = \emptyset$, that is, $\varphi(v) \leq \varphi(y)$ for all $y \in X$, or $\varphi(v) = \inf \varphi(X)$.

For the dual, $T(x) = \{y \in X : \varphi(y) > \varphi(x)\}$ works. □

For some more details on Theorem 6.1, see Cobzaş [7]. From his own principle, Takahashi deduced Caristi's fixed point theorem, Ekeland's ε-variational principle, Nadler's fixed point theorem, etc. Hence they also follow from the maximality of Theorem A*.

From Theorem F$(\alpha), (\gamma)$ and the Brøndsted-Jachymski Principle, we have the following;

Theorem G. *Let (X, d) be a complete metric space and a proper function $\varphi : X \to \overline{\mathbb{R}}$ be l.s.c. from above and bounded from below (resp. u.s.c. from below and bounded from above). If $f : X \to X$ is a map such that*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)) \quad (resp.\ d(x, f(x)) \leq \varphi(f(x)) - \varphi(x))$$

for any $x \in X$. Then we have

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preccurlyeq) \neq \emptyset \quad (resp.\ \mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preccurlyeq) \neq \emptyset).$$

7. Equivalents of Metric Completeness

A metric space is complete if every Cauchy sequence converges, by definition. Typical examples are Euclidean spaces, Hilbert spaces, Banach spaces, and many others. In this article, we showed many equivalent statements to the completeness of metric spaces as in Theorems D, E, F. Such equivalencies can be extended to more than one hundred generalizations of metric spaces.

In this section, we recall some history related to the equivalences of metric completeness as follows:

(1) Kirk [13] in 1976: Metric completeness is equivalent to the Caristi fixed point theorem.

(2) Park [17] in 1984: Historically well-known equivalences of metric completeness were collected. This extends [13]. Some editor comments like "Who dare use this kind of things to check the completeness?" Such characterizations show important properties of complete metric spaces, not only for to check their completeness.

(3) Takahashi [34] in 1991 showed

$$\text{Metric completenes} \Longrightarrow \text{Takahashi Principle} \Longrightarrow \text{Caristi theorem.}$$

Hence his principle is equivalent to metric completeness. This was also given in Theorem F.

(4) Ekeland [8-10] in 1972-77: His principle assumes completeness and implies the Caristi theorem. Hence it is equivalent to the metric completeness. Ansari [1] in 2014 proved the Ekeland principle implies completeness.

(5) Cobzaş [6] in 2020: Abstract. "The aim of this paper is to present various circumstances in which fixed point results imply completeness. For metric spaces, this is the case of Ekeland's variational principle and of its equivalent, Caristi's fixed point theorem. Other fixed point results having this property will also be presented in metric spaces, in quasi-metric spaces, and in partial metric spaces. A discussion on topology and order and on fixed points in ordered structures and their completeness properties is included as well."

This very informative article contains the contents of 201 articles on completeness of metric spaces and only some of generalized metric spaces.

8. Conclusion

Since the Ekeland variational principle in 1972 and the Caristi fixed point theorem in 1976 appeared, more than one thousand related papers were published. Most of them are related to certain maximal element principles in Nonlinear Analysis and belong to Ordered Fixed Point Theory [28].

As we have seen in Theorem A, the maximal (or minimal) element v in certain pre-ordered sets can be fixed point, stationary point, common fixed point, common stationary point, etc. of a family of maps or multimaps, and conversely. Some authors seem to be not recognized this fact yet.

In many fields of mathematical sciences, there are plentiful number of theorems concerning maximal points or various fixed points that can be applicable our Metatheorem and Theorem A. Some of such theorems can be seen in our previous works and the present article. Therefore, our Metatheorem and Theorem A are convenient machines to expand our knowledge easily. In this article we presented only small number of relatively old and well-known examples.

References

[1] Q.H. Ansari, Ekeland's variational principle and its extensions with applications, S. Almezel et al. (eds.), Topics in Fixed Point Theory, Chap. 3, 65–100, Springer Inter. Publ. Switzerland (2014) DOI 10.1007/978-3-319-01586-6-1

[2] Z. Boros, M. Iqbal, A. Száz, A relational improvement of a true particular case of Fierro's maximality theorem, (2022), manuscript.

[3] N. Brunner, Topologische Maximalprinzippen, Zeitschr. f. math. Logik und Grundlagen d. Math. 33 (1987), 135–139.

[4] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Trans. Amer. Math. Soc. 215 (1976), 241–251.

[5] Y. Chen, Y.J. Cho, L. Yang, Note on the results with lower semicontinuity. Bull. Korean Math. Soc. 39 (2002), 535–541.

[6] S. Cobzaş, Fixed points and completeness in metric and generalized metric spaces, J. Math. Sciences 250(3) (2020), 475–535. DOI 10.1007/s10958-020-05027-1

[7] S. Cobzaş, Ekeland, Takahashi and Caristi principles in preordered quasi-metric spaces, Quaestiones Mathematicae (2022), 1–22. DOI: 10.2989/16073606.2022.2042417

[8] I. Ekeland, Sur les problèmes variationnels, C.R. Acad. Sci. Paris 275 (1972), 1057–1059; 276 (1973), 1347–1348.

[9] I. Ekeland, On the variational principle, J. Math. Anal. Appl. 47 (1974), 324–353.

[10] I. Ekeland, Nonconvex minimization problems, Bull. Amer. Math. Soc. (N.S.) 1(3) (1979), 443–474.

[11] R. Fierro, Maximality, fixed points and variational principles for mappings on quasi-uniform spaces, Filomat 31:16 (2017), 5345–5355. https://doi.org/10.2298/FIL1716345F

[12] O. Kada, T. Suzuki, W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Math. Jan. 44 (1996), 381–391.

[13] W.A. Kirk, Caristi's fixed point theorem and metric convexity, Colloq. Math. XXXVI (1976), 81–86.

[14] W.A. Kirk, Contraction mappings and extensions, Chapter 1, Handbook of Metric Fixed Point Theory (W.A. Kirk and B. Sims, ed.), Kluwer Academic Publ. (2001), 1–34.

[15] W.A. Kirk, L. M. Saliga, The Brézis-Browder order principle and extensions of Caristi's theorem, Nonlinear Anal. 47 (2001), 2765-2778.

[16] L.-J. Lin, W.-S. Du, Ekeland's variational principle, minimax theorems and existence of nonconvex equilibria in complete metric spaces, J. Math. Anal. Appl. 323 (2006), 360–370.

[17] S. Park, Characterizations of metric completeness, Colloq. Math. 49 (1984), 21–26.

[18] S. Park, Some applications of Ekeland's variational principle to fixed point theory, Approximation Theory and Applications (S.P. Singh, ed.), Pitman Res. Notes Math. 133 (1985), 159–172.

[19] S. Park, Equivalent formulations of Ekeland's variational principle for approximate solutions of minimization problems and their applications, Operator Equations and Fixed Point Theorems (S.P. Singh et al., eds.), MSRI-Korea Publ. **1** (1986), pp.55–68.

[20] S. Park, Countable compactness, l.s.c. functions, and fixed points, J. Korean Math. Soc. 23 (1986), 61–66.

[21] S. Park, Equivalent formulations of Zorn's lemma and other maximum principles, J. Korea Soc. Math. Educ. **25** (1987), 19–24.

[22] S. Park, Partial orders and metric completeness, Proc. Coll. Natur. Sci., SNU 12 (1987), 11–17.

[23] S. Park, On generalizations of the Ekeland-type variational principles, Nonlinear Anal. 39 (2000), 881–889.

[24] S. Park, Equivalents of various maximum principles, Results in Nonlinear Analysis 5(2) (2022), 169–174.

[25] S. Park, Applications of various maximum principles, J. Fixed Point Theory (2022), 2022-3, 1–23. ISSN:2052–5338.

[26] S. Park, Equivalents of maximum principles for several spaces, Top. Algebra Appl. 10 (2022), 68–76. 10.1515/taa-2022-0113

[27] S. Park, Equivalents of ordered fixed point theorems of Kirk, Caristi, Nadler, Banach, and others, Adv. Th. Nonlinear Anal. Appl. 6(4) (2022), 420–432.

[28] S. Park, Foundations of ordered fixed point theory, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. 61(2) (2022), 1–51.

[29] S. Park, Applications of several minimum principles, Adv. Th. Nonlinear Anal. Appl. 7(1) (2023), 52–60. ISSN: 2587-2648

[30] S. Park, Equivalents of certain minimum principles, Letters Nonlinear Anal. Appl. 1(1) (2023), 1–11.

[31] S. Park, Variations of ordered fixed point theorems, Linear Nonlinear Anal. 8(3) (2022), 225–237.

[32] S. Park, Equivalents of some ordered fixed point theorems, J. Advances Math. Comp, Sci. 38(1) (2023), 52–67.

[33] S. Park, Remarks on the Metatheorem in Ordered Fixed Point Theory, Advanced Mathematical Analysis and Its Applications (Edited by P. Debnath, D. F. M. Torres, Y. J. Cho) CRC Press (2023), to appear.

[34] W. Takahashi, Existence theorems generalizing fixed point theorems for multivalued mappings, Fixed point theory and applications (Marseille, 1989), Pitman Res. Notes Math. Ser. 252, Longman Sci. Tech., Harlow, (1991), 397–406.